Heat Kernel on Noncomplete Manifolds

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ABSTRACT. Let M be a smooth manifolds and L a second order, nondegenerate elliptic operator on M. Let $p(t,x,y)$ be the heat kernel on M associated with the operator L. The basic asymptotic relation $\lim_{t\to 0} t \log p(t,x,y) = -d(x,y)^2/2$ (d is the Riemannian distance on M determined by L) may fail for noncomplete manifolds. We prove that the best condition under which this relation holds is $d(x,y) \le d(x,\infty) + d(y,\infty)$.

1. Introduction. Let M be a smooth manifold and L a second order, nondegenerate elliptic operator on M. The operator L determines a Riemannian metric d on M. Let Δ be the Laplace-Beltrami operator determined by the Riemannian metric. Then L can be written in the form $L = \frac{1}{2}\Delta + F$ with a smooth vector field F. The minimal heat kernel on M associated with the elliptic operator L is denoted by $p(t,x,y)$. Take any two points x, y on M. We are concerned with the behavior of $p(t,x,y)$ as $t \to 0$. The basic asymptotic result was proved by Varadhan [6]:

$$\lim_{t\to 0} t \log p(t,x,y) = -\frac{1}{2} d(x,y)^2. \tag{1.1}$$

This result and the subsequent extension to hypoelliptic operators (Léandre [5]) holds under the assumption that manifold M is complete in the Riemannian metric determined by the operator. The completeness of M is often implied by assumptions on the operators involved. However, it is sometimes neglected by some authors. It was first observed in Azencott et al [1] that the basic asymptotic relation (1.1) may fail for noncomplete manifolds. In the same work it was proved that (1.1) does hold if the condition

$$d(x,y) \le \max\{d(x,\infty), d(y,\infty)\} \tag{1.2}$$

is satisfied. Here $d(x,\infty)$ is the distance of x to the "infinity" defined by

$$d(x,\infty) = \sup_{\substack{K\subset M \\ K \text{ compact}}} d(x,K^c). \tag{1.3}$$

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Contrary to what was believed ([1], p. 149), (1.2) is not the best condition for (1.1). We prove in the present work that (1.1) holds under the weaker condition

$$d(x,y) \le d(x,\infty) + d(y,\infty). \tag{1.4}$$

In fact we have the following upper bound:

$$\varlimsup_{t\to 0} t \log p(t,x,y) \le -\frac{1}{2} \min\{d(x,y)^2, [d(x,\infty)+d(y,\infty)]^2\}. \tag{1.5}$$

Since the lower bound

$$\varliminf_{t\to 0} t \log p(t,x,y) \ge -\frac{1}{2} d(x,y)^2 \tag{1.6}$$

always holds whether M is complete or not (Azencott et al [1], p. 155), (1.1) follows from (1.5) and (1.6) under condition (1.4).

Condition (1.4) is the best condition of its kind. We take M to be the upper half-plane $R^2_+ = \{(x^1,x^2) : x^2 > 0\}$ with the usual Euclidean metric. We choose the vector field

$$F(x^1,x^2) = \left((x^2)^{-5/2},0\right).$$

Notice that F blows up along the boundary. We will show that with this choice of F, for each pair of points x, y such that (1.4) is violated, (1.1) does note hold.

Our approach to the problem is probabilistic; namely we think of the heat kernel $p(t,x,y)$ as the transition density function of the diffusion process determined by the elliptic operator L. We will use the fact that (1.1) is always true *locally*; i.e., for each compact subset K of M, there exists a positive constant r_K such that (1.1) holds uniformly whenever x, y lie in K and $d(x,y) \le r_K$. This local result can be proved by various methods, both probabilistic and differentio-geometric (see Chavel [2], or Azencott et al [1]). We pass from local results to global results by repeatedly using the strong Markov property of the diffusion process at judiciously chosen stopping times from both x and y.

