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Multiplicative Functional for the Heat Equation on Manifolds with Boundary

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1. Introduction

By the Weitzenböck formula relating the Hodge–de Rham Laplacian and the covariant Laplacian for differential forms on a Riemannian manifold, the heat equation for differential forms is naturally associated with a matrix-valued Feynman–Kac multiplicative functional determined by the curvature tensor. The case of a closed manifold (without boundary) is well known and will be briefly reviewed below. In constrast, the case of manifolds with boundary is not well known, and for good reasons. Because the absolute boundary condition on differential forms is Dirichlet in the normal direction and Neumann in the tangential directions, the associated multiplicative functional is discontinuous and much more difficult to handle. Ikeda and Watanabe [6; 7] have dealt with this situation by using an excursion theory (for reflecting Brownian motion) that seems to have been created especially for this problem. In this paper we suggest a different approach that is based on an idea of approximation due to Airault [1]. This construction has the advantage that a key property of the multiplicative functional (i.e., the attendant Itô's formula for this functional) follows almost automatically from the approximate multiplicative functional without resorting to excursion theory, thus greatly simplifying this part of the theory; see Theorem 3.7.

Before coming to another and more important raison d'être for the present work, we briefly review some relevant facts for a closed manifold. Let M be a compact, closed Riemannian manifold and let α_0 be a 1-form on M. Consider the following initial value problem:

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} = \dfrac{1}{2}\Box\alpha, \\ \alpha(\cdot, 0) = \alpha_0. \end{cases} \tag{1.1}$$

Here $\Box = -(d^*d + d^*d)$ is the Hodge–de Rham Laplacian on differential forms. Let $\Delta = \operatorname{trace}\nabla^2$ be the covariant Laplacian. Then we have the Weitzenböck formula

$$\Box\alpha = \Delta\alpha - \operatorname{Ric}\alpha,$$

where $\operatorname{Ric}_x : T_x^*M \to T_x^*M$ is the Ricci curvature transform. The solution can be represented probabilistically as follows. Let $\{x_t\}$ be a Brownian motion on M

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and let $\{u_t\}$ be its horizontal lift in the orthonormal frame bundle $O(M)$ starting from a frame $u_0 : \mathbb{R}^n \to T_x M$, which we will use to identify $T_x M$ with $\mathbb{R}^n$. Let $\mathrm{Ric}_u : \mathbb{R}^n \to \mathbb{R}^n$ be the Ricci curvature transform at the frame u and consider the matrix-valued multiplicative functional $\{M_t\}$ defined along each path X by

$$\frac{dM_t}{dt} + \frac{1}{2} M_t \,\mathrm{Ric}_{u_t} = 0, \qquad M_0 = I.$$

The solution of the heat equation can be represented as

$$\alpha(x,t) = \mathbb{E}_x\{M_t u_t^{-1} \alpha_0(x_t)\}. \tag{1.2}$$

Among many applications of this representation is the following. Consider the heat semigroup

$$P_t f(x) = \int_M p(t,x,y) f(y)\, dy, \quad f \in C^\infty(M).$$

Since the exterior differentiation commutes with the Hodge–de Rham Laplacian, it follows that $\alpha = d(P_t f)$ is a solution of (1.1) with the intial condition $\alpha_0 = df$; hence

$$|\nabla P_t f(x)|_2 \le \mathbb{E}_x\{|M_t|_{2,2} |\nabla f(x_t)|_2\}.$$

Let $\lambda(x)$ be the lower bound of the Ricci curvature at x. Then we have (obviously) that

$$|M_t|_{2,2} \le \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\, ds\right].$$

This gives the gradient estimate due to Elworthy [4]:

$$|\nabla P_t f(x)|_2 \le \mathbb{E}_x\left\{|\nabla f(x_t)|_2 \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\, ds\right]\right\}. \tag{1.3}$$

Other applications include explicit formulas of Bismut [2] and an integration-by-parts formula proved by Driver [D] (cf. Stroock and Zeitouni [10] and Hsu [5]).

The present work grows out of an attempt to generalize these and other interesting results to manifolds with boundary. As we will explain in this paper, such generalizations are by no means routine. In particular, we want to clarify the role of the Neumann boundary condition in the gradient estimate (1.3). We note that Qian [8] proved that (1.3) still holds if the boundary is convex. It is therefore natural to expect a general gradient estimate involving the second fundamental form integrated against the boundary local time of reflecting Brownian motion. In the course of our investigation, we find it necessary to give a different construction of the multiplicative functional, one where the second fundamental form is placed on a similar footing with the Ricci curvature. Based on this construction, we find the proper generalization of the gradient estimate (1.3):

$$|\nabla P_t f(x)|_2 \le \mathbb{E}_x\left\{|\nabla f(x_t)|_2 \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\, ds - \int_0^t h(x_s)\, dl_s\right]\right\},$$

where $\{x_s\}$ is a reflecting Brownian motion, L its boundary local time, and $h(x)$ the lower bound of the second fundamental form at $x \in \partial M$. If M is convex then we have $h \geq 0$ and the preceding inequality reduces to (1.3), thus recovering the result of Qian just mentioned.

2. Reflecting Brownian Motion

Throughout this paper, we assume that M is a compact Riemannian manifold of dimension n with boundary ∂M. The bundle of orthonormal frames is denoted by $O(M)$, with the canonical projection $\pi : O(M) \to M$. A frame $u \in O(M)$ is an isometry $u: \mathbb{R}^n \to T_x M$, the tangent space at $x = \pi u$. A curve $\{u_t\}$ in $O(M)$ is horizontal if, for any $e \in \mathbb{R}^n$, the vector field $\{u_t e\}$ is parallel along the curve $\{\pi u_t\}$. A vector on $O(M)$ is horizontal if it is the tangent vector of a horizontal curve. For each $v \in T_x M$ and a frame $u \in O(M)$ such that $\pi u = x$, there is a unique horizontal vector V, called the horizontal lift of v, such that $\pi_* V = v$. For each $i = 1, \ldots, n$, let $H_i(u)$ be the horizontal lift of $ue_i \in T_x M$. Each H_i is a horizontal vector field on $O(M)$, and $H_1, \ldots, H_n$ are the fundamental horizontal vector fields on $O(M)$. Bochner's horizontal Laplacian is $\Delta_{O(M)} = \sum_{i=1}^n H_i^2$.

