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Integration by parts in loop spaces

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1. Introduction

We assume throughout this paper that M is a n-dimensional compact Riemannian manifold and $O(M)$ its orthonormal frame bundle. We use $\mathbb{H}$ to denote the $\mathbb{R}^d$-valued Cameron-Martin space over the interval $[0, 1]$ with zero initial values and $\mathbb{H}_0$ the subspace of $\mathbb{H}$ with zero values at 1. We fix a point $o \in M$ and a frame $u_o \in O(M)$ over o. We use $W_o(M)$ to denote the set of M-valued paths (of time length 1) starting from o and $L_o(M)$ the set of loops at o, i.e., the set of paths γ in $W_o(M)$ such that $\gamma(1) = o$.

The Levi-Civita connection determines a Laplace-Beltrami operator Δ on M. We use ν to denote the Wiener measure on $W_o(M)$ generated by $\Delta/2$. The measure ν_o defined intuitively by

$$\nu_o(\cdot) = \nu(\cdot|\omega(1) = o)$$

is a measure on the loop space $L_o(M)$, which we call the Wiener measure on $L_o(M)$.

For a smooth or a typical Brownian $\gamma \in W_o(M)$ or $L_o(M)$, let $U(\gamma)$ be the horizontal lift of γ such that $U(\gamma)_o = u_o$. Fix an $h \in \mathbb{H}$ (or $\mathbb{H}_0$) , the "vector field" D_h on $W_o(M)$ (or $L_o(M)$) is defined by

$$D_h(\gamma)_s = U(\gamma)_s h_s. \tag{1}$$

There is a complete theory of integration by parts for D_h on $W_o(M)$, developed by Driver[1] and supplemented by Hsu[5]. See also Enchev and Stroock[3] for

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another approach. In the case of the loop space $L_o(M)$, Driver[2] proved an integration by parts formula for vector fields D_h with lipschitzian h and the complete result for all Cameron-Martin vector fields was proved in Enchev and Stroock[4]. The purpose of this paper is to give an alternative approach to integration by parts in loop spaces. Armed with an upper estimate on the $\nabla^2 \log p(s, x, y)$ due to Sheu[8] (see (7) below), we prove an integration by parts formula in the loop space $L_o(M)$ through the corresponding formula for the path space $W_o(M)$. Such an approach avoids the quasi-invariance of the Wiener measure in the loop spaces thus providing a more direct route to the result.

2. Integration by parts in path spaces

Let μ be the usual Wiener measure on the path $W_o(\mathbb{R}^n)$. Let $\{U_s\}$ be the solution of the stochastic differential equation on $O(M)$

$$dU_s = H_{U_s} \circ d\omega_s, \qquad U_0 = u_o. \tag{2}$$

Here $H = \{H_i, i = 1, \ldots, d\}$ are the canonical horizontal vector fields on $O(M)$ and $\{\omega_s\}$ is the coordinate process on $W_o(\mathbb{R}^n)$. Let $\gamma_s = \pi(U_s)$ be the projection of U in $W_o(M)$. The Itô map $J : W_o(\mathbb{R}^n) \to W_o(M)$ is defined by $J\omega = \gamma$. It is well known that the law of γ is ν, the Wiener measure on $W_o(M)$, i.e., J carries the Wiener measure μ on $W_o(\mathbb{R}^d)$ to the Wiener measure ν on $W_o(M)$. The inverse $J^{-1} : W_o(M) \to W_o(\mathbb{R}^d)$ is the stochastic development map.

A function $F : W_o(M) \to \mathbb{R}^1$ is called cylindrical if there is a positive integer l, a set of l points $0 \le s_1 < \cdots < s_l \le 1$ and a smooth function $f : M \times \cdots \times M \to \mathbb{R}^1$ such that

$$F(\gamma) = f\left(\gamma_{s_1}, \cdots, \gamma_{s_l}\right). \tag{3}$$

The set of cylindrical functions on $W_o(M)$ is denoted by $\mathscr{C}$.

We will use $L^2(\nu)$ to denote the Hilbert space of measurable functions F on $W_o(M)$ such that

$$\|F\|^2_{L^2(\nu)} = \int_{W_o(M)} |F(\gamma)|^2_{\mathbb{E}} \nu(d\gamma) < \infty.$$

The inner product on $L^2(\nu)$ is denoted by $(\cdot,\cdot)_{L^2(\nu)}$ or simply $(\cdot,\cdot)$.

