Probab. Theory Relat. Fields 117, 183–200 (2000)
Digital Object Identifier (DOI) 10.1007/s004400000049

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Pathwise uniqueness for reflecting Brownian motion in Euclidean domains

Received: 3 February 1999 / Revised version: 2 September 1999 /
Published online: 11 April 2000

Abstract. For a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ we show that there exists a strong solution to the multidimensional Skorokhod equation and that weak uniqueness holds for this equation. These results imply that pathwise uniqueness and strong uniqueness hold for the Skorokhod equation.

1. Introduction

Let D be a domain in $\mathbb{R}^d$ and ν the inward-pointing unit normal vector field on ∂D, the boundary of D. Let B be a d-dimensional Brownian motion starting at the origin. Consider the Skorokhod equation for a pair of processes (X, L):

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \nu(X_s)dL_s \quad , \tag{1}$$

where X is a $\overline{D}$-valued continuous process, X_0 is a point in $\overline{D}$, and L is a continuous nondecreasing process which increases only when $X_t \in \partial D$. When D is a C^2 domain it was proved in Lions and Sznitman[13] and Hsu[9] that pathwise uniqueness holds for the equation. In fact, given an $f \in C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ (the space of continuous functions from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}^d$ starting from a point in $\overline{D}$) there is a unique solution (g, l) to the deterministic Skorokhod equation

$$g_t = f_t + \frac{1}{2}\int_0^t \nu(g_s)dl_s \quad .$$

(We often write f_t for $f(t)$.) The map $F : C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d) \to C(\mathbb{R}_+, \overline{D}) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$ given by $F(f) = (g, l)$ is (progressively) measurable and is the unique strong solution to the Skorokhod equation (1). This means that if (B, X, L) are related by

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The research of the first author was supported in part by NSF grant DMS9700721. The research of the second author was supported in part by NSF grant DMS9706910.

Mathematics Subject Classification (1991): 60D58

(1) and B is a Brownian motion (with initial value zero) independent of X_0 then we must have $(X, L) = F(B+X_0)$. Furthermore, X has the law of reflecting Brownian motion. As an application of Itô's formula the process L can be recovered by the formula

$$L_t = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_0^t I_{S_\lambda}(X_s) ds \quad ,$$

where $S_\lambda = \left\{x \in \overline{D} : \mathrm{dist}\,(x, \partial D) \le \lambda\right\}$.

Dupuis and Ishii[5] showed that pathwise uniqueness for Brownian motion with oblique reflection holds for C^1 domains; however they required the angle of reflection to vary in almost a C^2 manner. For normal reflection this implies the domains must be nearly C^2.

For an arbitrary domain, we define reflecting Brownian motion on D to be a $\overline{D}$-valued diffusion process (strong continuous Markov process with continuous sample paths) whose transition density function is the one determined by the Dirichlet form

$$\mathscr{E}(f, f) = \frac{1}{2} \int_D |\nabla f(x)|^2 dx, \qquad f \in H^1(D);$$

(see Fukushima[6]). If D has a rough boundary, such a process does not always exist. But it was proved in Bass and Hsu[3],[4] that if D is Lipschitz, then such an X exists, X is a reflecting Brownian motion as defined above, and the Skorokhod equation holds. The process L in this case is just the continuous additive functional determined by the surface measure of ∂D. This means that if X is a reflecting Brownian motion on a Lipschitz domain D, then there exists a Brownian motion starting at a point in $\overline{D}$ such that (1) holds. More recently, Bass[1] proved that under certain additional conditions on L weak uniqueness holds for the Skorokhod equation on a Lipschitz domain. This means that for such domains if B is a Brownian motion starting from the origin, X_0 is a point in $\overline{D}$, and (B, X, L) satisfies (1), then X is a reflecting Brownian motion.

C^2 domains are smooth enough so that reflecting Brownian motion in such a domain shares many properties with reflecting Brownian motion in a half space, and this fact can be exploited in proving pathwise results. This is no longer the case in less smooth domains such as $C^{1,\alpha}$ domains. (This is analogous to the situation for the Neumann problem in analysis, where there is an extensive literature attempting to extend results known to hold in C^2 domains to less smooth ones.)

The main result of the present paper is Theorem 5.1, which states that in a $C^{1,\alpha}$ domain the solution to the Skorokhod equation is pathwise unique. The method we use is quite different from existing techniques for proving pathwise uniqueness and consists primarily of a measurability argument. First, we prove that for C^1 domains, there exists a strong solution. Second, for $C^{1,\alpha}$ domains we remove the technical conditions imposed in Bass[1], that is, we prove that weak uniqueness holds for $C^{1,\alpha}$ domains. We put these two results together to imply, by a measure-theoretic argument whose origins can be traced back to Knight[11], Perkins, and Girsanov, that there exists a unique strong solution for the Skorokhod equation on $C^{1,\alpha}$ domains and that the solution is pathwise unique.

It is tempting to conjecture that there exists a unique pathwise solution for the Skorokhod equation on Lipschitz domains, but we do not know how to prove this.

2. Deterministic Skorokhod equation

In this section we show that if D is a bounded C^1 domain, then there is a solution to the deterministic Skorokhod equation. Recall that a C^1 function is one whose first partial derivatives are continuous and a $C^{1,\alpha}$ function is one whose first derivatives are Hölder continuous of order α. A domain D is a C^1 domain if for all $z \in \partial D$ there exists a coordinate system CS_z, an $r_z > 0$, and a C^1 function φ_z such that

$$D \cap B(z, r_z) = \{x = (x_1, \ldots, x_d) \text{ in } CS_z : x_d > \varphi_z(x_1, \ldots, x_{d-1})\} \cap B(z, r_z) \ ,$$

i.e., locally D looks like the region above the graph of a C^1 function. Similar definitions apply to $C^{1,\alpha}$ or C^2 domains.

