The Annals of Probability
1989, Vol. 17, No. 1, 144–150

SMOOTHNESS OF THE CONVEX HULL OF PLANAR BROWNIAN MOTION

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In this article we prove that for each $t > 0$, almost surely $\partial C(t)$, the boundary of the convex hull of two dimensional Brownian motion up to time t, is a C^1 curve in the plane. We also prove that if η is a modulus of continuity such that $x\eta(x)$ is convex and $\int_0^1 \eta(x)\,dx/x < \infty$ then for each $t > 0$, almost surely $\partial C(t)$ is not a $C^{1,\eta}$ curve in the plane.

1. Introduction. Let $B = \{B(t), \mathscr{F}_t, P; t \geq 0\}$ be a standard Brownian motion in R^2 starting from the origin and let $C(t) = \operatorname{conv}\{B(s); 0 \leq s \leq t\}$ be the closed convex hull of B up to time t. Recent results of Shimura (1984, 1985) and Burdzy (1985) imply that for any angle $\alpha \in (\pi/2, \pi)$, there are random times τ such that $C(\tau)$ has corners of opening α. In fact, from the work of Le Gall (1987) and Evans (1985) the Hausdorff dimension of the set of time points t such that $B(t)$ visits a corner of $C(t)$ of opening α is known to be $1 - \pi/2\alpha$ almost surely. Some results by Varadhan–Williams (1985) and Williams (1985a, b, 1986) on reflected Brownian motion in wedges are also relevant to this topic. While they do not directly concern the so-called *cone points*, some of these results have been used by Le Gall in his work cited above.

All this is in contrast with the observation of Lévy (1948) that for each fixed t, almost surely $C(t)$ has no corners at all, and so $\partial C(t)$ is a C^1 curve in the plane. A proof of this fact has been supplied in El Bachir (1983). In this article we prove this fact anew, namely

$$\forall\, t > 0, \qquad P\big[\partial C(t) \text{ is a } C^1 \text{ curve}\big] = 1.$$

In the negative direction, we will prove that if η is a modulus of continuity (defined below) such that $x\eta(x)$ is convex then

$$\forall\, t > 0, \qquad P\big[\partial C(t) \text{ is a } C^{1,\eta} \text{ curve}\big] = 0 \quad \text{if } \int_0^1 \eta(x)\frac{dx}{x} < \infty.$$

In particular, the slope of the tangent to $\partial C(t)$ almost surely does not vary in a Hölder continuous fashion. Our method is to consider curves based at certain

Received October 1987; revised May 1988.

[1]Part of this research was carried out while the author was a visitor at the Institute of Mathematics and Its Applications in Minneapolis, with support provided by the National Science Foundation.

[2]Research supported in part by NSF Grant DMS-85-03332.

[3]Research supported in part by NSF Grant DMS-86-00233.

[4]Research supported in part by an NSERC Canada grant.

AMS 1980 *subject classification.* Primary 60G65.

Key words and phrases. Planar Brownian motion, convex hull, excursion.

extreme points of $C(t)$ which are not hit immediately by excursions of the Brownian path away from its convex hull.

2. Main result. Fix $t > 0$ and for each angle $\theta \in [0, 2\pi]$ let B^θ and $B^{\theta'}$ be the components of the planar Brownian motion B in the directions θ and $\theta' = \theta + \pi/2$. It is clear that B^θ and $B^{\theta'}$ are independent one dimensional Brownian motions. Let

$$M^\theta(t) = \sup_{0 \le s \le t} B^\theta(s)$$

and define

$$\sigma = \sup\{s < t\colon B^\theta(s) = M^\theta(t)\}.$$

Let $D(\theta)$ be the support line to $C(t)$ which is perpendicular to the ray $\{re^{i\theta}; r \ge 0\}$. Then $\partial C(t) \cap D(\theta)$ contains at least one and at most two extreme points of the convex set $C(t)$. By Rockefellar [(1970), Corollary 18.3.1] the point $B(\sigma)$ is one of them. We denote this point by $Q(\theta)$.