Let $F \in \mathscr{C}$ be given by (3). From the defintion of the vector field D_h in (1) it is natural to define

$$D_h F(\gamma) = \sum_{p=1}^{l} \langle \nabla^{(p)} F(\gamma), U(\gamma)_{s_p} h_{s_p} \rangle, \tag{4}$$

where $\nabla^{(p)} F$ denotes the gradient of f with respect to the pth variable.

Let $h \in \mathbb{H}$, define

$$l_h(\gamma) = \int_0^1 \langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U_s} h_s,\ d\omega_s \rangle,$$

where $\omega = J^{-1}\gamma$, $U = U(\gamma)$ is the horizontal lift of γ to $O(M)$, and $\mathrm{Ric}_u : \mathbb{R}^n \to \mathbb{R}^n$ is the Ricci transform at $u \in O(M)$.

Theorem 2.1. *(Integration by parts in path space) Let F, G be two cylindrical functions on $W_o(M)$. Then*

$$(D_h F, G) = \left(F, D_h^* G\right), \tag{5}$$

where

$$D_h^* = -D_h + l_h.$$

The assumption that $h \in \mathbb{H}$ implies that there exists a constant $c > 0$ such that $E_\nu e^{c|l_h|^2} < \infty$. By a standard functional analysis argument, the integration by parts formula implies that D_h is closable and the adjoint D_h^* is densely defined (the closability of D_h requires only $l_h \in L^2(\nu)$). There are plenty of functions in Dom (D_h^*). More precisely, we have the following result. Let $L^{2+}(\nu) = \bigcup_{p>2} L^p(\nu)$.

Theorem 2.2. *Let $h \in \mathbb{H}$. Then $D_h : \mathscr{C} \to L^2(\nu)$ is closable in $L^2(\nu)$ and has a densely defined adjoint D_h^*. Furtheremore,*

$$\mathrm{Dom}\,(D_h) \cap L^{2+}(\nu) \subset \mathrm{Dom}\,\left(D_h^*\right)$$

and for all $G \in \mathrm{Dom}\,(D_h) \cap L^{2+}(\nu)$ we have

$$D_h^* G = -D_h G + l_h G.$$

3. Some preliminary results

In this section we collect some results which will be used in the proof of integration by parts formula on the loop space in the next section.

We denote by $p(s, x, y)$ the heat kernel of the half Laplacian $\Delta/2$ on M.

Proposition 3.1. *There exists a constant depending only on M such that for all $(s, x, y) \in (0, 1) \times M \times M$,*

$$|\nabla \log p(s, x, y)| \le C \left\{ \frac{d(x, y)}{s} + \frac{1}{\sqrt{s}} \right\} \tag{6}$$

$$|\nabla^2 \log p(s, x, y)| \le C \left\{ \frac{d(x, y)^2}{s^2} + \frac{1}{s} \right\}. \tag{7}$$

Proof. As far as we know, these results are due to Sheu[8]. See also Hsu[7] and Stroock and Trubetsky[9] for further discussions. □

Lemma 3.2. *For each positive integer N there is a constant C_N depending only on N and M such that*

$$E_{\nu_o} d(\gamma_s, o)^N \le C_N \min\left\{ s^{N/2}, (1-s)^{N/2} \right\}.$$

Proof. This inequality is intuitively clear and can be proved based on the estimate (6). See Driver[2] or Hsu[6] for details. □

Lemma 3.3. *(Hardy's inequality) Let $h \in \mathbb{H}_0$, then*

$$\int_0^1 \left| \frac{h_s}{1-s} \right|^2 ds \le 4 \int_0^1 |\dot{h}_s|^2 \, ds.$$

Proof. We have for any $t \in (0,1)$,

$$\begin{aligned} \int_0^t \left| \frac{h_s}{1-s} \right|^2 ds &= \int_0^t |h_s|^2 d\left[\frac{1}{1-s} \right] \\ &= 2\int_0^t \frac{h_s \cdot \dot{h}_s}{1-s} ds + \frac{|h_t|^2}{1-t} \\ &\le \frac{1}{2} \int_0^t \left| \frac{h_s}{1-s} \right|^2 ds + 2\int_0^t |\dot{h}_s|^2 ds + \frac{|h_t|^2}{1-t}. \end{aligned}$$

In the last step we have used inequality

$$2ab \le \frac{1}{2} a^2 + 2b^2.$$

Therefore

$$\int_0^t \left| \frac{h_s}{1-s} \right|^2 ds \le 4 \int_0^t |\dot{h}_s|^2 ds + \frac{2|h_t|^2}{1-t}.$$

The desired inequality follows by letting $t \to 1$ in the above inequality because

$$\frac{|h_t|^2}{1-t} = \frac{1}{1-t} \left| \int_t^1 \dot{h}_s ds \right|^2 \le \int_t^1 |\dot{h}_s|^2 ds \to 0. \qquad \square$$