Let S_λ be the shell of width λ around the boundary ∂D:

$$S_\lambda = \left\{x \in \overline{D} : \text{dist}\,(x, \partial D) \le \lambda\right\} \ .$$

For a bounded C^1 domain D, the inward-pointing unit normal vector field $\nu : \partial D \to \mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ is uniformly continuous. Let

$$\theta(\lambda) = \sup\{|\nu(x) - \nu(y)| : x, y \in \partial D, |x - y| \le \lambda\}$$

be the modulus of continuity of ν. Then $\theta(\lambda) \downarrow 0$ as $\lambda \downarrow 0$.

Lemma 2.1. *Let D be a bounded C^1 domain in $\mathbb{R}^d$. Then there exists a positive λ_0 depending only on the modulus of continuity θ of the normal vector field ν such that*

(a) For all $(x, y) \in \partial D \times \partial D$, $|x - y| \le \lambda_0$,

$$|\nu(y) - (\nu(y) \cdot \nu(x))\nu(x)| \le \frac{1}{3}\nu(y) \cdot \nu(x) \ .$$

(b) Let $z \in \partial D$ and $\lambda \le \lambda_0$. Let F be the right circular cylinder which is centered at z with height 6λ, base radius 3λ, and axis parallel to $\nu(z)$. Then the two bases of F lie entirely outside the shell $S_{2\lambda}$.

Proof. (a) It is easy to check that

$$|\nu(y) - (\nu(y) \cdot \nu(x))\nu(x)| \le 2\theta(|x - y|), \quad |\nu(y) \cdot \nu(x) - 1| \le \theta(|x - y|) \ .$$

Thus it is enough to choose λ_0 such that $\theta(\lambda_0) \le 1/7$.

(b) Choose a coordinate system CS_z centered at z such that the unit vector along the x_d-axis is $\nu(z)$. Choose λ_0 such that $\theta(10\lambda_0) \le 1/400$. Since $F \subseteq B(z, 5\lambda)$, it is clear that there is a C^1 function φ defined on $B(z, 10\lambda_0) \cap L$ (where L is the hyperplane perpendicular to $\nu(z)$) such that $D \cap B(z, 10\lambda)$ is the region above the graph of φ.

Suppose that $x \in S_{2\lambda} \cap F$. Then there is a point $y \in \partial D$ such that $|x - y| = \text{dist}\,(x, \partial D)$ and $x = y + |x - y|\nu(y)$; hence $|x_d| \le |y_d| + 2\lambda$. On the other hand,

$|y| \leq |y - x| + |x - z| \leq 2\lambda + 5\lambda = 7\lambda$ and $y \in \partial D$, hence there is a point $y = (w, \varphi(w))$ for some $w \in B(z, 7\lambda) \cap L$. Therefore

$$|y_d| = |\varphi(w)| \leq 7\lambda \sup_{|u| \leq 7\lambda} |\nabla\varphi(u)| \ .$$

Now $\nu(u) = (\nabla\varphi(u), 1)/\sqrt{1 + |\nabla\varphi(u)|^2}$. Comparing the components in the direction of the x_d axis, we have

$$|\nabla\varphi(u)| = \sqrt{\frac{1}{(\nu(u) \cdot \nu(z))^2} - 1} \ .$$

But for $u \in B(z, 7\lambda)$ we have $|\nu(u) \cdot \nu(z) - 1| \leq 1/400$, and the above relation gives $|\nabla\varphi(u)| \leq 1/14$. It follows that $|y_d| \leq \lambda/2$ and hence $|x_d| \leq (5/2)\lambda$. Thus we have shown that $x \in S_{2\lambda} \cap F$ implies $|x_d| < 3\lambda$. Finally, if x is on either of the two bases of F, then $|x_d| = 3\lambda$; this means that x cannot be in $S_{2\lambda}$. □

Definition 2.2. *Let D be a bounded C^1 domain in $\mathbb{R}^d$ and ν its inward-pointing unit normal vector field on ∂D. Let $f \in C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ such that $f_0 \in \overline{D}$. We say that a pair of function (g, l) is a solution to the Skorokhod equation*

$$g_t = f_t + \frac{1}{2}\int_0^t \nu(g_s)dl_s$$

if $g \in C(\mathbb{R}_+, \overline{D})$ and l is a continuous nondecreasing function on $\mathbb{R}_+$ (with initial value $l_0 = 0$) which increases only when $g_t \in \partial D$.

Our strategy for proving the solvability of the Skorokhod equation for a bounded C^1 domain is to approximate D from outside by a sequence of bounded C^2 domains. The existence and uniqueness for the solutions to the Skorokhod equation for C^2 domains are well known. Later we will need the fact that the map $f \mapsto (g, l)$ is continuous for C^2 domains (see Lemma 3.2). This is the content of the next theorem.

Theorem 2.3. *Let D be a bounded C^2 domain. Then for any $f \in C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ there is a unique solution (g, l) to the Skorokhod equation. Furthermore, the map $f \mapsto (g, l)$ is continuous from $C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ to $C(\mathbb{R}_+, \overline{D}) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$.*

Proof. The existence and uniqueness are proved in Lions and Sznitman[13] and Hsu[9]. The continuity of $f \mapsto g$ is proved in [13], Theorem 2.2 on p. 521, so we only need to prove the continuity of $f \mapsto l$.