Consider processes in R^2 defined as follows:

$$W^+(s) = \{B^{\theta'}(\sigma) - B^{\theta'}(\sigma + s), B^\theta(\sigma) - B^\theta(\sigma + s)\}, \qquad 0 \le s \le t - \sigma;$$

$$W^-(s) = \{B^{\theta'}(\sigma) - B^{\theta'}(\sigma - s), B^\theta(\sigma) - B^\theta(\sigma - s)\}, \qquad 0 \le s \le \sigma.$$

Evidently $W^\pm$ are the excursions of B away from $C(t)$ at $Q(\theta)$ as viewed in the coordinate system in which $C(t)$ lies in the upper-half plane and $Q(\theta)$ is at the origin. Finally introduce the process

$$Z(s) = \{-B^{\theta'}(s), |B^\theta(s)|\}, \qquad s \ge 0,$$

and the time $S = \sup\{s < t\colon |B^\theta(s)| = 0\}$. Then

$$e_t(s) = Z(S + s) - Z(S), \qquad 0 \le s \le t - S,$$

is the initial fragment of the excursion of Z above the abscissa which straddles t.

REMARK. Itô and McKean [(1965), Problem 2, page 44] say there are at most two time points in $[0, t]$ such that B^θ attains its maximum $M^\theta(t)$. Using the fact that $P[B^\theta(t) = M^\theta(t)] = 0$, the same argument there shows that *for fixed* $t > 0$, almost surely there is only one such point. This remark will be used in Lemma 1 and Lemma 2 below.

LEMMA 1. *For fixed* $t > 0$, *the processes* W^+, W^- *and* e_t *are identical in distribution.*

PROOF. Define

$$X(s) = \{X^1(s), X^2(s)\} = \{B^{\theta'}(t - s) - B^{\theta'}(t), B^\theta(t - s) - B^\theta(t)\},$$
$$0 \le s \le t.$$

Then X is a standard Brownian motion up to time t. Let

$$\tau = \sup\Big\{s < t\colon X^2(s) = \max_{0 \le u \le t} X^2(u)\Big\}.$$

Then it is clear that

$$\{B^{\theta'}(s), B^{\theta}(s); 0 \le s \le \sigma\} \equiv \{X(s); 0 \le s \le \tau\}.$$

(Here $\equiv$ means "equivalent in law".)

For a continuous path $\omega = (\omega^1, \omega^2)\colon R^+ \to R^2$, let

$$T(\omega) = \Big\{s \le t\colon \omega^2(s) = \max_{0 \le u \le t} \omega^2(s)\Big\}.$$

Then we have $\sigma = t - \min T(X)$ and $\tau = \max T(X)$. But by the remark preceding this lemma,

$$P\{T(X) \text{ is a singleton set}\} = 1,$$

so that $\sigma = t - \tau$ almost surely. Using this fact we have

$$\begin{aligned}
&\{W^-(s); 0 \le s \le \sigma\}\\
&\quad \equiv \{X(\tau) - X(\tau - s); 0 \le s \le \tau\}\\
&\quad = \{B^{\theta'}(t-\tau) - B^{\theta'}(t - \tau + s), B^{\theta}(t-\tau) - B^{\theta}(t-\tau+s); 0 \le s \le \tau\}\\
&\quad = \{B^{\theta'}(\sigma) - B^{\theta'}(\sigma + s), B^{\theta}(\sigma) - B^{\theta}(\sigma + s); 0 \le s \le t - \sigma\}\\
&\quad = \{W^+(\sigma); 0 \le s \le t - \sigma\}.
\end{aligned}$$

On the other hand, it is well known that

$$\{Z(s); s \ge 0\} \equiv \{B^{\theta'}(s), M^{\theta}(s) - B^{\theta}(s); s \ge 0\}.$$

Thus we have

$$\begin{aligned}
&\{W^+(s); 0 \le s \le t - \sigma\}\\
&\quad = \{B^{\theta'}(\sigma) - B^{\theta'}(\sigma + s), M^{\theta}(\sigma) - B^{\theta}(\sigma + s); 0 \le s \le t - \sigma\}\\
&\quad \equiv \{Z(S + s) - Z(S); 0 \le s \le t - S\}\\
&\quad = \{e_t(s); 0 \le s \le t - S\},
\end{aligned}$$

and the lemma is proved. □

LEMMA 2. *Fix $t > 0$ and $\theta \in [0, 2\pi]$. For each $\varepsilon > 0$ the following event occurs with probability* 1: *There exists a neighborhood $U \subset R^2$ of $Q(\theta)$ such that if $0 \le s \le t$ and $B(s) \in U$ then $|\sigma - s| \le \varepsilon$.*

PROOF. Let

$$\delta(\varepsilon) = \inf_{\substack{|\sigma - s| \ge \varepsilon\\ 0 \le s \le t}} \|Q(\theta) - B(s)\|.$$