Using Brownian motion, we can give an intuitive explantion of the counterexample mentioned above. Brownian motion particles traveling from x to y tend to follow paths φ whose action functional

$$\frac{1}{2}\int_0^t |\dot\varphi(s)|^2\, ds - \int_0^t \left\langle \dot\varphi(s), F\big(\varphi(s)\big)\right\rangle ds$$

is as small as possible (Hsu [3] contains a more precise statement to this effect). Assume that (1.4) does not hold. Since the drift near the boundary is very large along the direction of the boundary, the preferred paths for the diffusion particles are not those whose action functional is close to $d(x,y)^2/2t$, but rather those paths which start from x, reach the neighborhood of the boundary as soon as possible, travel along the boundary in no time (because of the large drift), and then head for point y at an optimal moment. The calculations in Section 3 put this intuitive reasoning in evidence.

We note that operators of the form $\frac{\Delta}{2} + F$ cover all second order nondegenerate elliptic operators on smooth manifolds. In our counterexample, the vector field F plays an essential part. $F = 0$ is a geometrically more interesting case. Our results do not exclude the possibility that (1.1) holds for the heat kernel associated with the Laplace-Beltrami operator and for *all* x, y regardless of the noncompleteness of M. It is believed that this is not the case, but so far we are unable to produce a counterexample.

2. Proof of the main result. Let $L = \frac{1}{2}\Delta + F$ and $p(t,x,y)$ be as before. Let $\{X_t,\ t \geq 0\}$ be the diffusion process on M determined by L and P_x the probability measure of the process starting at x. The goal of this section is to prove inequality (1.5).

Our starting point is the following two local results about the heat kernel. Let K be any compact set on M. There exists a positive constant r_K such that:

(L1) Uniformly on $(x,y) \in K \times K$ and $d(x,y) \leq r_K$, we have

$$\lim_{t\to 0} t \log p(t,x,y) = -\frac{1}{2}\, d(x,y)^2.$$

(L2) Define the stopping time

$$\tau_r = \inf\{t > 0 : d(X_t,X_0) = r\}.$$

Uniformly on $x \in K$, $r \in [0,r_K]$, we have

$$\lim_{t\to 0} t \log P_x\{\tau_r < t\} = -\frac{1}{2} r^2.$$

For the proof of (L1) and (L2) see Azencott et al [1], p. 173 and p. 185. Since our proof is solely based on these two properties of the heat kernel, the results of this section apply to more general situations in which a distance function $d(\cdot\,,\cdot)$ can be found such that the heat kernel (with respect to some smooth measure) satisfies (L1) and (L2). In particular, our results apply to heat kernels associated with hypoelliptic operators (see Léandre [5]).

We will need two simple lemmas.

Lemma 2.1. *Let τ be a nonnegative random variable such that*

$$P\{\tau < t\} \le e^{-a^2/2t}$$

for some positive constant a and all $t \le t_0$. Then for any positive b and $\varepsilon \in (0,1)$, there exists $t_1 = t_1(t_0,b,\varepsilon)$ such that for all $t \le t_1$

$$E\{e^{-b^2/2(t-\tau)};\ \tau < t\} \le e^{-(1-\varepsilon)(a+b)^2/2t}.$$

Proof. Integrating by parts, we have

$$\begin{aligned} E\{e^{-b^2/2(t-\tau)};\ \tau < t\} &= \int_0^t e^{-b^2/2(t-s)}\, dP\{\tau < s\} \\ &= \int_0^t P\{\tau < s\} \frac{b^2}{2(t-s)^2} e^{-b^2/2(t-s)}\, ds \\ &\le \frac{c(\varepsilon)}{b^2} \int_0^t \exp\left\{-\frac{1-\varepsilon}{2}\left(\frac{a^2}{s} + \frac{b^2}{t-s}\right)\right\} ds. \end{aligned}$$

Since

$$\frac{a^2}{s} + \frac{b^2}{t-s} \ge \frac{(a+b)^2}{t},$$

the last integral is bounded by $\exp\{-(1-\varepsilon)(a+b)^2/2\}$ if $t \le \min\{b^2/c(\varepsilon),t_0\}$. This proves the Lemma. □

Lemma 2.2. *Let K be a compact set on M. For each $r > 0$, there exists a positive number t_0, depending on K and r, such that $p(t,z,y) \le 1$ for $y \in K$, $z \in M$, $d(z,y) \ge r$, $t \le t_0$.*