For a point $x \in \partial M$, we denote by $\nu(x)$ the inward unit normal vector at x. Its horizontal lift at u is denoted by $N(u)$. Thus, N is a vector field on the boundary

$$\partial O(M) = \{u \in O(M) : \pi u \in \partial M\}.$$

Let $w = \{w_t\}$ be a Euclidean Brownian motion and consider the following stochastic differential equation on $O(M)$ with normally reflecting boundary condition:

$$du_t = \sum_{i=1}^n H_i(u_t) \circ dw_t^i + N(u_t)\, dl_t. \tag{2.1}$$

By general theory, there is a unique solution to this equation starting from any given initial frame u_0. The process $\{u_t\}$ is a horizontal reflecting Brownian motion. Let $x_t = \pi u_t$. Then it is well known that $\{x_t\}$ is a reflecting Brownian motion on M, that is, a diffusion process on M generated by the Laplace–Beltrami operator $\Delta_M/2$ with the Neumann boundary condition. Its transition density function is the Neumann heat kernel $p(t, x, y)$. The nondecreasing process l is the boundary local time, which increases only when $u_t \in \partial O(M)$ or, equivalently, when $x_t \in \partial M$.

We denote the space of $n \times n$ matrices by $\mathcal{M}_n$. Now suppose that we have two smooth functions

$$R: O(M) \to \mathcal{M}_n, \qquad A: \partial O(M) \to \mathcal{M}_n.$$

Define the $\mathcal{M}_n$-valued, continuous multiplicative functional $\{M_t\}$ by

$$dM_t + M_t \left\{ \frac{1}{2} R(u_t)\, dt + A(u_t)\, dl_t \right\} = 0, \quad M_0 = I.$$

The following lemma shows that $\{M_t\}$ is the multiplicative functional associated with the operator

$$\mathcal{L} = \frac{\partial}{\partial s} - \frac{1}{2}[\Delta_{O(M)} - R]$$

with the boundary condition

$$(N - A)F = 0 \text{ on } \partial O(M). \tag{2.2}$$

Let $\nabla^H F = \{H_1 F, H_2 F, \dots, H_n F\}$ be the horizontal gradient of a function F on $O(M)$.

LEMMA 2.1. *Let $F\colon O(M) \times \mathbb{R}_+ \to \mathbb{R}^n$ be a smooth function. Then*

$$\begin{aligned} M_t F(u_t, T - t) = F(u_0, T) &+ \int_0^t \langle M_s \nabla^H F(u_s, T - s), dw_s\rangle \\ &+ \int_0^t M_s \mathcal{L} F(u_s, T - s)\, ds \\ &+ \int_0^t M_s [N - A] F(u_s, T - s)\, dl_s. \end{aligned}$$

Proof. Apply Itô's formula to $M_t F(u_t, T - t)$ and use equation (2.1) for the horizontal reflecting Brownian motion u. □

3. Discontinuous Multiplicative Functional

In Section 4 we will show that the heat equation on 1-forms with the absolute boundary condition is equivalent to the following heat equation on an $O(n)$-invariant function $F\colon O(M) \times \mathbb{R}_+ \to \mathbb{R}^n$:

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{2}[\Delta_{O(M)} - R]F, \\ F(\cdot, 0) = f, \\ [QN - (H + P)]F = 0. \end{cases}$$

Here $R = \text{Ric}$ is the Ricci transform. Let's explain the notation in the boundary condition. For each $x \in \partial M$, let $P(x)\colon T_x M \to T_x M$ be the projection onto the 1-dimensional normal subspace spanned by the normal vector $n(x)$, and let $P(u) = u^{-1}P(x)u\colon \mathbb{R}^n \to \mathbb{R}^n$ be its lift to the frame space $O_u(M)$. Thus $P(u)$ is the projection onto the 1-dimensional subspace spanned by $N(u)$. Let $Q(u) = I - P(u)$. Let $H(x)\colon T_x\partial M \to T_x\partial M$ be the second fundamental form of the boundary ∂M at x. We can regard it as a linear transform on $T_x M$ by letting $H(x)\nu(x) = 0$. Let $H(u) = u^{-1}H(x)u\colon \mathbb{R}^n \to \mathbb{R}^n$ be its lift to $O_u(M)$. The boundary condition in the heat equation just displayed consists of two independent components:

$$Q[N - H]F = 0, \qquad PF = 0. \tag{3.1}$$

In contrast with (2.2), this is a degenerate boundary condition, because Q is a degenerate matrix. Our goal in this section is to construct the matrix-valued multiplicative functional associated with this heat equation. The main idea, which goes back to [1], is to replace the Q in (3.1) by $Q + \varepsilon I$ and rewrite the boundary condition as

$$\left[N - H - \frac{P}{\varepsilon}\right]F = 0.$$

According to Lemma 2.1, the multiplicative functional for this approximate boundary condition is given by

$$dM_t^\varepsilon + M_t\left\{\frac{1}{2}R(u_t)\,dt + \left[\frac{1}{\varepsilon}P(u_t) + H(u_t)\right]dl_t\right\} = 0. \tag{3.2}$$

The technical part of this work is to show that M^ε converges to a discontinuous multiplicative functional M (as $\varepsilon \downarrow 0$) that is the right one for the boundary condition (3.1). In order not to interrupt our exposition, we will move some proofs to the last section.