4. Integration by parts on loop spaces

Recall that in path space $W_o(M)$ the adjoint of D_h is given by

$$D_h^* = -D_h + l_h, \tag{8}$$

where $l_h : W_o(M) \to \mathbb{R}^n$ is defined by

$$l_h(\gamma) = \int_0^1 \langle \dot{h}_s + \frac{1}{2} \mathrm{Ric}_{U_s} h_s, d\omega_s \rangle.$$

Here U is the horizontal lift of γ and $\omega = J^{-1}\gamma$ is the stochastic development of γ. On the loop space $L_o(M)$, we define $D_h F$ for a cylindrical function by the same formula (4) as in the path space. The next proposition shows that l_h is well defined under the measure ν_o. This step is necessary because ν and ν_o are mutually singular.

Let $\{l_{h,s}\}$ be the ν-martingale

$$l_{h,s} = \int_0^s \langle \dot{h}_\tau + \frac{1}{2} \operatorname{Ric}_{U_\tau} h_\tau, d\omega_\tau \rangle.$$

Let $\{\mathscr{B}_s, 0 \le s \le 1\}$ be the standard filtration of σ-fields on $W_o(M)$. Then the measures ν_o and ν are mutually absolutely continuous on $\mathscr{B}_s$ for all $s < 1$. Hence the process $\{l_{h,s}, 0 \le s < 1\}$ is well defined under the measure ν_o. The next lemma concerns the limit of $l_{h,s}$ as $s \to 1$ under the measure ν_o.

Proposition 4.1. *The limit $l_{h,s} \to l_h$ exists in $L^1(\nu_o)$ as $s \to 1$. Furthermore $l_h \in L^2(\nu_o)$.*

Proof. Under the measure ν, the stochastic development $\omega = J^{-1}\gamma$ is a Brownian motion. Under the measure ν_o, it is a local semimartingale before time 1 and its martingale part $\{b_s\}$ is a Brownian motion. The measure ν_o is characterized by the fact that

$$\omega_s = b_s + \int_0^s U_\tau^{-1} \nabla \log p(1-\tau, \gamma_\tau, o) d\tau.$$

Let

$$Q_s = U_s^{-1} \nabla \log p(1-s, \gamma_s, o),$$

$$F_s = h_s - \frac{1}{2} \int_s^1 \operatorname{Ric}_{U_\tau} h_\tau d\tau$$

for simplicity. We have for $s < 1$

$$\begin{aligned} l_{h,s} &= \int_0^s \langle \dot{F}_\tau, db_\tau \rangle + \int_0^s \langle \dot{F}_\tau, Q_\tau d\tau \rangle \\ &= \int_0^s \langle \dot{F}_\tau, db_\tau \rangle - \int_0^s \langle F_\tau, dQ_\tau \rangle + \langle Q_s, F_s \rangle - \langle Q_0, F_0 \rangle \\ &= I_{1,s} - I_{2,s} + I_{3,s} - \langle Q_0, F_0 \rangle. \end{aligned}$$

Now $\operatorname{Ric}_u h_\tau$ is uniformly bounded, and $\dot{h} \in L^2[0,1]$. These facts imply that the limit $I_{1,s} \to I_1$ exists in $L^2(\nu_o)$ and

$$I_1 = \int_0^1 \langle \dot{F}_s, db_s \rangle.$$

For $I_{3,s}$ we have $|F_s| \le C\{|h_s| + (1-s)\}$ and using (6) and Lemma 3.2 we have

$$\begin{aligned} E_{\nu_o}|\langle F_s, Q_s\rangle| &\le C\,\{|h_s| + (1-s)\}\, E_{\nu_o}|\nabla \log p(1-s,\gamma_s,o)| \\ &\le C_1\,\{|h_s| + (1-s)\}\left\{\frac{E_{\nu_o} d(\gamma_s,o)}{1-s} + \frac{1}{\sqrt{1-s}}\right\} \\ &\le C_2\left\{\sqrt{1-s}\sqrt{\int_s^1 |\dot h_\tau|^2 d\tau} + (1-s)\right\}\frac{1}{\sqrt{1-s}} \\ &\to 0. \end{aligned}$$

This shows that $I_{3,s} \to 0$ in $L^1(\nu_o)$.