Proof. Fix $r > 0$. Let U be a relatively compact open set which contains K. Let $r_0 = \min\{d(K,U^c)/2, r_{\bar{U}}\}$. Without loss of generality, we may assume that $r \le r_0$. If $d(z,y) = r$, then $p(t,z,y) \le 1$ follows from (L1) (with K there replaced by $\bar{U}$), because $z \in \bar{U}$, $y \in \bar{U}$, $d(z,y) \le r_{\bar{U}}$. If $d(z,y) > r$, then

$$\begin{aligned} p(t,z,y) &= E_z[p(t-\tau,X_\tau,y);\ \tau < t] \\ &\le \max_{\substack{0\le s\le t \\ d(y_1,y)=r}} p(s,y_1,y) \\ &\le 1. \end{aligned}$$

Here $\tau = T_{\partial B_r(y)} = \inf\{t \ge 0 : X_t \in \partial B_r(y)\}$. □

The distance of point x to the "infinity" is defined as in (1.3). It is not difficult to show that there are only two possibilities: either $d(\cdot,\infty) \equiv 0$, in which case M is complete (Hopf-Rinow Theorem), or $d(\cdot,\infty)$ is an everywhere finite, continuous function on M, in which case M is not complete. We assume the latter case.

Let U be an open set with compact closure and $r < r_{\bar{U}}$. Define a sequence of stopping times:

$$s_0 = 0$$

$$s_1 = \tau_r = \inf\{t > 0 : d(X_t,X_0) = r\}$$

$$s_n = s_{n-1} + \tau_r \circ \vartheta_{s_{n-1}} = \inf\{t > s_{n-1} : d(X_t,X_{s_{n-1}}) = r\}.$$

Consider the probability

$$I_n(t,x) = P_x\{X_{s_k} \in \bar{U},\ k \leq n-1;\ s_n < t\}.$$

Lemma 2.3. *Fix n. For any $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon,r,n)$ such that for any $x \in \bar{U}$ and $t \leq t_0$,*

$$I_n(t,x) \leq e^{-(1-\varepsilon)(nr)^2/2t}.$$

Proof. We prove by induction. Since $\bar{U}$ is assumed to be compact, the case $n = 1$ is just (L2). For $n > 1$, we have by the Markov property,

$$I_n(t,x) = E_x\{I_{n-1}(t-\tau_r,X_{\tau_r});\ X_{\tau_r} \in \bar{U},\ \tau_r < t\}.$$

Thus the induction step follows by using Lemma 2.1 and (L2). □

We now prove inequality (1.5). Let K be a fixed compact set on M and ε a given small positive number. Let U be a relatively compact open set containing K, which will be fixed later and will be dependent on ε. Let $r \leq \min\{r_{\bar{U}},d(K,U^c)\}/4$. Let x, y be two points in K. We need only consider those x, y for which the local result (L1) does not apply. So we assume $d(x,y) \geq 4r$. Define

$$\nu = \inf\{n : X_{s_n} \notin U\}.$$

Let

$$n_1 = \left[\frac{d(x,y)}{r}\right] \qquad n_2 = \left[\frac{d(x,U^c)}{r}\right].$$

Clearly

$$\{X_t = y,\ s_\nu \geq t\} \subseteq \{X_t = y;\ X_{s_k} \in \bar{U},\ k \leq n_1 - 2;\ s_{n_1-1} < t\}.$$

By considering two possibilities $s_\nu \geq t$ and $s_\nu < t$, and using the Markov property at times s_{n_1} and s_ν respectively, we see that

$$
\begin{aligned}
(2.1) \quad p(t,x,y) &\leq E_x\{p(t - s_{n_1-1},X_{s_{n_1}-1},y);\ X_{s_k} \in \bar{U},\ k \leq n_1 - 2;\ s_{n_1-1} < t\} \\
&\quad + E_x\{p(t - s_\nu,X_{s_\nu},y);\ s_{n_2} < t\} \\
&\stackrel{\text{def}}{=} J_1(t,x,y) + J_2(t,x,y).
\end{aligned}
$$

Note that we have used $s_\nu \geq s_{n_2}$.