Let $f^n \to f$ uniformly on bounded intervals. Then $g^n \to g$ does the same. Let $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function with compact support such that $\psi(x) = \nu(x)$ for $x \in \partial D$. We can show that $\{l_t^n\}$ is uniformly bounded just as in part (a) of the proof of Theorem 2.6 below. Let $t_i = it/N$, $i = 0, \ldots, N$. We have

$$\begin{aligned}
l_t &= \int_0^t \psi(g_s)\cdot\nu(g_s)dl_s \\
&= \sum_{i=1}^N \psi(g_{t_i})\cdot\int_{t_{i-1}}^{t_i}\nu(g_s)dl_s + \sum_{i=1}^N\int_{t_{i-1}}^{t_i}\left\{\psi(g_s)-\psi(g_{t_{i-1}})\right\}\cdot\nu(g_s)dl_s \\
&= 2\sum_{i=1}^N \psi(g_{t_{i-1}})\cdot\left\{g_{t_i}-g_{t_{i-1}}-f_{t_i}+f_{t_{i-1}}\right\} \\
&\quad+\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\left\{\psi(g_s)-\psi(g_{t_{i-1}})\right\}\cdot\nu(g_s)dl_s \\
&= I_t^1+I_t^2.
\end{aligned}$$

Similarly we have $l_t^n = I_t^{n,1}+I_t^{n,2}$. Since $g_t^n \to g_t$ uniformly on $[0,T]$, it is clear that for any ϵ, there exists N such that for all $n \geq N$,

$$\sup_{0\le t\le T}|I_t^{n,2}| \le l_T^n\epsilon, \qquad \sup_{0\le t\le T}|I_t^2| \le l_T\epsilon \quad .$$

Fix this N. Again by the uniform convergence of $g_t^n \to g_t$ on $[0,T]$, we see that there exists n_0 depending on N and ϵ such that for all $n \geq n_0$,

$$\sup_{0\le t\le T}|I_t^{n,1}-I_t^1| \le \epsilon \quad .$$

It follows that

$$\sup_{0\le t\le T}|l_t^n - l_t| \le (1+l_T^n+l_T)\epsilon \quad .$$

This shows that $l_t^n \to l_t$ uniformly on $[0,T]$. □

The next result shows that the modulus of continuity of the solution of the Skorokhod equation is completely controlled by that of f and the number λ_0 in Lemma 2.1. For our later application, it is important that the proof of this result depends on D only through the modulus of continuity θ of the normal vector field on the boundary.

For a continuous function taking values in $\mathbb{R}^d$, let

$$\omega_T(\delta;h) = \sup\{|h_s-h_t| : 0\le s,t\le T, |t-s|\le\delta\} \quad .$$

We denote the range of a path h over a time interval $[s,t]$ by $h[s,t]$.

Proposition 2.4. *Let D be a bounded C^2 domain in $\mathbb{R}^d$ and $f\in C_{\overline{D}}(\mathbb{R}_+,\mathbb{R}^d)$. Let (g,l) be the solution of the Skorokhod equation for D with the driving path f. For each fixed $T>0$, there exists a $\delta_0=\delta_0(\theta,f)>0$ such that $\omega_T(\delta;g)\le 9\omega_T(\delta;f)$ for all $\delta\le\delta_0$,*

Proof. Set $\lambda = \omega_T(\delta;f)$. We can choose δ_0 small such that $\delta \le \delta_0$ implies $\lambda \le \lambda_0/5$ for the λ_0 in Lemma 2.1. Suppose that $s,t\in[0,T]$ and $|t-s|\le\delta$. One case can be dismissed quickly, namely when the path $g[s,t]$ lies entirely in D

and does not intersect ∂D. In this case l does not increase on $[s,t]$. We then have $g_s - g_t = f_s - f_t$. Hence

$$|g_s - g_t| \le \omega_T(\delta; f) \le \lambda \ .$$

So it is enough to consider the case when there is a point $u_0 \in [s,t]$ such that $g_{u_0} \in \partial D$. Fix such a u_0.

We first show that the assumption that $g_{u_0} \in \partial D$ implies that the whole path $g[s,t]$ lies within a narrow shell around the boundary; more precisely, $g[s,t] \subseteq S_{2\lambda}$. If this were not the case, then there must be a time $u \in [s,t]$ such that $g_u \in D\backslash S_{2\lambda}$. Assume without loss of generality that $u < u_0$. Let $w \in [u, u_0]$ be the first time such that $g_w \in \partial D$. Then $g[u,w)$ lies entirely in D. This means that l does not increases on the time interval $[u,w]$. Hence

$$2\lambda \le |g_u - g_w| = |f_u - f_w| \le \omega_T(\delta; f) \le \lambda \ .$$

This is a contradiction.

We now choose a coordinate system centered at $z = g_{u_0} \in \partial D$ such that the unit vector along the x^d-axis $e_d = \nu(z)$. Let F be the right circular cylinder whose axis is parallel to $\nu(z)$ and which is centered at the origin with height 6λ and base radius 3λ. We will show that $g[s,t] \subseteq F$. Since $F \subseteq B(0, 9\lambda/2)$, this will imply that $|g_s - g_t| \le 9\lambda$ and the proof of the equicontinuity of $\{g^n\}$ will be completed.

Obviously $g_{u_0} \in F$. Let

$$\tau = \sup\{u \le u_0 : g_u \notin F\}, \quad \sigma = \inf\{u \ge u_0 : g_u \notin F\} \ .$$

What we want amounts to showing $\tau < s$ and $\sigma > t$. The two cases being similar we prove the first statement.

Suppose on the contrary that $\tau \in [s, u_0]$. By our assumption $\lambda \le \lambda_0$, the bases of F lie entirely outside the shell $S_{2\lambda}$ (see Lemma 2.1). We have shown that the shell $S_{2\lambda}$ contains the entire path $g[s,t]$. Hence the exit position g_τ must be on the side surface of F. This implies that the horizontal part (the component perpendicular to $\nu(z)$, the axis of the cylinder) of g has to travel a distance at least 3λ from time τ to u_0. This is not possible, because the displacement of g between these times is the sum of that of f, which is at most λ, and the integral $\int_\tau^{u_0} \nu(g_u)dl_u$, which is almost along the vertical direction $\nu(z)$. The rest of this proof is to make precise this intuition.