Let's start with a few properties of M^ε. Let

$$\begin{cases} \lambda(x) = \inf_{v\in T_xM, |v|=1}\langle R(x)v, v\rangle, \\ h(x) = \inf_{v\in T_x\partial M, |v|=1}\langle H(x)v, v\rangle. \end{cases} \tag{3.3}$$

LEMMA 3.1. *For all positive ε such that $\varepsilon^{-1} \geq \min_{x\in\partial M} h(x)$, we have*

$$|M_t^\varepsilon|_{2,2} \leq \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\,ds - \int_0^t h(x_s)\,dl_s\right].$$

Here $|\cdot|_{2,2}$ denotes the norm of a matrix as a linear map on $\mathbb{R}^n$ with the standard Euclidean norm.

Proof. In this proof we drop the superscript ε for simplicity. Since $|M_t^\dagger|_{2,2} = |M_t|_{2,2}$, it is enough to show the inequality for $M_t^\dagger$, the transpose of M_t. Let $v \in \mathbb{R}^n$ and consider the function

$$f(t) = |M_t^\dagger v|^2 = v^\dagger M_t M_t^\dagger v.$$

Differentiating with respect to t, we have

$$d\{f(t)\} = -2v^\dagger M_t\left\{\frac{1}{2}R(u_t)\,dt + \left[\frac{1}{\varepsilon}P(u_t) + H(u_t)\right]dl_t\right\}M_t^\dagger v.$$

For the terms involving the boundary local time, by our assumption on ε we have

$$v^\dagger M_t\left[\frac{P(u_t)}{\varepsilon} + H(u_t)\right]M_t^\dagger v \geq h(x_s)|M_t^\dagger v|^2.$$

Hence we obtain the inequality

$$df(t) \leq -f(t)\{\lambda(x_t)\,dt + 2h(x_t)\,dl_t\}.$$

Solving this differential inequality yields

$$f(t) \leq f(0)\exp\left[-\int_0^t \lambda(x_s)\,ds - 2\int_0^t h(x_s)\,dl_s\right].$$

The desired result follows immediately. □

In view of the inequality in Lemma 3.1, we need the following integrability result concerning the boundary local time.

LEMMA 3.2. *For any positive constant C, there is a constant C_1 dependent on C but independent of x such that*

$$\mathbb{E}_x e^{Cl_t} \le C_1 e^{C_1 t}.$$

Proof. By the definition of the boundary local time,

$$\mathbb{E}_x l_t = \int_0^t ds \int_{\partial M} p(s, x, y)\sigma(dy),$$

where σ is the Riemannian volume measure of the boundary ∂M. The Neumann heat kernel $p(s, x, y)$ can be constructed by the method of parametrix (see Sato and Ueno [9]), and we have a Gaussian type upper bound for $(s, x, y) \in (0, 1] \times M \times M$:

$$p(s, x, y) \le \frac{C}{t^{d/2}} e^{-d(x,y)^2/Ct}.$$

Hence, by a simple calculation we have the inequality

$$\mathbb{E}_x l_t \le C_2 \sqrt{t}$$

for some constant C_2 independent of x and $t \in [0, 1]$.

We now proceed inductively. Suppose that

$$\mathbb{E}_x l_t^n \le K_n t^{n/2} \quad \text{for all } \ x \in M.$$

From

$$l_t^n = n \int_0^t [l_t - l_s]^{n-1}\, dl_s,$$

we have

$$\begin{aligned}
\mathbb{E}_x l_t^n &= n\mathbb{E}_x \int_0^t \{\mathbb{E}_{x_s} L_{t-s}^{n-1}\}\, dl_s \\
&\le nK_{n-1}\mathbb{E} \int_0^t (t-s)^{(n-1)/2}\, dl_s \\
&= \frac{1}{2} n(n-1)K_{n-1}\mathbb{E} \int_0^t (t-s)^{(n-3)/2} l_s\, ds \\
&\le \frac{1}{2} n(n-1)K_{n-1}C_2 \int_0^t s^{1/2}(t-s)^{(n-3)/2}\, ds \\
&\le \sqrt{n} K_{n-1} C_3 t^{n/2}.
\end{aligned}$$

We can afford to be generous and choose K_n such that

$$K_n = nC_3 K_{n-1} \quad \text{or} \quad K_n = n!\, C_3^n.$$

Now it is clear that, if $t \le 1/2CC_3 \overset{\text{def}}{=} t_0$, then

$$\mathbb{E}_x e^{Cl_t} \le \sum_{n=0}^{\infty} \frac{C^n}{n!} \mathbb{E} l_t^n \le \sum_{n=0}^{\infty} (CC_3 t)^n = \frac{1}{1 - CC_3 t} \le 2.$$

For any $t \geq t_0$, let $k = [t/t_0]$. Then

$$\mathbb{E}_x e^{Cl_t} \leq \left[\sup_{z \in M} \mathbb{E}_z e^{Cl_{t_0}} \right]^k \sup_{z \in M} \mathbb{E}_z e^{Cl_{t-kt_0}} \leq 2^{k+1}.$$

It is easy to verify that $k + 1 \leq 4CC_3 t$, hence

$$\mathbb{E}_x e^{Cl_t} \leq 2^{4CC_3 t}.$$

This completes the proof. □

Define

$$\begin{aligned} T_{\partial M} &= \inf\{s \geq 0 : x_s \in \partial M\} \\ &= \text{the first hitting time of } \partial M. \end{aligned}$$

A point $t \geq T_{\partial M}$ such that $l_t - l_{t-\delta} > 0$ for all positive $\delta \leq t$ is called a *left support point* of the boundary local time L.