By the definition of n_1, we have $d(X_{s_1-1},y) \geq r$. Hence according to Lemma 2.2, we have $p(t - s_{n_1-1},X_{s_{n_1}-1},y) \leq 1$. It follows from Lemma 2.1 that

$$
(2.2) \qquad J_1(t,x,y) \leq I_{n_1-1}(t,x) \leq \exp\left\{-\frac{1-\varepsilon}{2t}[d(x,y) - 2r]^2\right\}.
$$

We now estimate $J_2(t,x,y)$. The adjoint operator of L (with respect to the Riemannian volume measure) is $L^* - \operatorname{div} F = \frac{1}{2}\Delta - F - \operatorname{div} F$. Let $X^* = \{X_t^*,\ t \geq 0\}$ be the diffusion associated with the operator $L^* = \frac{1}{2}\Delta - F$. By the Feynman-Kac formula we have

$$
p(t,z,y) = E_y\left[\exp\left\{-\int_0^t \operatorname{div} F(X_s^*)\, ds\right\};\ X_t^* \in dz\right] \Big/ dz.
$$

Let $z \notin U$ and $\sigma = \inf\{t > 0 : d(X^*,U^c) = r\}$. Then using the strong Markov property at σ, we have

$$
p(t,z,y) = E_y\left[p(t - \sigma,z,X_\sigma^*)\exp\left\{-\int_0^\sigma \operatorname{div} F(X_s^*)\, ds\right\};\ \sigma \leq t\right].
$$

Now it is clear that $X_\sigma^* \in \bar{U}$ and $d(z,X_\sigma^*) \geq r$. Hence by Lemma 2.2 (with K there replaced by $\bar{U}$), we see that $p(t - \sigma,z,X_\sigma^*) \leq 1$. Now we have

$$
p(t,z,y) \leq e^{c_0 t} P_y[\sigma \leq t] \leq e^{c_0 t} I_{n_3-1}^*(t,y),
$$

where $I_n^*(t,y)$ is defined for the process X^* in the same way as $I_n(t,y)$ for the process X, and

$$
n_3 = \left[\frac{d(y,U^c)}{r}\right], \qquad c_0 = \max_{x \in U} |\operatorname{div} F(x)|.
$$

It follows now from Lemma 2.3 that if $z \notin U$, then

$$
p(t,z,y) \leq 2\exp\left\{-\frac{1-\varepsilon}{2t}[d(y,U^c) - 2r]^2\right\}.
$$

In the above inequality, we can take $z = X_{s_\nu} \notin \bar{U}$ and replace t by $t - s_\nu$. Noting that $s_\nu \geq s_{n_2}$, we obtain

$$(2.3) \qquad p(t - s_\nu, X_{s_\nu}, y) \leq 2\exp\left\{-\frac{1-\varepsilon}{2(t-s_{n_2})}[d(y,U^c) - 2r]^2\right\}.$$

Substitute (2.3) into the definition of $J_2(t,x,y)$. Using Lemma 2.3 and a proof similar to that of Lemma 2.1 we obtain

$$(2.4) \qquad J_2(t,x,y) \leq 2\exp\left\{-\frac{1-2\varepsilon}{2t}[d(y,U^c) + d(x,U^c) - 3r]^2\right\}.$$

It now follows from (2.1) and the estimates (2.2) and (2.4) that

$$(2.5) \qquad t \log p(t,x,y) \leq -\frac{1-3\varepsilon}{2}[\min\{d(x,y), d(x,U^c) + d(y,U^c)\} - 3r]^2$$

for any $(x,y) \in K \times K$, $d(x,y) \geq 4r$, $r \leq r_0(K,U)$ and $t \leq t_1(\varepsilon,r,K,U)$.