For a vector γ, we denote its vertical and horizontal components by $\gamma^V = (\gamma \cdot \nu(z))\nu(z)$ and $\gamma^H = \gamma - \gamma^V$, respectively. Since the exit position g_τ is on the side surface of F, we have $|g_\tau^H| \ge 3\lambda$. Hence

$$\frac{1}{2}\left|\int_\tau^{u_0} \nu(g_u)^H dl_u\right| \ge |g_{u_0}^H - g_\tau^H| - |f_{u_0}^H - f_\tau^H| \ge 3\lambda - \lambda = 2\lambda \ .$$

The path $g[\tau, u_0]$ lies entirely in $F \subseteq B(0, 9\lambda/2) \subset B(0, \lambda_0)$. Hence by Lemma 2.1(a), for $u \in [\tau, u_0]$,

$$|\nu(g_u)^H| \le \frac{1}{3}\nu(g_u) \cdot e_d \ .$$

It follows that

$$\frac{1}{2}\left|\int_\tau^{u_0} \nu(g_u)^V dl_u\right| = \frac{1}{2}\left|\int_\tau^{u_0} \{\nu(g_u)\cdot e_d\}\, dl_u\right|$$
$$\geq \frac{3}{2}\int_\tau^{u_0} |\nu(g_u)^H| dl_u$$
$$\geq 6\lambda \ .$$

This in turn implies that

$$|g_\tau^V| \geq |g_{u_0}^V - g_\tau^V| - |g_{u_0}^V|$$
$$\geq \frac{1}{2}\left|\int_\tau^{u_0} \nu(g_u)^V dl_u\right| - |f_{u_0}^V - f_\tau^V| - \lambda$$
$$\geq 6\lambda - \lambda - \lambda$$
$$= 4\lambda \ .$$

This is a contradiction because $g_\tau \in F$ implies $|g_\tau^V| \leq 3\lambda$, the half-height of the cylinder F. □

Let D be a bounded C^1 domain. We can choose a sequence $\{D_n\}$ of bounded C^2 domains with the following properties:

(i) $D \subseteq D_n$, $D_n \downarrow \overline{D}$, and $\partial D_n \to \partial D$;
(ii) $\nu^n(x_n) \to \nu(x)$ if $x_n \in \partial D_n$, $x \in \partial D$, and $x_n \to x$; here ν^n is the inward-pointing unit normal vector field on ∂D_n;
(iii) the set of functions $\{\nu^n\}$ is equicontinuous; therefore there is an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\theta(0) = 0$ such that

$$|\nu^n(x) - \nu^n(y)| \leq \theta(|x - y|)$$

for all $n \geq 1$ and all $x, y \in \partial D_n$; and there is a positive λ_0 such that (a) and (b) of Lemma 2.1 hold for every D_n.

For star-like domains D this can be done as in [1], Prop. 3.4; for the general case one can use a partition of unity.

Let $f \in C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ and (g^n, l^n) the solution to the Skorokhod equation on D_n with driving path f:

$$g_t^n = f_t + \frac{1}{2}\int_0^t \nu^n(g_s^n) dl_s^n \ . \tag{2}$$

Theorem 2.5. *The sequence $\{g^n\}$ is equicontinuous on each finite interval.*

Proof. By our choice of $\{D_n\}$ and Proposition 2.4 there exists $\delta_0 = \delta_0(\theta, f)$ such that $\omega_T(\delta; g^n) \leq 9\omega_T(\delta; f)$ for all $\delta \leq \delta_0$. □

Theorem 2.6. *Let (g^n, l^n) be as above. Suppose that a subsequence $\{g^{n_j}\}$ of $\{g^n\}$ converges to g. Then l^{n_j} converges uniformly on every finite interval to a continuous, nondecreasing function l which increases only when $g_t \in \partial D$. Furthermore the Skorokhod equation holds:*

$$g_t = f_t + \frac{1}{2}\int_0^t \nu(g_s)dl_s \quad .$$

Proof. In this proof, we assume that a subsequence of integers $\{n_j\}$ has been fixed such that $\{g^{n_j}\}$ converges to g. When we say a sequence converges as n goes to infinity we always mean it converges through the subsequence $\{n_j\}$. It is enough to consider a fixed interval $[0, T]$. For simplicity we let

$$h_t^n = g_t^n - f_t, \quad h_t = g_t - f_t \quad .$$

Also, in this proof, $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function supported in a narrow neighborhood of ∂D such that $\psi(x) = \nu(x)$ for $x \in \partial D$. It is clear that $g \in C(\mathbb{R}_+, \overline{D})$.

(a) We first show that $\{l_T^n\}$ is uniformly bounded. Note first that l^n increases only when $g_t^n \in \partial D_n$. If n is sufficiently large and s lies in the support of the measure on $[0, T]$ determined by l^n, then $\psi(g_s) \cdot \nu^n(g_s^n) \geq 1/2$. Since $\psi(g_t)$ is continuous in t, there exists a positive γ such that $\psi(g_t) \cdot \nu^n(g_s^n) \geq \frac{1}{3}$ if $s, t \in [0, T]$ and $|t - s| \leq \gamma$. Fix an $N \geq T/\gamma$ and let $t_l = lT/N$. Then for sufficiently large n,

$$\begin{aligned} l_T^n &= \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} dl_u^n \\ &\leq 3 \sum_{l=0}^{N-1} \int_{t_l}^{t_{l+1}} \psi(g_{t_l}) \cdot \nu^n(g_u^n) dl_u^n \\ &= 3 \sum_{l=0}^{N-1} \psi(g_{t_l}) \cdot \int_{t_l}^{t_{l+1}} \nu^n(g_u^n) dl_u^n \\ &= 6 \sum_{l=0}^{N-1} \psi(g_{t_l}) \cdot \left\{h^n(t_{l+1}) - h^n(t_l)\right\} \\ &\to 6 \sum_{l=0}^{N-1} \psi(g_{t_l}) \cdot \{h(t_{l+1}) - h(t_l)\} \quad . \end{aligned}$$

It follows that $\{l_T^n\}$ is uniformly bounded.