LEMMA 3.3. *We have $M_t^\varepsilon P(u_t) \to 0$ for all left support points $t \geq T_{\partial M}$.*

Proof. See Lemma 6.1 (in Section 6). □

We now come to the main result of this section—namely, the limit $\lim_{\varepsilon \to 0} M_t^\varepsilon = M_t$ exists. The first thing to do is identify the limit. From the definition of M_t^ε we see that, if t is such that $x_t \notin \partial M$, then

$$dM_t^\varepsilon + \tfrac{1}{2} M_t^\varepsilon R(u_t)\, dt = 0.$$

Let $\{e(s,t),\ t \geq s\}$ be the solution of

$$\frac{d}{dt} e(s,t) + \frac{1}{2} e(s,t) R(u_t) = 0, \qquad e(s,s) = I.$$

Then, for $t \geq T_{\partial M}$,

$$M_t^\varepsilon = M_{t_*}^\varepsilon e(t_*, t),$$

where for each $t \geq T_{\partial M}$ we have

$$\begin{aligned} t_* &= \sup\{s \leq t : x_s \in \partial M\} \\ &= \text{the last exit time from } \partial M \text{ before } t. \end{aligned}$$

Now we extend $P : \partial O(M) \to \mathcal{M}_n$ to a smooth, projection matrix-valued function on the whole bundle $O(M)$ and define

$$Y_t^\varepsilon = M_t^\varepsilon P(u_t), \qquad Z_t^\varepsilon = M_t^\varepsilon Q(u_t).$$

Note that $Y_t^\varepsilon + Z_t^\varepsilon = M_t^\varepsilon$. We have

$$Y_t^\varepsilon = I_{\{t < T_{\partial M}\}} M_t^\varepsilon P(u_t) + I_{\{t \geq T_{\partial M}\}} M_t^\varepsilon P(u_t).$$

If $t \leq T_{\partial M}$, then $M_t^\varepsilon = e(0,t)$; otherwise,

$$M_t^\varepsilon = \{Z_{t_*}^\varepsilon + Y_{t_*}^\varepsilon\} e(t_*, t).$$

Hence we can write

$$Y_t^\varepsilon = I_{\{t\le T_{\partial M}\}} e(0,t)P(u_t) + I_{\{t>T_{\partial M}\}} Z_{t_*}^\varepsilon e(t_*,t)P(u_t) + \alpha_t^\varepsilon, \tag{3.4}$$

where

$$\alpha_t^\varepsilon = I_{\{t\ge T_{\partial M}\}} Y_{t_*}^\varepsilon e(t_*,t)P(u_t). \tag{3.5}$$

If $t > T_{\partial M}$, then t_* is a left support point of the boundary local time. By Lemma 3.3, $Y_{t_*}^\varepsilon \to 0$ as $\varepsilon \downarrow 0$; hence $\alpha_t^\varepsilon \to 0$. On the other hand, by equation (3.2) for M_t^ε we have

$$\begin{aligned} Z_t^\varepsilon &= Q(u_0) + \int_0^t dM_s^\varepsilon Q(u_s) + \int_0^t M_s^\varepsilon \, dQ(u_s) \\ &= Q(u_0) + \int_0^t [Y_s^\varepsilon + Z_s^\varepsilon]\, d\chi_s, \end{aligned} \tag{3.6}$$

where

$$d\chi_s = -H(u_s)\,dl_s - \tfrac{1}{2}R(u_s)Q(u_s)\,ds + dQ(u_s). \tag{3.7}$$

Note—and this is an important point—that the term involving $1/\varepsilon$ disappears because $P(u_s)Q(u_s) = 0$.

From the equations for Y^ε and Z^ε, we expect that the limit (Y_t, Z_t) is the solution of the following equations:

$$\begin{cases} Y_t = I_{\{t\le T_{\partial M}\}} e(0,t)P(u_t) + I_{\{t>T_{\partial M}\}} Z_{t_*} e(t_*,t)P(u_t), \\ Z_t = Q(u_0) + \int_0^t (Y_s + Z_s)\, d\chi_s. \end{cases} \tag{3.8}$$

Substituting the first equation into the second, we obtain an equation for Z itself in the form

$$Z_t = Q(u_0) + \int_0^t \Phi(Z)_s \, d\chi_s, \tag{3.9}$$

where

$$\Phi(Z)_s = Z_s + I_{\{s\le T_{\partial M}\}} e(0,s)P(u_s) + I_{\{s>T_{\partial M}\}} Z_{s_*} e(s_*,s)P(u_s).$$

THEOREM 3.4. *Equation* (3.9) *has a unique solution Z. Define Y by the first equation in* (3.8) *and let $M_t = Y_t + Z_t$. Then $\{M_t\}$ is right continuous with left limits. Furthermore $M_t P(u_t) = 0$ whenever $x_t \in \partial M$.*

Proof. See Theorem 6.2. □

We now come to the main convergence result. For a stochastic process $V = \{V_t\}$, we define

$$|V|_{\infty,t} = \sup_{0\le s\le t} |V_s|.$$

THEOREM 3.5. *We have, as $\varepsilon \downarrow 0$,*

$$\mathbb{E}|Z^\varepsilon - Z|_{\infty,t}^2 \to 0, \qquad \mathbb{E}|Y_t^\varepsilon - Y_t|^2 \to 0.$$

Hence $\mathbb{E}|M_t^\varepsilon - M_t|^2 \to 0$ as $\varepsilon \to 0$.

Proof. See Theorem 6.3. □

COROLLARY 3.6. *$\{M_t\}$ is a multiplicative functional, and*

$$|M_t|_{2,2} \le \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\,ds - \int_0^t h(x_s)\,dl_s\right].$$

Proof. The first assertion follows because $\{M_t^\varepsilon\}$ is a multiplicative functional. Letting $\varepsilon \to 0$ in Lemma 3.1, we obtain the second assertion. □

We are now in a position to prove the following important property of the multiplicative functional just constructed. Recall that N denotes the horizontal lift of the inward normal vector field on ∂M.