Finally, let $\varepsilon_0 = \min\{d(K,\infty) + d(K,\infty), r_K\}/5$. For any $\varepsilon \leq \varepsilon_0$, we can find an open set U containing K with compact closure such that

$$d(z,U^c) \geq d(z,\infty) - \varepsilon$$

for all $z \in K$. The existence of such U follows from the uniform continuity of the family of functions $\{d(\cdot, U^c),\ U \supset K\}$ on the compact set K. Letting $r \leq \min\{r_0(K,u),\varepsilon\}$ in (2.5), we obtain for all $(x,y) \in K \times K$, $d(x,y) \geq 5\varepsilon$, $t \leq t_2(\varepsilon,K)$,

$$t \log p(t,x,y) \leq -\frac{1-3\varepsilon}{2}[\min\{d(x,y), d(x,U^c) + d(y,U^c)\} - 5\varepsilon]^2.$$

The case $d(x,y) < 5\varepsilon$ being taken care of by the local result (L1), we therefore have proved the following result.

Theorem 2.4. *For any compact K, we have uniformly on $(x,y) \in K \times K$,*

$$\varlimsup_{t\to 0} t \log p(t,x,y) \leq -\frac{1}{2}\min\{d(x,y)^2, [d(x,\infty) + d(y,\infty)]^2\}.$$

As mentioned in Section 1, the lower bound (1.6) always holds. The following result is then immediate.

Theorem 2.5. *If K is a compact subset of M and for any x, $y \in K$ we have $d(x,y) \le d(x,\infty) + d(y,\infty)$, then uniformly on $(x,y) \in K \times K$,*

$$\lim_{t \to 0} t \log p(t,x,y) = -\frac{1}{2} d(x,y)^2.$$

Corollary 2.6. *Let x, y be two points on M such that $d(x,y) < d(x,\infty) + d(y,\infty)$, then there exist constants c_1, c_2, t_0 such that for any $t \le t_0$, we have*

$$c_1 t^{-d/2} e^{-d(x,y)^2/2t} \le p(t,x,y) \le c_2 t^{-(2d+1)/2} e^{-d(x,y)^2/2t}. \tag{2.6}$$

Proof. Compare with [1], pp. 156–157. Take any

$$r < \frac{1}{2}[d(x,\infty) + d(y,\infty) - d(x,y)].$$

There exists an open set U with compact closure such that

$$d(x,y) \le d(x,U^c) + d(y,U^c) - 2r. \tag{2.7}$$

Let $p_U(t,x,y)$ denote the heat kernel on U with Dirichlet boundary condition, then the Markov property implies

$$p(t,x,y) = p_U(t,x,y) + E_x\{p(t - t_U, X_{\tau_U}, y);\ \tau_U < t\}, \tag{2.8}$$

where $\tau_U = \inf\{t > 0 : X_t \in U^c\}$. Since $\min\{d(y,X_{\tau_U}), d(y,\infty) + d(X_{\tau_U},\infty)\} \ge d(y,U^c)$ we have by Theorem 2.4

$$\begin{aligned} &E_x\{p(t - \tau_U, X_{\tau_U}, y);\ \tau_U < t\} \\ &\le E_x\left\{\exp\left\{-\frac{1-\varepsilon}{2(t-\tau_U)} d(y,U^c)^2\right\};\ \tau_U < t\right\}. \end{aligned} \tag{2.9}$$

On the other hand, by Lemma 2.3,

$$P_x\{\tau_U < t\} \le I_{n_2}(t,x) \le \exp\left\{-\frac{1-\varepsilon}{2t}[d(x,U^c) - r]^2\right\}. \tag{2.10}$$

It follows from (2.7), (2.9), (2.10) and Lemma 2.1 that

$$E_x\{p(t - \tau_U, X_{\tau_U}, y);\ \tau_U < t\} \le \exp\left\{-\frac{1-\varepsilon}{2t}[d(x,y) + r]^2\right\}. \tag{2.11}$$

Since ε can be arbitrarily small, we conclude from (2.8), (2.11), and the lower bound (1.6) that

$$\lim_{t \to 0} \frac{p(t,x,y)}{p_U(t,x,y)} = 1.$$

Next, we can modify the metric outside U so that the resulting manifold under the new metric is complete. Let $\tilde{p}(t,x,y)$ be the new heat kernel. Then the same argument as above leads to

$$\lim_{t \to 0} \frac{\tilde{p}(t,x,y)}{p_U(t,x,y)} = 1.$$

It follows that for the purpose of proving (2.6), we can assume that M is complete. We may then proceed as in Azencott et al [1], p. 178, to complete the proof. □