By the remark before Lemma 1, σ is the unique point $s \in [0, t]$ such that $B(s) = M^{\theta}(t)$. This implies that $\delta(\varepsilon)$ is strictly positive for positive ε, since $B(s)$ is continuous. We can take U to be the ball of radius $\delta(\varepsilon)$ centered at $Q(\theta)$. □

Lemma 3. *Let f_1, f_2 and f_3 be finite, positive convex functions such that $f_i(0) = 0$, $i = 1,2,3$ and $f_1(x) \le f_2(x) \le f_3(x)$ on $|x| \le R$ for some $R > 0$. If $0 < |x| < R/2$, then we have*

$$\left|\frac{f_1(x/2)}{x}\right| \le |D^{\pm} f_2(x)| \le \left|\frac{f_3(2x)}{x}\right|,$$

where $D^{\pm}$ denote one-sided derivatives.

Proof. First consider the case $x > 0$. By convexity we have

$$0 \le D^{\pm} f_2(x) \le \inf_{x<u<R} \frac{f_2(u) - f_2(x)}{u - x} \le \frac{f_2(2x) - f_2(x)}{x} \le \frac{f_3(2x)}{x}.$$

Similarly one proves $D^{\pm} f_1(x/2) \le 2 f_2(x)/x$. On the other hand, for any positive convex function f vanishing at 0, one has $f(x)/x \le D^{\pm} f(x)$. It follows that

$$\frac{f_1(x/2)}{x} \le \frac{1}{2} D^{\pm} f_1\left(\frac{x}{2}\right) \le \frac{f_2(x)}{x} \le D^{\pm} f_2(x) \le \frac{f_3(2x)}{x}.$$

The case $x < 0$ can be argued symmetrically. The lemma is proved. □

We now turn to the proof of the main result. Consider the rectangular coordinate system (x, y) in which $C(t)$ lies in the upper half plane and $Q(\theta)$ is at the origin. Since $C(t)$ is almost surely not a line segment, it is not difficult to see that we can choose the coordinates (x, y) suitably so that $\partial C(t)$ lies on the upper-half plane, $Q(\theta)$ is at the origin and the boundary $\partial C(t)$ is locally the graph of some function, say f_θ. Thus f_θ: $R \to R$ is a random, positive convex function with $f_\theta(0) = 0$ whose graph, as a subset of R^2 coincides near 0 with a portion of $\partial C(t)$. Now let η be a modulus of continuity, that is, $\eta(0) = 0$ and η is increasing.

Theorem 1. *Fix $t > 0$. Then:*

(a) *We have*

$$P\left[\lim_{|x|\to 0} D^{\pm} f_\theta(x) = D^{\pm} f_\theta(0) = 0 \text{ for all rationals } \theta \in [0, 2\pi]\right] = 1.$$

(b) *If $x\eta(x)$ is convex, then*

$$P\left[\liminf_{|x|\to 0} \frac{|D^{\pm} f_\theta(x)|}{\eta(|x|)} = \infty \text{ for all rationals } \theta \in [0, 2\pi]\right] = 1$$

if and only if $\int_0^1 \eta(x)\, dx/x$ converges.

Proof. Introduce the notation

$$\tau_A(V) = \inf\{s > 0 : V(s) \in A\}$$

for the first passage time of a process V to a set A. Let h: $R_+ \to R_+$ be a nonnegative, increasing function such that $h(0) = 0$ and $h(x)/x$ decreases to 0 as $x \downarrow 0$. Consider the sets

$$A_h = \{x \in R^2 : 0 \le |x_2| \le h(|x_1|)\}.$$

By Lemma 1 we have

$$P\{\tau_{A_h}(W^{\pm}) = 0\} = P\{\tau_{A_h}(e_t) = 0\}$$

and this latter probability is, by Corollary 3.1 of Burdzy (1986a), either 0 or 1 according as

$$\int_0^1 \frac{h(x)}{x^2}\,dx \quad \text{converges or diverges.}$$

This test can be applied to the functions $h(x) = \lambda x \eta(x)$, $\lambda > 0$ to the effect that if $\int_0^1 \eta(x)\,dx/x < \infty$ then almost surely the processes $W^{\pm}$ do not hit the set A_h until some strictly positive time. Now, assume h is convex. By Lemma 2, the above fact implies that $f_\theta(x) \ge \lambda|x|\eta(|x|)$ over some open interval containing $x = 0$. It is also clear from the same integral test (as applied, say, to $\tilde{h}(x) = \mu x/\log x^{-1}$, $\mu > 0$) that if $h(x) = \mu x$ for some $\mu > 0$ then $P[\tau_{A_h}(W^{\pm}) = 0] = 1$, i.e., there are times $t_n^{\pm} \downarrow 0$ such that $W^{\pm}(t_n^{\pm}) \in A_h$. In turn this implies there exist sequences of points $x_n^1 \downarrow 0$ and $x_n^2 \uparrow 0$ such that $f_\theta(x_n^i) \le \mu|x_n^i|$, $i = 1, 2$. Because $h(x) = \mu x$ is linear it must be true that $f_\theta(x) \le \mu|x|$ for all x in a neighborhood of the origin. Thus for some positive R, we have