THEOREM 3.7. *Let $F\colon O(M) \times \mathbb{R}_+ \to \mathbb{R}$ be a smooth function such that $P(u)F(u,t) = 0$ for all $u \in \partial O(M)$ and $t \ge 0$. Then we have*

$$\begin{aligned} M_t F(u_t, T-t) = F(u_0, T) &+ \int_0^t \langle M_s \nabla^H F(u_s, T-s), dw_s\rangle \\ &+ \int_0^t M_s \mathcal{L} F(u_s, T-s)\,ds \\ &+ \int_0^t M_s [QN - H] F(u_s, T-s)\,dl_s. \end{aligned}$$

Proof. From Theorem 2.1, we have

$$\begin{aligned} M_t^\varepsilon F(u_t, T-t) = F(u_0, T) &+ \int_0^t \langle M_s^\varepsilon \nabla^H F(u_s, T-s), dw_s\rangle \\ &+ \int_0^t M_s^\varepsilon \mathcal{L} F(u_s, T-s)\,ds \\ &+ \int_0^t M_s^\varepsilon \left[N - \frac{1}{\varepsilon}P - H\right] F(u_s, T-s)\,dl_s. \end{aligned}$$

The terms involving $1/\varepsilon$ vanish because $P(u_s)F(u_s, T-s) = 0$ for $u_s \in \partial O(M)$. Using Theorem 3.5, we take the limit as $\varepsilon \to 0$ to obtain the desired equality. Note that we can insert a $Q(u_s)$ before N in the local time integral because, on the support of the local time, we have $x_s \in \partial M$ and $M_s = M_s Q(u_s)$ by Theorem 3.4. □

REMARK 3.8. The existence of the multiplicative functional $\{M_t\}$ and the probabilistic representation of the solution of the heat equation (see the next section) were proved in Ikeda and Watanabe [6; 7]. Our approach is different. By using the approximate multiplicative functional suggested by Airault [1], we are able to prove Theorem 3.7 without recourse to excursion theory. Also, by not localizing the argument, we have clarified the role of the second fundamental form. More importantly, we are able to obtain Corollary 3.6 without any extra effort. As we mentioned in Section 1, this inequality was one of the main reasons that motivated the current investigation.

4. Heat Equation on 1-Forms

A probabilistic representation of the solution of the initial boundary value problem for the heat equation associated with the Hodge–de Rham Laplacian on differential forms with absolute boundary condition can be obtained easily once we identify the boundary condition in the form discussed in the previous section.

Let α be a k-form M. At each point $x \in \partial M$, let $Q(x)\colon T_xM \to T_xM$ be the projection to the tangent space $T_x\partial M \subseteq T_xM$. The tangential component $\alpha_{\tan}$ is defined by

$$\alpha_{\tan}(v_1, \dots, v_k) = \alpha(Qv_1, \dots, Qv_k), \quad v_i \in T_xM.$$

The normal component is defined as

$$\alpha_{\text{norm}} = \alpha - \alpha_{\tan}.$$

The form α is said to satisfy the absolute boundary condition if

$$\alpha_{\text{norm}} = 0 \quad \text{and} \quad (d\alpha)_{\text{norm}} = 0.$$

Let Λ_x^*M be the space of differential forms at $x \in M$. If $x \in \partial M$, we will use $P(x)\colon \Lambda_x^*M \to \Lambda_x^*M$ to denote the orthogonal projection to the normal component; that is, $P(x)\alpha = \alpha_{\text{norm}}$. Let $Q(x) = I - P(x)$.

An orthonormal frame $u \in O(M)$ at $x = \pi u$ can be regarded canonically as an isometry $u\colon \Lambda^*\mathbb{R}^n \to \Lambda_x^*M$. For a differential form α on M, its scalarization $F_\alpha\colon O(M) \to \Lambda^*\mathbb{R}^n$ is defined by $F_\alpha(u) = u^{-1}\alpha(\pi u)$. As such, it is an $\mathbb{R}^n$-valued function on $O(M)$, which is $O(n)$-invariant: $F_\alpha(ug) = gF_\alpha(u)$ for all $g \in O(n)$. Conversely, any $O(n)$-invariant, $\mathbb{R}^n$-valued function on $O(M)$ is the scalarization of a differential form on M.

For simplicity, from now on we consider only 1-forms. Parallel discussion can be made for forms of higher degrees. The covariant Laplacian $\Delta = \operatorname{trace}\nabla^2$ on M is related to Bochner's horizontal Laplacian $\Delta_{O(M)} = \sum_{i=1}^n H_i^2$ on $O(M)$ by

$$\Delta_{O(M)}F_\alpha(u) = F_{\Delta\alpha}(u).$$

Let $\Box = -(dd^* + d^*d)$ be the Hodge–de Rham Laplacian on differential forms. The Hodge–de Rham Laplacian $\Box$ and the covariant Laplacian Δ are related by the Weitzenböck formula

$$\Box\alpha = \Delta\alpha - \operatorname{Ric}\alpha,$$

where $\operatorname{Ric}(x)\colon T_x^*M \to T_x^*M$ is the Ricci curvature transform (tensor). This formula can be lifted to $O(M)$ to read

$$\Box_{O(M)}F_\alpha = \Delta_{O(M)}F_\alpha - \operatorname{Ric} F_\alpha,$$

where now $\operatorname{Ric}(u)\colon \mathbb{R}^n \to \mathbb{R}^n$ is the lift of the Ricci transform $\operatorname{Ric}(x)$.

We will express the absolute boundary condition in terms of scalarizations on $O(M)$. As before, let $N(u)$ be the horizontal lift of the inward unit normal vector $n(x)$ to a frame u at x. The second fundamental form $H\colon T_x\partial M \otimes_{\mathbb{R}} T_x\partial M \to \mathbb{R}$ is defined by

$$H(x)(X, Y) = \langle \nabla_X Y, \nu\rangle, \quad X, Y \in T_x\partial M.$$

By duality, $H(x)$ can also be regarded as a linear map $H(x)\colon T_x\partial M \to T_x\partial M$ via the relation

$$\langle HX, Y\rangle = H(X, Y).$$

We extend H to the whole tangent space T_xM by letting $H\nu = 0$. We denote the dual of H still by $H\colon T_x^*M \to T_x^*M$. As usual, at each frame u we can lift H to a linear map:

$$H(u) = u^{-1}H(\pi u)u\colon \mathbb{R}^n \to \mathbb{R}^n.$$

LEMMA 4.1. *A* 1*-form* α *on* M *satisfies the absolute boundary condition if and only if*

$$[QN - H]F_\alpha - PF_\alpha = 0 \ \text{ on } \partial O(M).$$

Proof. It is enough to show that

$$\alpha_{\text{norm}} = 0 \iff PF_\alpha = 0$$

and that, if $\alpha_{\text{norm}} = 0$, then

$$(d\alpha)_{\text{norm}} = 0 \iff [QN - H]F_\alpha = 0.$$

Let $\theta \in T_x^*M$ be defined by $\theta(X) = \langle X, \nu\rangle$. Then $\alpha_{\text{norm}} = \langle\alpha,\theta\rangle\theta$. Thus $\alpha_{\text{norm}} = 0$ if and only if $\langle\alpha,\theta\rangle = 0$. On the other hand,

$$PF_\alpha(u) = u^{-1}P(x)\alpha = u^{-1}(\langle\alpha,\theta\rangle\theta) = \langle\alpha,\theta\rangle u^{-1}\theta.$$

Thus $PF_\alpha = 0$ if and only if $\alpha_{\text{norm}} = 0$.

Next, let $\{\theta^i\}$ be an orthonormal basis for T_x^*M with $\theta^1 = \theta$, and let $\{f_i\}$ be the dual basis. Then $f_1 = \nu$, the inward unit normal vector. The condition $[QN - H]F_\alpha = 0$ on $\partial O(M)$ is equivalent to

$$Q\nabla_\nu\alpha - H\alpha = 0 \ \text{ on } \partial M.$$

Because $f_2, \dots, f_n$ span $T_x^*\partial M$, this in turn is equivalent to

$$(\nabla_\nu\alpha)(f_i) - (H\alpha)(f_i) = 0, \quad i = 2, \dots, n.$$

On the other hand,

$$d\alpha = \sum_{1\le i<j\le n} d\alpha(f_i, f_j)\theta^i\wedge\theta^j.$$

By definition,

$$(d\alpha)_{\text{norm}} = \sum_{i=2}^n d\alpha(f_i,\nu)\theta^1\wedge\theta^i;$$

hence $(d\alpha)_{\text{norm}} = 0$ is equivalent to

$$d\alpha(\nu, f_i) = 0, \quad i = 2, \dots, n.$$

The left side is equal to

$$\begin{aligned} d\alpha(\nu, f) &= \nu\alpha(f_i) + f_i\alpha(\nu) - \alpha([\nu, f_i]) \\ &= (\nabla_\nu\alpha)(f_i) + \alpha(\nabla_\nu f_i) - \alpha([\nu, f_i]) \\ &= (\nabla_\nu\alpha)(f_i) + \alpha(\nabla_{f_i}\nu) \\ &= (\nabla_\nu\alpha)(f_i) - (\nabla_{f_i}\alpha)(\nu) \\ &= (\nabla_\nu\alpha)(f_i) - (H\alpha)(f_i). \end{aligned}$$

Here in the second and fourth steps we have used $\alpha(\nu) = 0$, which follows from $\alpha_{\text{norm}} = 0$. It follows that, under the condition $\alpha_{\text{norm}} = 0$, $(d\alpha)_{\text{norm}} = 0$ if and only if $[QN - H]F_\alpha = 0$. □

Let α_0 be a 1-form on M and consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial\alpha}{\partial t} = \dfrac{1}{2}\Box\alpha, \\ \alpha(\cdot, 0) = \alpha_0, \\ \alpha_{\text{norm}} = 0, \ \ (d\alpha)_{\text{norm}} = 0. \end{cases} \tag{4.1}$$

Let $F = F_\alpha$ be the scalarization of the solution. Then, by Lemma 4.1, system (4.1) is equivalent to the following system on an $\mathbb{R}^n$-valued function on $O(M) \times \mathbb{R}_+$:

$$\begin{cases} \dfrac{\partial F}{\partial t} = \dfrac{1}{2}(\Delta_{O(M)} - \text{Ric})F, \\ F(\cdot, 0) = F_f, \\ [QN - H]F - PF = 0. \end{cases}$$

We have the following probabilistic representation of the solution. Let $\{M_t\}$ be the discountinuous multiplicative functional defined in Section 3.

THEOREM 4.2. *Let F be the scalarization of the solution of the initial boundary value problem* (4.1). *Then*

$$F(u, t) = \mathbb{E}_u\{M_t F_{\alpha_0}(u_t)\}.$$

Equivalently, the solution is given by

$$\alpha(x, t) = \mathbb{E}_x\{M_t u_t^{-1}\alpha_0(x_t)\} \tag{4.2}$$

for any $u \in O(M)$ such that $\pi u = x$.

Proof. We have $P(u)F(u, t - s) = 0$ for all $u \in \partial O(M)$ because F satisfies the absolute boundary condition. That F is a solution implies, by Theorem 3.7, that $\{M_s F(u_s, t - s), 0 \le s \le t\}$ is a martingale. Equating the expected values at $s = 0$ and $s = t$ yields the formula for $F(u, t)$. □

5. A Gradient Inequality

Let $P_t f(x) = \int_M p(t, x, y) f(y)\, dy$ for $f \in C^\infty(M)$, where $p(t, x, y)$ is the Neumann heat kernel on M. We have the following gradient inequality.