(b) Next we show that $\{l^n\}$ converges to a nondecreasing, continuous function which increases only when $g_t \in \partial D$. By definition,

$$h_t^n = \frac{1}{2}\int_0^t \nu^n(g_u^n) dl_u^n \quad .$$

Hence $\left\{h_t^n, 0 \le t \le T\right\}$ is a sequence of functions with uniformly bounded variations $\left\{l_T^n\right\}$ which at the same time converges uniformly on $[0, T]$ to h. Because $t \mapsto \psi(g_t)$ is uniformly continuous on $[0, T]$, the limit

$$2\int_0^t \psi(g_u)\cdot dh_u^n \to 2\int_0^t \psi(g_u)dh_u = l_t \tag{3}$$

exists for each fixed $t \le T$ and defines a function l on $[0, T]$.

We claim that l_t^n converges to l_t. This is clear from

$$2\int_0^t \psi(g_u)\cdot dh_u^n = l_t^n + \int_0^t \left\{\psi(g_u) - \nu^n(g_u^n)\right\}\cdot \nu^n(g_u^n)dl_u^n$$

because the second term on the right-hand side converges to zero as $n \to \infty$ and $\left\{l_t^n\right\}$ is uniformly bounded.

It is clear from (3) that l increases only when $g_t \in \operatorname{supp}\psi$. From $l^n \to l$ we know that l is independent of the choice of ψ. We can choose ψ to be supported in an arbitrarily narrow neighborhood of ∂D. It follows that l increases only when $g_t \in \partial D$.

The continuity of l can be proved as follows. We note that

$$\int_s^t \psi(g_u)\cdot dh_u^n = \psi(g_s)\cdot\left\{h_t^n - h_s^n\right\} + \int_s^t \left\{\psi(g_u) - \psi(g_s)\right\}\cdot dh_u^n \ .$$

Hence

$$\left|\int_s^t \psi(g_u)dh_u^n\right| \le \|\psi\|_\infty |h_t^n - h_s^n| + \omega_T(|s-t|;\psi\circ g)l_T^n \ .$$

Taking the limit as $n \to \infty$, we have

$$l_t - l_s \le 2\|\psi\|_\infty |h_t - h_s| + \omega_T(|s-t|;\psi\circ g)l_T \ .$$

Thus l is continuous.

(c) Finally we show that the pair (g, l) satisfies the Skorokhod equation. We have

$$g_t^n - f_t = \frac{1}{2}\int_0^t \nu^n(g_s^n)dl_s^n = \frac{1}{2}\int_0^t \psi(g_s)dl_s^n + \frac{1}{2}\int_0^t \left\{\nu^n(g_s^n) - \psi(g_s)\right\}dl_s^n \ .$$

The second term goes to zero because $\nu^n(g_s^n) \to \nu(g_s)$ uniformly on $[0, T]$ and $\left\{l_T^n\right\}$ is uniformly bounded. Hence

$$g_t - f_t = \frac{1}{2}\int_0^t \psi_\lambda(g_s)dl_s = \frac{1}{2}\int_0^t \nu(g_s)dl_s \ .$$

The last equality holds because l increases only when $g_t \in \partial D$ and $\psi(x) = \nu(x)$ for $x \in \partial D$. □

3. Existence of strong solutions

We will use the method of measurable selection to show the existence of a strong solution to the Skorokhod equation for C^1 domains. Let us state some general facts concerning this method.

Let Y be a separable metric space and $K(Y)$ the space of compact subsets of Y. Then $K(Y)$ is a separable metric space with a distance function defined by

$$d(C_1, C_2) = \inf\left\{\epsilon > 0 : C_1 \subseteq C_2^\epsilon, C_2 \subseteq C_1^\epsilon\right\} \quad ,$$

where C^ϵ denotes the ϵ-neighborhood of C. The proof of the following result can be found in Stroock and Varadhan[14], Section 12.1.

Proposition 3.1. *Suppose that X and Y are separable metric spaces and $C : X \to K(Y)$ a measurable map. Then there is a measurable map $\psi : X \to Y$ such that $\psi(x) \in C(x)$ for every $x \in X$.*

The following result gives a simple way of producing a measurable map from X to $K(Y)$.

Lemma 3.2. *Suppose that $\phi_n : X \to Y$ is a sequence of continuous maps such that for each $x \in X$, the set $\{\phi_n(x)\}$ is precompact. Let $C(x)$ be the set of the accumulation points of the sequence $\{\phi_n(x)\}$. Then the map $C : X \to K(Y)$ given by $x \mapsto C(x)$ is measurable.*

Proof. First of all, it is clear that $C(x)$ is compact for every x. We will use $K(A)$ to denote the collection of compact subsets of $A \subseteq Y$. It is known that $K(F)$ is closed for each closed $F \subseteq Y$ and the class $\{K(F) : F \text{ closed in } Y\}$ generates the Borel σ-field of $K(Y)$. Hence it is enough to show that for each closed $F \subseteq Y$, the set

$$C^{-1}[K(F)] = \{x \in X : C(x) \subseteq F\}$$

is measurable in X.

Let G_N be the $1/N$-neighborhood of F. Then G_N is open and $G_N \downarrow F$. It is easy to verify that $K(G_N) \downarrow K(F)$ and

$$C^{-1}[K(F)] = \bigcap_{N=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x \in X : \phi_k(x) \in G_N\} \quad .$$

Note that for the above relation to hold we need the condition that $\{\phi_n(x)\}$ is precompact for each $x \in X$. The set $\{x \in X : \phi_n(x) \in G_N\}$ is open because G_N is open and ϕ_n is continuous. Hence $C^{-1}[K(F)]$ is measurable. □

We now apply the above lemma to our situation.