3. Optimality of the condition. We show in this section that the condition $d(x,y) \le d(x,\infty) + d(y,\infty)$ is the best possible for the relation

$$\lim_{t\to 0} t \log p(t,x,y) = -\frac{d(x,y)^2}{2}.$$

Let M be the upper-half plane $R^2_+ = \{(x^1,x^2) : x^2 > 0\}$ with the usual Euclidean metric. Let $F(x^1,x^2) = \big((x^2)^{-5/2},0\big)$. Thus our diffusion process $\{X_t = (X^1_t,X^2_t),\ t \ge 0\}$ is determined by the following stochastic differential equation:

$$\begin{aligned} dX^1_t &= dB^1_t + (X^2_t)^{-5/2}\,dt, \\ dX^2_t &= dB^2_t, \end{aligned} \tag{3.1}$$

where $B_t = (B^1_t,B^2_t)$ is a standard two-dimensional Brownian motion.

This diffusion provides a counterexample to the basic asymptotic relation whenever $d(x,y) > d(x,\infty) + d(y,\infty)$.

Proposition 3.1. *Let $p(t,x,y)$ be the transition density function of the diffusion* (3.1). *Then for any $x = (x^1,x^2)$ and $y = (y^1,y^2)$ such that $x^1 < y^1$, we have*

$$\lim_{t\to 0} t \log p(t,x,y) \ge -\frac{1}{2}[x^2 + y^2]^2.$$

Proof. We define three rectangles near points $(x^1,0)$, $(y^1,0)$ and $y = (y^1,y^2)$, respectively:

$S_1 =$ the rectangle centered at $(x^1,\sqrt{t})$ with horizontal length $2(C+1)t^{1/2}$ and vertical length $2t^{3/4}$,

$S_2 =$ the rectangle centered at $(y^1,\sqrt{t})$ with horizontal length $4(C+1)t^{1/4}$ and vertical length $4t^{3/4}$,

$S_3 =$ the rectangle centered at $y = (y^1,y^2)$ with horizontal length $6(C+1)t^{1/4}$ and vertical length $6t^{3/4}$;

C is a constant depending on x, y which will be determined in the course of the proof.

Let

$$\Delta t = (y^1 - x^1)t^{5/4},$$

$$t_1 = \frac{x^2}{x^2 + y^2} t - \Delta t,$$

$$t_2 = \frac{y^2}{x^2 + y^2} t - \Delta t.$$

For two paths φ, ψ we set

$$\|\varphi - \psi\|_t = \max_{0 \le s \le t} |\varphi(s) - \psi(s)|.$$

We compute three probabilities: $P_x[X_{t_1} \in S_1]$, $P_z[X_{\Delta t} \in S_2]$ $(z \in S_1)$, and $P_z[X_{t_2} \in S_3]$ $(z \in S_2)$.

(a) Let

$$\varphi(s) = \frac{x^2 - \sqrt{t}}{t_1} s, \qquad 0 \le s \le t_1.$$

If $\|B^2 + \varphi\|_{t_1} \le t^{3/4}$, then there is a constant C depending on x, y such that

$$\int_0^{t_1} [X_s^2]^{-5/2} \, ds \le \int_0^{t_1} \left[x^2 - \frac{x^2 - \sqrt{t}}{t_1} s - t^{3/4} \right]^{-5/2} ds \le C t^{1/4}.$$

Thus $\|B^2 + \varphi\|_{t_1} \le t^{3/4}$ and $|B_{t_1}^1| \le t^{1/4}$ imply $X_{t_1} \in S_1$. We have then by the independence of B^1 and B^2,

$$P_x[X_{t_1} \in S_1] \ge P\big[\|B^2 + \varphi\|_{t_1} \le t^{3/4}\big] P\big[|B_t^1| \le t^{1/4}\big].$$

The second factor tends to 1 as $t \to 0$. To compute the first factor we use the Girsanov transform:

$$\text{(3.1)} \qquad P\big[\|B^2 + \varphi\|_{t_1} \le t^{3/4}\big] = E\left[\exp\left\{ \int_0^{t_1} \dot\varphi(s)\, dB_s^2 - \frac{1}{2} \int_0^{t-1} |\dot\varphi(s)|^2 \, ds \right\};\ \|B^2\|_{t_1} \le t^{3/4} \right].$$