$$\lambda|x|\eta(|x|) \le f_\theta(x) \le \mu|x| \quad \text{for } |x| \le R \text{ a.s.}$$

By Lemma 3 we have

$$\frac{\lambda}{2}\eta\left(\frac{|x|}{2}\right) \le |D^{\pm}f_\theta(x)| \le 2\mu \quad \text{for } 0 < |x| < R/2 \text{ a.s.}$$

In particular, by taking $\mu \downarrow 0$ through rational values, one sees that f_θ is differentiable at 0 almost surely and $D^{\pm}f_\theta(0) = 0$. Moreover, by taking $\lambda \uparrow \infty$ through integers, one sees that $|D^{\pm}f_\theta(x)|$ must vanish more slowly than $\eta(|x|)$ as $|x| \downarrow 0$. Thus part (a) and the sufficiency of part (b) have been proved. To see the necessity, note that if

$$\liminf_{|x| \to 0} \frac{|D^{\pm}f_\theta(x)|}{\eta(|x|)} = +\infty$$

almost surely then for all sufficiently small $x_0 > 0$, we have

$$D^{\pm}f_\theta(x) > \eta(x), \qquad 0 < x \le x_0,$$

which implies

$$g(x) \equiv \int_0^x \eta(s)\,ds < f_\theta(x).$$

Now

$$\int_\varepsilon^{x_0} \eta(u)\frac{du}{u} = \frac{g(u)}{u}\bigg|_\varepsilon^{x_0} + \int_\varepsilon^{x_0} g(u)\frac{du}{u^2}.$$

In our present situation, $P\{\tau_{A_\varepsilon}(e_t) = 0\} = 0$ so by the above integral test, $\int_\varepsilon^{x_0} g(u)\, du/u^2$ converges as $\varepsilon \downarrow 0$. Thus by the above equality, $\int_\varepsilon^x \eta(u)\, du/u$ converges as $\varepsilon \downarrow 0$. This finishes the proof. □

THEOREM 2. *Fix $t > 0$. Then*

(a) *We have*

$$P\left[\partial C(t) \text{ is a } C^1 \text{ curve in } R^2\right] = 1.$$

(b) *If $x\eta(x)$ is convex and $\int_0^1 \eta(x)\, dx/x < \infty$, then*

$$P\left[\partial C(t) \text{ is a } C^{1,\eta} \text{ curve in } R^2\right] = 0.$$

PROOF. Since $C(t)$ is convex, $\partial C(t)$ can fail to be C^1 only if it has a corner at some extreme point Q. However any such point has the form $Q(\theta)$ for all θ in some subinterval of $[0, 2\pi]$. Because this possibility is ruled out by Theorem 1(a), there can be no such Q. Part (a) follows. Part (b) is easily implied by Theorem 1(b).

3. Miscellaneous remarks. We have noticed some interesting results on the convex hull of Brownian motion, and we wish to note them briefly here.

(a) Let $p(t)$ be the perimeter of $C(t)$ and let $a(t)$ be its area. Then $E[p(t)] = \sqrt{8\pi t}$ [Takács (1980)] and $E[a(t)] = \pi t/2$ [El Bachir (1983)].

(b) A remarkable paper of Lévy (1955) contains a functional law of the iterated logarithm of which

$$P\left[\limsup_{t\to\infty} \frac{a(t)}{2t \log\log t} = \frac{1}{2\pi}\right] = 1$$

is a special case. We want to thank F. Knight for pointing out this reference to us.

(c) Let $e(t)$ be the set of extreme points of $C(t)$. Then almost surely, $e(t)$ is a closed set of Hausdorff dimension 0 and $\partial C(t) \setminus e(t)$ is a countable union of straight line segments [Evans (1985)].

(d) Burdzy (1986b) has recently announced interesting results about the geometry of the complement of the Brownian path which are closely related to the present work.

Acknowledgment. Thanks are due to an unnamed referee whose suggestions improved both the content and presentation of our work.

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