Proposition 3.3. *There exists a measurable map $F : C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d) \to C(\mathbb{R}_+, \overline{D}) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$ with the following property: For each $f \in C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$, we have $F(f) = (g, l)$, where l is a continuous nondecreasing function which increases only when $g_t \in \partial D$ and*

$$g_t = f_t + \frac{1}{2}\int_0^t \nu(g_s)dl_s \quad .$$

Proof. Let (g^n, l^n) as defined in the previous section. By Theorem 2.3 the map ϕ_n : $f \mapsto (g^n, l^n)$ is a continuous map from $C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ to $C(\mathbb{R}_+, \overline{D}_1) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$. By Theorems 2.5 and 2.6 the conditions of Lemma 3.2 are satisfied. The existence of a F with the desired properties follows immediately from Lemma 3.2 and Proposition 3.1. □

It is now easy to obtain a strong solution for the stochastic Skorokhod equation:

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \nu(X_s)dL_s, \quad \forall t \geq 0 \ . \tag{4}$$

We make a formal definition. For a probability measure μ on $\overline{D}$, we use $\mathbb{P}^\mu$ to denote the law of d-dimensional Brownian motion with initial distribution μ.

Definition 3.4. *We say that a Borel measurable map*

$$F : C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d) \to C(\mathbb{R}_+, \overline{D}) \times C(\mathbb{R}_+, \mathbb{R}_+)$$

is a strong solution to the Skorokhod equation if it satisfies the following condition: whenever B is a Brownian motion defined on a probability space, X_0 an $\overline{D}$-valued random variable independent of B and $F(B + X_0) = (X, L)$, then the nondecreasing process L increases only when $X_t \in \partial D$ and the Skorokhod equation (4) holds. We say that the equation has a unique strong solution if for any other strong solution G we have $F(\omega) = G(\omega)$, $\mathbb{P}^\mu$-almost surely on $C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}^d)$ for every probability measure μ on $\overline{D}$.

Theorem 3.5. *Let D be a bounded C^1 domain in D. There exists a strong solution to the Skorokhod equation in D.*

Proof. Take F to be the one defined in Proposition 3.3. □

The uniqueness of the strong solution will be proved in Section 5.

4. Weak uniqueness

In this section we show how the arguments in Bass[1] can be modified to prove the weak uniqueness for the Skorokhod equation on bounded $C^{1,\alpha}$ domains.

We will occasionally use polar coordinates: $x = (r, \theta)$, where $r = |x|$ and $\theta = x/|x| \in \partial B(0, 1)$, the boundary of $B(0, 1)$. We write $\sigma(dx)$ for surface measure on ∂D. We use $\partial_i f$ and $\partial_{ij} f$ to denote $\partial f/\partial x_i$ and $\partial^2 f/\partial x_i \partial x_j$, respectively. A $C^{1,\alpha}$ domain D is star-like (relative to 0) if there exists a $C^{1,\alpha}$ function $\gamma : \partial B(0, 1) \to (0, \infty)$ such that $D = \{(r, \theta) : 0 \leq r < \gamma(\theta)\}$.

Let us suppose for the moment that the dimension d is greater than or equal to 3. Let D be a star-like $C^{1,\alpha}$ domain with $K = \overline{B(0, \rho)}$, where $\rho < \inf \gamma/4$. In Bass and Hsu[3] a strong Markov process $(\mathbb{Q}^x, X_t)$, $x \in \overline{D}$, was constructed that represents reflecting Brownian motion in $\overline{D}$ with absorption at K. We recall a few properties; see Bass and Hsu[3] for details. Let

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}$$

be the first hitting time of a set A. Reflecting Brownian motion in $\overline{D}$ with absorption in K has a Green function $g(x,y)$ that is symmetric in x and y for $x, y \in D-K$, harmonic in y in $D-K-\{x\}$, harmonic in x in $D-K-\{y\}$, vanishes as x or y tends to the boundary of K, and there exists c_1 such that

$$g(x,y) \le c_1|x-y|^{2-d} \quad . \tag{5}$$

The constant c_1 depends only on ρ, $\|\nabla\gamma\|_\infty$, $\inf\gamma$, and $\sup\gamma$. In particular, for each $\rho' > 0$, $g(x,\cdot)$ is bounded in $\overline{D} - K - B(x,\rho')$.

A consequence of (5) is that

$$\mathbb{E}^x T_K = \int_{\overline{D}-K} g(x,y)\,dy \le c_2, \qquad x \in \overline{D} \ .$$

In Bass and Hsu[3] it is proved that there exists a continuous additive functional L_t corresponding to the measure $\sigma(dy)$:

$$\mathbb{E}^x L_{T_K} = \int_{\partial D} g(x,y)\,\sigma(dy), \qquad x \in \overline{D} \ ,$$

and L_t increases only when X_t is in the support of σ, namely ∂D. It follows from (5) that $\mathbb{E}^x L_{T_K} \le c_3$, $x \in \overline{D}$, where c_3 depends on ρ, $\|\nabla\gamma\|_\infty$, $\inf\gamma$, and $\sup\gamma$.

We now suppose that $d \ge 2$ and that D is an arbitary bounded $C^{1,\alpha}$ domain. In Bass and Hsu[4] and Fukushima, Oshima, and Takeda[7], Ex. 5.2.2, it was shown that the $(\mathbb{Q}^x, X_t)$ constructed in Bass and Hsu[3] satisfies the Skorokhod equation: there exists a d-dimensional Brownian motion W_t such that

$$X_t = X_0 + W_t + \frac{1}{2}\int_0^t \nu(X_t)\,dL_t \quad . \tag{6}$$

We want to show that the solution to (6) is unique in law. In the following definition, we use X to denote the coordinate process on $C(\mathbb{R}_+, \overline{D})$, namely, $X_t(\omega) = \omega_t$ for $\omega \in C(\mathbb{R}_+, \overline{D})$.