THEOREM 5.1. *Suppose that the smallest eigenvalue of the Ricci curvature at x is $\lambda(x)$ and the smallest eigenvalue of the second fundamental form at x is $h(x)$. Then we have the gradient estimate*

$$|\nabla P_t f(x)| \le \mathbb{E}_x\left\{\left|\nabla f(x_t)\right| \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\,ds - \int_0^t h(x_s)\,dl_s\right]\right\}.$$

Proof. Let $\alpha(x,t) = dP_t f(x)$. Then α satisfies the absolute boundary condition because $\partial P_t f/\partial\nu = 0$. Now the Hodge–de Rham Laplacian $\Box$ commutes with d, hence

$$\frac{\partial\alpha}{\partial t} = d\left(\frac{\partial P_t f}{\partial t}\right) = \frac{1}{2}\,d\Box P_t f = \frac{1}{2}\Box\, dP_t f = \frac{1}{2}\Box\alpha.$$

Thus $\alpha = dP_t f$ is a solution to the heat equation (4.1). By Theorem 4.2, we have the following generalization of Bismut's formula (see [2]):

$$\nabla P_t f(x) = \mathbb{E}_x\{M_t u_t^{-1}\nabla f(x_t)\}.$$

The desired inequality follows this and Corollary 3.6. $\Box$

REMARK 5.2. If M is closed (without boundary) or the boundary is convex, we have

$$|\nabla P_t f(x)| \le \mathbb{E}_x\left\{|\nabla f(x_t)| \exp\left[-\frac{1}{2}\int_0^t \lambda(x_s)\,ds\right]\right\}. \tag{5.1}$$

These two special cases were proved by Elworthy [4] and Qian [8], respectively.

6. Some Proofs

This section contains the proofs of the technical results used in Section 3. We retain the notation used throughout the paper. The results are restated for easy reference.

LEMMA 6.1. *We have $M_t^\varepsilon P(u_t) \to 0$ for all left support points $t \ge T_{\partial M}$.*

Proof. For simplicity we write M^ε as M in this proof. Let $a \in \mathbb{R}^n$ and differentiate the function

$$f(s) = |P(u_t)M_s^\dagger a|^2 = a^\dagger M_s P(u_t) M_s^\dagger a.$$

Using the equation for M_s yields

$$df(s) = -\frac{2}{\varepsilon} f(s)\,dl_s + dN_s, \tag{6.1}$$

where the stochastic differential dN_s is equal to

$$\begin{aligned}
&\frac{1}{\varepsilon} a^\dagger M_s\{2P(u_t) - P(u_s)P(u_t) - P(u_t)P(u_s)\}M_s^\dagger a\,dl_s \\
&\quad - a^\dagger M_s\{H(u_s)P(u_t) + P(u_t)H(u_s)\}M_s^\dagger a\,dl_s \\
&\quad - a^\dagger M_s\{R(u_s)P(u_t) + P(u_t)R(u_s)\}M_s^\dagger a\,ds.
\end{aligned}$$

By continuity we have, as $s \uparrow t$ with $x_s \in \partial M$,

$$P(u_t)P(u_s) \to P(u_t)^2 = P(u_t), \qquad P(u_s)P(u_t) \to P(u_t).$$

Hence, by Lemma 3.1, for any $\eta > 0$ there exists a $\delta > 0$ such that, for all $s \in [t-\delta, t]$ with $x_s \in \partial M$,

$$|M_s\{2P(u_t) - P(u_s)P(u_t) - P(u_t)P(u_s)\}M_s^\dagger| \le \eta.$$

Also by Lemma 3.1, there is a constant C such that, for all $s \in [t-\delta, t]$ with $x_s \in \partial M$,

$$|M_s\{H(u_s)P(u_t) + P(u_t)H(u_s)\}M_s| \le C$$

and

$$|M_s\{R(u_s)P(u_t) + P(u_t)R(u_s)\}M_s| \le C.$$

It follows that

$$|dN_s| \le |a|^2\left[\left(\frac{\eta}{\varepsilon} + C\right) dl_s + C\, ds\right].$$

Now, from (6.1) we have

$$f(t) = e^{-2(l_t - l_{t-\delta})/\varepsilon} f(t-\delta) + \int_{t-\delta}^t e^{-2(l_t - l_s)/\varepsilon}\, dN_s.$$

Using the definition of $f(t)$ and the estimate for dN_s, we find that this equation gives

$$\begin{aligned} |M_t P(u_t)|^2 \le e^{-2(l_t - l_{t-\delta})/\varepsilon}|M_{t-\delta}|^2 + \frac{\eta + C\varepsilon}{2}\{1 - e^{-2(l_t - l_{t-\delta})/\varepsilon}\} \\ + C\int_{t-\delta}^t e^{-2(l_t - l_s)/\varepsilon}\, ds. \end{aligned} \tag{6.2}$$

Because t is a left support point, $l_t - l_s > 0$ for all $s < t$. Letting $\varepsilon \to 0$ and then $\eta \to 0$ in (6.2), we have $M_t P(u_t) \to 0$. □

THEOREM 6.2. *Equation* (3.9) *has a unique solution Z. Define Y by the first equation in* (3.8) *and let $M_t = Y_t + Z_t$. Then $\{M_t\}$ is right continuous with left limits. Furthermore, $M_t P(u_t) = 0$ whenever $x_t \in \partial M$.*

Proof. The unique solvability of (3.9) is a consequence of the following three facts.

(1) Φ is Lipschitz in the norm $|Z|_t = \sup_{0\le s\le t}|Z_s|$; that is, there exists a constant independent of Z and t such that

$$|\Phi(Z^1) - \Phi(Z^2)|_t \le Ce^{Ct}|Z^1 - Z^2|_t.$$

(2) If Z is adapted, then $\Phi(Z)$ is also adapted.

(3) The semimartingale differential $d\chi_s$ has the form

$$d\chi_s = a(u_s)\, dw_s + b(u_s)\, ds + c(u_s)\, dl_s, \tag{6.3}$$

with uniformly bounded smooth functions a, b, c on $O(M)$.

By the standard Picard iteration, we know that (3.9) has a unique solution that is a continuous semimartingale. We now define Y by the first equation in (3.8). It is clear that Y is right continuous with left limits, hence so is $M_t = Y_t + Z_t$.