For $I_{2,s}$ we use Itô's formula on the $\mathbb{R}^n$-valued function

$$Q_s = U_s^{-1}\nabla \log p(1-s,\gamma_s,0) = \nabla^H \log P(1-s,U_s)$$

of $(s, U_s) \in (0,1) \times O(M)$, where $P(s,u) = p(s,\pi u, o)$. Using the stochastic differential equation (2) for U_s we have for the ith component

$$\begin{aligned} (9)\qquad & dH_i \log P(1-s,U_s) \\ = \;& \langle H_i \nabla^H \log P(1-s,U_s), db_s\rangle + \frac{1}{2}\langle \mathrm{Ric}_{U_s} e_i, \nabla^H \log P(1-s,U_s)\rangle ds \\ & + H_i\left\{\Box^H \log P(1-s,U_s) + \frac{1}{2}|\nabla^H \log P(1-s,U_s)|^2\right\} ds, \end{aligned}$$

where

$$\Box^H = \frac{1}{2}\Delta^H + \frac{\partial}{\partial s},$$

and $\Delta^H = \sum_{j=1}^n H_j^2$ is Bochner's horizontal Laplacian. Note that in the above computation we need to use the second structural equation

$$[H_i, H_j] = \Omega(H_i, H_j)^*$$

to exchange H_i and H_j (Ω^* is the canonical vertical vector field corresponding to $\Omega \in o(n)$). The last term in (9) vanishes because $p(t,x,y)$ satisfies the heat equation. Hence we have

$$\begin{aligned} I_{2,s} \;=\; & \int_0^s \langle F_\tau, U_\tau^{-1}\nabla^2 \log p(1-\tau,\gamma_\tau,o), db_\tau\rangle \\ & + \frac{1}{2}\int_0^s \langle \mathrm{Ric}_{U_\tau} F_\tau, U_\tau^{-1}\nabla \log p(1-\tau,\gamma_\tau,o) d\tau\rangle. \end{aligned}$$

To show that the limit $I_{2,s} \to I_2$ exists in $L^2(\nu_o)$ it is enough to show that

$$(10)\qquad E_{\nu_o}\int_0^1 |F_s|^2 \cdot |\nabla^2 \log p(1-s,\gamma_s,o)|^2 ds < \infty$$

and

$$(11)\qquad E_{\nu_o}\int_0^1 |F_s|^2 \cdot |\nabla \log p(1-s,\gamma_s,o)|^2 ds < \infty.$$

From the definition of F_s there exists a constant C such that

(12) $$|F_s| \leq C\left\{|h_s| + (1-s)\right\}.$$

Using the estimate (7) and Lemma 3.2 we see that there exists a constant C such that

$$E_{\nu_o}|\nabla^2 \log p(1-s,\gamma_s,o)|^2 \leq \frac{C}{(1-s)^2}.$$

It follows from Lemma 3.3 that

$$\begin{aligned} & E_{\nu_o}\int_0^1 |F_s|^2 \cdot |\nabla^2 \log p(1-s,\gamma_s,o)|^2 ds \\ \leq\ & C\int_0^1 \left\{\frac{|h_s| + (1-s)}{(1-s)}\right\}^2 ds \\ \leq\ & 8C\left\{\int_0^1 |\dot{h}_s|^2 ds + 1\right\}. \end{aligned}$$

This proves (10). From (6) and Lemma 3.2 there is a constant C such that

$$E_{\nu_o}|\nabla \log p(1-s,\gamma_s,o)|^2 \leq \frac{C}{1-s}.$$

Using this inequality and (12) we have

$$\begin{aligned} & E_{\nu_o}\int_0^1 |F_s|^2 |\nabla \log p(1-s,\gamma_s,o)|^2 ds \\ \leq\ & C\int_0^1 \frac{\{|h_s| + (1-s)\}^2}{1-s} ds \\ \leq\ & 8C\left\{\int_0^1 |\dot{h}_s|^2 ds + 1\right\}. \end{aligned}$$

This proves (11). It follows that the limit $I_{2,s} \to I_2$ exists in $L^2(\nu_o)$ and

$$\begin{aligned} I_2 = & \int_0^1 \langle F_s, U_s^{-1}\nabla^2 \log p(1-s,\gamma_s,o), db_s\rangle \\ & + \frac{1}{2}\int_0^1 \langle \mathrm{Ric}_{U_s} F_s, U_s^{-1}\nabla \log p(1-s,\gamma_s,o) ds\rangle. \end{aligned}$$

To summarize, we have

$$l_{h,s} = I_{1,s} - I_{2,s} + I_{3,s} - \langle Q_0, F_0\rangle;$$

$I_{1,s} \to I_1, I_{2,s} \to I_2$, both in $L^2(\nu_o)$, and $I_{3,s} \to 0$ in $L^1(\nu_o)$. It follows that the stochastic integral

$$l_h = \int_0^1 \langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U_s} h_s, d\omega_s\rangle$$

($\omega = J^{-1}\gamma$) exists as the $L^1(\nu_o)$-limit of $l_{h,s}$ as $s \to 1$ and $l_h \in L^2(\nu_o)$. □

We now prove the main theorem.