Definition 4.1. *Let D be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ with $d \ge 2$. For $x_0 \in \overline{D}$, let $\mathscr{M}(x_0)$ be the collection of probability measures $\mathbb{P}$ on $C(\mathbb{R}_+, \overline{D})$ such that*

(a) $\mathbb{P}(X_0 = x_0) = 1$,
(b) there exists a continuous nondecreasing process L_t which increases only when $X_t \in \partial D$, and
(c) there exists a continuous process W which under $\mathbb{P}$ is a d-dimensional Brownian motion adapted to the filtration of X such that

$$X_t = X_0 + W_t + \frac{1}{2}\int_0^t \nu(X_s)\,dL_s \quad .$$

An element of $\mathscr{M}(x_0)$ is called a (weak) solution of the Skorokhod equation.

By our discussion so far there exists at least one element of $\mathscr{M}(x_0)$, namely $\mathbb{Q}^{x_0}$. Saying that W_t is a Brownian motion adapted to the filtration generated by X means that $W_t - W_s$ has the same distribution as that of a normal random variable with mean 0 and variance $t - s$ and $W_t - W_s$ is independent of $\sigma\{X_r; r \le s\}$ whenever $s < t$.

The main result of this section is the following.

Theorem 4.2. *If D is a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$, $d \ge 2$, then there is exactly one weak solution to the Skorokhod equation.*

The condition (b) in Definition 4.1 is slightly weaker than the one given in Bass[1], which essentially requires that the local time L_t be an additive functional corresponding to surface measure on the boundary.

We will need the following proposition. Let θ_t be shift operators so that $X_s \circ \theta_t = X_{s+t}$. By Bass[2], Section I.2, we may always suppose such θ_t exist.

Proposition 4.3. *Let $\mathbb{P} \in \mathscr{M}(x_0)$, and S a finite stopping time, and let $\mathbb{P}_S(\omega, d\omega')$ be a regular conditional probability for the law of $X_\cdot \circ \theta_S$ under $\mathbb{P}[\cdot \mid \mathscr{F}_S]$. Then $\mathbb{P}$-almost surely, $\mathbb{P}_S \in \mathscr{M}(X_S(\omega))$.*

Proof. This is the strong Markov property for $\mathbb{P}$. See Bass[1], Proposition 2.3. □

We will need the following.

Proposition 4.4. *Let us suppose that $d \ge 3$ and D is a $C^{1,\alpha}$ domain that is star-like. Let h be a C^∞ function with support in $D - K$. Let u be the solution to the problem: $\Delta u = -2h$ in $D - K$, $u = 0$ on ∂K, and $\partial u/\partial \nu = 0$ on ∂D. Suppose $\gamma \in C^2$. Then u is $C^{1,\alpha}$ in a neighborhood of ∂D with $C^{1,\alpha}$ norm that depends only on the $C^{1,\alpha}$ norm of γ, $\|h\|_\infty$, and the distance from the support of h to ∂D (and not on any further smoothness of γ).*

Proof. This follows from Lieberman[12], Theorem VI.6.46 on p. 141. □

Proposition 4.5. *Let D, h, and K be as above. Suppose $x_0 \in D$. There exists a sequence of C^2 functions u_n on D such that $u_n(x_0)$ converges, $\Delta u_n = -2h$ in D, $u_n = 0$ on K, and $\partial u_n/\partial \nu$ converges to 0 uniformly on ∂D.*

Proof. Let D_n be a sequence of C^2 domains, all star-like with respect to the same point, such that the D_n decrease to D and the closure of D is contained in D_n for each n. Moreover, let us arrange matters such that if $D_n = \{(r, \theta) : 0 \le r < \gamma_n(\theta)\}$, then γ_n converges to γ in $C^{1,\alpha}$ norm. Let u_n be the solution to the problem

$$\begin{aligned} \Delta u_n &= -2h \quad \text{in} \quad D_n - K, \\ u_n &= 0 \quad \text{on} \quad K, \\ \frac{\partial u_n}{\partial \nu_n} &= 0 \quad \text{on} \quad \partial D_n \ . \end{aligned}$$

Here ν_n is the unit normal vector on ∂D_n. By Theorem 4.4 there exists a subsequence n_j such that u_{n_j} and ∇u_{n_j} converge uniformly on $\overline{D}$. By relabeling, we may assume the full sequence u_n converges. Since $\partial u_n/\partial \nu_n = 0$ on ∂D_n and the γ_n converge to γ in $C^{1,\alpha}$ norm, it follows that $\partial u_n/\partial \nu \to 0$ uniformly on ∂D. □

For the next proposition let us suppose that $\mathbb{P} \in \mathscr{M}(x_0)$, where $x_0 \in \partial D$. We need to show $\mathbb{P}(T_D = 0) = 1$, that is, starting at the boundary, we leave the boundary immediately.

Proposition 4.6. *Suppose D is a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$, $d \geq 2$. Suppose $x_0 \in \partial D$ and $\mathbb{P} \in \mathscr{M}(x_0)$. Then $\mathbb{P}(T_D = 0) = 1$.*

Proof. Choose a coordinate system such that $x_0 = 0$ and the hyperplane $\{y_d = 0\}$ is tangent to D at x_0. Let $\beta = 2/(2+\alpha)$ and let

$$V = \{y \in D : |(y_1, \dots, y_{d-1})| < \epsilon^\beta, y_d < \epsilon\}.$$

Thus V is the intersection of a right circular cylinder and D. Set $U = \{y \in \partial V : y_d = \epsilon\}$ (the top base) and $S = \partial V - \partial D - U$ (the side surface). For ϵ sufficiently small, $U \subseteq D$.