Let us now prove that $M_t P(u_t) = 0$ if $x_t \in \partial M$. This is to be expected because we have imposed the Dirichlet condition in the normal direction. If $x_t \in \partial M$, then $t = t_*$ and we have

$$Y_t = Z_{t_*} Q(u_{t_*}) P(u_{t_*}) = 0.$$

It remains to show that $Z_t = Z_t Q(u_t)$ for all $t \geq 0$, for if this holds then $Z_t P(u_t) = 0$ and this implies

$$M_t P(u_t) = Y_t P(u_t) + Z_t P(u_t) = 0.$$

In the rest of the proof we will abbreviate $Q(u_t)$ as Q_t. By Itô's formula and (3.9),

$$\begin{aligned} d\{Z_t Q_t\} &= M_t\, d\chi_t Q_t + Z_t\, dQ_t + M_t\, d\langle Q, Q\rangle_t \\ &= M_t \{d\chi_t Q_t + d\langle Q, Q\rangle_t\} + Z_t\, dQ_t. \end{aligned}$$

From (3.7) and the fact that Q_t is a projection matrix, we have

$$d\chi_t Q_t = d\chi_t - dQ_t + dQ_t Q_t.$$

Thus, the stochastic differential after M_t is

$$d\chi_t - dQ_t + dQ_t Q_t + d\langle Q, Q\rangle_t = d\chi_t - Q_t\, dQ_t.$$

Hence

$$\begin{aligned} d\{Z_t Q_t\} &= M_t \{d\chi_t - Q_t\, dQ_t\} + Z_t\, dQ_t \\ &= M_t\, d\chi_t + \{Z_t - M_t Q_t\}\, dQ_t. \end{aligned}$$

Using $M_t\, d\chi_t = dZ_t$ and $M_t Q_t = Z_t Q_t$ we have, for $\Sigma_t = Z_t Q_t - Q_t$,

$$d\Sigma_t = -\Sigma_t\, dQ_t, \qquad \Sigma_0 = Q_0^2 - Q_0 = 0.$$

It follows that $\Sigma_t = 0$, and the proof is completed. □

THEOREM 6.3. *We have, as* $\varepsilon \downarrow 0$,

$$\mathbb{E}|Z^\varepsilon - Z|_{\infty,t} \to 0, \qquad \mathbb{E}|Y_t^\varepsilon - Y_t| \to 0.$$

Hence $\mathbb{E}|M_t^\varepsilon - M_t|^2 \to 0$ *as* $\varepsilon \to 0$.

Proof. We write $|Z^\varepsilon - Z|_{\infty,t}$ as $|Z^\varepsilon - Z|_t$ to simplify the notation. From (3.4) and (3.6), we have

$$Z_t^\varepsilon = Q(u_0) + \int_0^t \Phi(Z^\varepsilon)_s\, d\chi_s + \theta_t^\varepsilon,$$

where

$$\theta_t^\varepsilon = \int_0^t \alpha_s^\varepsilon\, d\chi_s. \tag{6.4}$$

Subtracting from this the equation for Z yields

$$Z_t^\varepsilon - Z_t = \int_0^t \{\Phi(Z^\varepsilon)_s - \Phi(Z)_s\}\, d\chi_s + \theta_t^\varepsilon.$$

Let $t \mapsto \tau_t$ be the inverse function of $s \mapsto s + l_s$. Then each τ_t is a stopping time. Switching to the new time scale τ_t and using the Lipschitz property of Φ and standard moment estimates for stochastic integrals, we have

$$\mathbb{E}|Z^\varepsilon - Z|_{\tau_t}^2 \le C_2 \int_0^t \mathbb{E}|Z^\varepsilon - Z|_{\tau_s}^2\, ds + 2\mathbb{E}|\theta_{\tau_t}^\varepsilon|^2.$$

This implies that

$$\mathbb{E}|Z^\varepsilon - Z|_{\tau_t}^2 \le 2\int_0^t e^{C_2(t-s)} \mathbb{E}|\theta_{\tau_s}^\varepsilon|^2\, ds. \tag{6.5}$$

From (6.3), (3.5), (6.4), and the inequalities $\tau_s \le s$ and $l_{\tau_s} \le s$, we now have

$$\mathbb{E}|\theta_{\tau_s}^\varepsilon|^2 \le C\mathbb{E}\int_0^s |M_{s_*}^\varepsilon P(u_{s_*})|^2 \{ds + dl_s\}.$$

It is well known that reflecting Brownian motion does not spend time on the boundary. For a time point s such that $x_s \in M$, s_* is a left support point of the boundary local time. On the other hand, as a measure on $\mathbb{R}_+$, the boundary local time is supported on the set of left support points (and also on the set of right support points, for that matter). Hence, Lemma 3.3 shows that the integrand tends to zero, and Lemma 3.1 shows that the integrand is dominated by an integrable random variable. Therefore, $\mathbb{E}|\theta_{\tau_s}^\varepsilon|^2$ tends to zero boundedly. Now we can let $\varepsilon \to 0$ in (6.5) and obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}|Z^\varepsilon - Z|_{\tau_t}^2 = 0.$$

To show that we can replace τ_t by t, we note first that, since $\tau_s \uparrow \infty$ as $s \uparrow \infty$, this implies in particular that Z_t also satisfies the inequality in Lemma 3.1. Hence, for a fixed $T \ge 0$,

$$|Z^\varepsilon - Z|_{\tau_t \wedge T}^2 \le |Z^\varepsilon - Z|_T^2$$

is bounded by an integrable random variable independent of t. Now,

$$\mathbb{E}|Z^\varepsilon - Z|_T^2 \le \mathbb{E}|Z^\varepsilon - Z|_{\tau_t}^2 + \mathbb{E}\{|Z^\varepsilon - Z|_T^2;\ \tau_t \le T\}.$$

Letting $\varepsilon \to 0$ and then $t \uparrow \infty$, we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}|Z^\varepsilon - Z|_T^2 = 0.$$

Finally, from Lemma 3.1, Lemma 3.3, (3.4), (3.5), and what we have just proved, it follows that $\lim_{\varepsilon\to 0} \mathbb{E}|Y_t^\varepsilon - Y_t|^2 = 0$ for all t. □

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