Theorem 4.2. *(Integration by parts formula in loop space) Let F, G be two cylindrical functions on $L_o(M)$. Then*

$$(D_h F, G)_{L^2(\nu_o)} = \left(F, D_h^* G\right)_{L^2(\nu_o)},$$

where

$$D_h^* = -D_h + l_h$$

and $l_h \in L^2(\nu_o)$ is defined by

$$l_h(\gamma) = \int_0^1 \langle \dot{h}_s + \frac{1}{2}\mathrm{Ric}_{U_s} h_s,\, d\omega_s\rangle.$$

Here $\omega = J^{-1}\gamma$ is the stochastic development of γ and U is the horizontal lift of γ.

Proof. Suppose that F and G depend on the path up to time $s^* < 1$. Then we have for all $s \in (s^*, 1)$,

$$(D_h F, G)_{L^2(\nu_o)} = C_o\,(D_h F, Gp(1-s, \gamma_s, o))_{L^2(\nu)},$$

where $C_o = p(1, o, o)^{-1}$. By the integration by parts formula (5) for the path space, we have

$$\begin{aligned}
& (D_h F, G)_{L^2(\nu_o)} \\
= \ & C_o\left(F, D_h^*\left\{Gp(1-s,\gamma_s,o)\right\}\right)_{L^2(\nu)} \\
= \ & -C_o\,(F, D_h Gp(1-s,\gamma_s,o))_{L^2(\nu)} \\
& -C_o\,(F, GD_h p(1-s,\gamma_s,o))_{L^2(\nu)} \\
& +C_o\,(F, l_h Gp(1-s,\gamma_s,o))_{L^2(\nu)} \\
= \ & -(F, D_h G)_{L^2(\nu_o)} - (F, GD_h \log p(1-s,\gamma_s,o))_{L^2(\nu_o)} \\
& + \left(F, l_{h,s} G\right)_{L^2(\nu_o)}.
\end{aligned}$$

Since F and G are uniformly bounded, by Proposition 4.1 we have

$$\left(F, l_{h,s} G\right)_{L^2(\nu_o)} \to (F, l_h G)_{L^2(\nu_o)}.$$

It is therefore enough to show

$$(F, GD_h \log p(1-s,\gamma_s,o))_{L^2(\nu_o)} \to 0.$$

This is implied by

$$E_{\nu_o}|D_h \log p(1-s,\gamma_s,o)| \to 0. \tag{13}$$

We have

$$D_h \log p(1-s,\gamma_s,o) = \langle h_s, U_s^{-1}\nabla \log p(1-s,\gamma_s,o)\rangle.$$

By (6) and Lemma 3.2 we have

$$\begin{aligned}
E_{\nu_o}|D_h \log p(1-s,\gamma_s,o)| &= |h_s|\cdot E_{\nu_o}|\nabla p(1-s,\gamma_s,o)| \\
&\le C|h_s|\left\{\frac{E_{\nu_o}d(\gamma_s,o)}{1-s}+\frac{1}{\sqrt{1-s}}\right\} \\
&\le \frac{C_1|h_s|}{\sqrt{1-s}} \\
&= \frac{C_1}{\sqrt{1-s}}\left|\int_s^1 \dot h_\tau d\tau\right| \\
&\le C_1\sqrt{\int_s^1 |\dot h_\tau|^2 d\tau} \\
&\to 0.
\end{aligned}$$

This shows (13) and the theorem is proved. □

As a consequence of the above integration by parts formula and the fact that $l_h \in L^2(\nu_o)$, we have the following result parallel to Theorem 2.2. Let $B(\nu_o)$ be the space of ν_o-essentially bounded measurable functions on $L_o(M)$.

Theorem 4.3. *Let $h \in \mathbb{H}_0$. Then $D_h : \mathscr{C} \to L^2(\nu_o)$ is closable in $L^2(\nu_o)$ and has a densely defined adjoint D_h^*. Furthermore*

$$\mathrm{Dom}(D_h)\cap B(\nu_o) \subset \mathrm{Dom}(D_h^*)$$

and for all $G \in \mathrm{Dom}(D_h)\cap B(\nu_o)$ we have

$$D_h^* G = -D_h G + l_h G. \tag{14}$$

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