If $\|B^2\|_{t_1} \le t^{3/4}$, then

$$\int_0^{t_1} \dot\varphi(s)\,dB_s^2 - \frac{1}{2}\int_0^{t_1} |\dot\varphi(s)|^2\,ds = \frac{x^2-\sqrt{t}}{t_1}B_{t_1}^2 - \frac{1}{2}\left[\frac{x^2-\sqrt{t}}{t_1}\right]^2 t_1 \tag{3.2}$$
$$\ge -\left(1+\frac{\varepsilon}{2}\right)\frac{(x^2+y^2)x^2}{2t}.$$

We also have

$$P\big[\|B^2\|_{t_1} \le t^{3/4}\big] \sim c_1 \exp[-c_2 t^{-1/2}]. \tag{3.3}$$

See Ikeda-Watanabe [4], p. 432. From (3.1) and (3.3) we conclude that

$$P[X_{t_1} \in S_1] \ge \exp\left\{-(1+\varepsilon)\frac{(x^2+y^2)x^2}{2t}\right\}.$$

(b) Let z be any point in the rectangle S_1. If $\|B^2\|_{\Delta t} \le t^{3/4}$, then $|X_s^2 - \sqrt{t}| \le 2t^{3/4}$ for all $0 \le s \le \Delta t$. Thus there is a constant C depending on x, y such that

$$\left|\int_0^{\Delta t} [X_s^2]^{-5/2}\,ds - (y^1 - x^1)\right| \le Ct^{1/4}.$$

If we assume further that $|B_{\Delta t}^1| \le t^{1/4}$, then the above inequality implies

$$|X_{\Delta t}^1 - y^1| \le 2(C+1)t^{1/4}.$$

Therefore we have shown that $|B_{\Delta t}^1| \le t^{1/4}$ and $\|B^2\|_{\Delta t} \le t^{3/4}$ imply $X_{\Delta t} \in S_2$. It follows that for some constant c_3 depending on x, y,

$$P_z[X_{\Delta t} \in S_2] \ge P\big[\|B^2\|_{\Delta t} \le t^{3/4}\big]P\big[|B_{\Delta t}^1| \le t^{1/4}\big] \ge \exp\{-c_3 t^{-1/4}\}.$$

(c) A similar argument as in (a) gives the following estimate: For any $z \in S_2$,

$$P_z[X_{t_2} \in S_3] \ge \exp\left\{-(1+\varepsilon)\frac{(x^2+y^2)y^2}{2t}\right\}.$$

Now using (a), (b) and (c) and the Markov property at time t_1, $t_1 + \Delta t$ respectively, we have

$$\begin{aligned} P_x[X_{t-\Delta t} \in S_3] &\geq P_x[X_{t_1} \in S_1] \cdot \min_{z \in S_1} P_z[X_{\Delta t} \in S_2] \cdot \min_{z \in S_2} P_z[X_{t_2} \in S_3] \\ &\geq \exp\left\{ -(1+\varepsilon)\frac{(x^2+y^2)^2}{2t} - c_r t^{-1/4} \right\}. \end{aligned}$$

Finally, by a precise local asymptotic formula for the heat kernel, we have for any $z \in S_3$,

$$p(\Delta t, z, y) \geq \exp\{-c_5 t^{-3/4}\}.$$

It follows that

$$\begin{aligned} p(t,x,y) &= \int_{R_+^2} p(t-\Delta t, x, z) p(\Delta t, z, y)\, dz \\ &\geq P_x[X_{t-\Delta t} \in S_3] \cdot \min_{z \in S_3} p(\Delta t, z, y) \\ &\geq \exp\left\{ -(1+\varepsilon)\frac{(x^2+y^2)^2}{2t} - c_6 t^{-1/4} \right\}. \end{aligned}$$

The desired result follows immediately. □

Remark. The idea of using shrinking rectangles to estimate the probability as above is due to Azencott et al [1]. Note that in the above diffusion, we have $d(x,\infty) + d(y,\infty) = x^2 + y^2$.

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