Let $t_0 = \epsilon^{(4+\alpha)/(2+\alpha)}$, $R = \inf\{t : L_t > \epsilon^\beta/2\}$,

$$A_1 = A_1(\epsilon) = \{\sup_{s \leq t_0} W_s^d < \epsilon\},$$
$$A_2 = A_2(\epsilon) = \{\sup_{s \leq t_0} |(W_s^1, \dots, W_s^{d-1})| > \epsilon^\beta/2\},$$
$$A_3 = A_3(\epsilon) = \{\sup_{s \leq t_0} |W_s^d| > \epsilon^\beta/8\} \ .$$

By the scaling property of Brownian motion, $\mathbb{P}(A_1(\epsilon))$, $\mathbb{P}(A_2(\epsilon))$, and $\mathbb{P}(A_3(\epsilon))$ all tend to 0 as $\epsilon \to 0$.

Write $\nu = (\nu_1, \dots, \nu_d)$. If δ is sufficiently small, $\nu_d \geq 1/2$ and $|\nu_1|^2 + \cdots + |\nu_{d-1}|^2 \leq 1/(4d)$ in $\partial D \cap \{|(y_1, \dots, y_{d-1})| < \delta\}$. Let us restrict attention to ϵ such that $\epsilon^\beta < \delta$. As $\epsilon \to 0$, then $t_0 \to 0$, so by the continuity of the paths of X_t we see that $\mathbb{P}(A_4(\epsilon)) \to 0$ as $\epsilon \to 0$, where

$$A_4 = A_4(\epsilon) = \{\sup_{s \leq t_0} |X_s - x_0| > \delta\} \ .$$

Note that on the set A_4^c

$$X_s^d = W_s^d + \int_0^s \nu_d(X_r) dL_r \geq W_s^d \ .$$

So on $A_1^c \cap A_4^c$ we have $\sup_{s \leq t_0} X_s^d \geq \epsilon$.

Consider the set $A_1^c \cap A_2^c \cap A_3^c \cap A_4^c$. Observe that for $i \leq d-1$,

$$\sup_{s \leq t_0} |X_s^i| \leq \sup_{s \leq t_0} |W_s^i| + \left(\frac{1}{4d}\right) L_{t_0} \ .$$

So if $R > t_0$, then $T_U < T_S$ and $T_U < t_0$. Hence $T_D < t_0$. On the other hand, on the set $A_1^c \cap A_2^c \cap A_3^c \cap A_4^c$, if $R \leq t_0$, then

$$\left|(W_R^1, \dots, W_R^{d-1})\right| < \epsilon^\beta/2$$

and

$$\left|\left(\int_0^R \nu_1(X_r)dL_r, \ldots, \int_0^R \nu_{d-1}(X_r)dL_r\right)\right| \le \frac{1}{4}L_R \le \epsilon^\beta/2 \ .$$

Also, $|W_R^d| < \epsilon^\beta/8$ and

$$\int_0^R \nu_d(X_r)dL_r \ge \frac{1}{2}L_R \ge \epsilon^\beta/4 \ .$$

So $X_R^d \ge \epsilon^\beta/8 \ge \epsilon$, and hence $T_U \le R \le t_0$, and again $T_D \le t_0$. Now letting $\epsilon \to 0$ shows $\liminf_{\epsilon\to 0} \mathbb{P}(T_D \le \epsilon^{(4+\alpha)/(2+\alpha)}) \ge 1$, which implies $\mathbb{P}(T_D = 0) = 1$. □

We obtain the following corollary.

Corollary 4.7. *Suppose that $x_0 \in \partial D$ and $\mathbb{P} \in \mathscr{M}(x_0)$. For each n there exists a stopping time ξ_n such that* $\sup_{s\le\xi_n} |X_s - x_0| \le 1/n$*, $\xi_n \le 1/n$, and $\mathbb{P}(X_{\xi_n} \in \partial D) \le 1/n$.*

Proof. Fix n. Let $\zeta_1 = \inf\{t : |X_t - x_0| \ge 1/n\} \wedge 1/n$. By the continuity of paths of X_t, we have $\zeta_1 > 0$, a.s. Choose m large so that if $\zeta_2(m) = \inf\{t : \text{dist}\,(X_t, \partial D) \ge 1/m\}$, then $\mathbb{P}(\zeta_2(m) > \zeta_1) \le 1/n$; this is possible by Proposition 4.6. Now let $\xi_n = \zeta_1 \wedge \zeta_2(m)$. □

We now turn to the proof of Theorem 4.2, the main result of this section. Suppose first that D is a star-like $C^{1,\alpha}$ domain, K is as above, and $x_0 \in D - K$. As in the proof of Proposition 4.1 of Bass[1] and the discussion immediately preceding that proposition, we may restrict attention to probability measures $\mathbb{P} \in \mathscr{M}(x_0)$ such that $\mathbb{E}_\mathbb{P} L_{T_K} < \infty$ and $\mathbb{E}_\mathbb{P} T_K < \infty$.

We apply Itô's formula to the process X_t and the functions u_n defined in Proposition 4.5. We obtain

$$u_n(X_{T_K}) - u_n(X_0) = \int_0^{T_K} \nabla u_n(X_s) \cdot dW_s + \int_0^{T_K} \nabla u_n(X_s) \cdot \nu(X_s)dL_s + \frac{1}{2}\int_0^{T_K} \Delta u_n(X_s)ds \ .$$

Taking the expectation with respect to $\mathbb{P}$ we have

$$-u_n(x_0) = \mathbb{E}_\mathbb{P} \int_0^{T_K} \frac{\partial u_n}{\partial \nu}(X_s)dL_s - \mathbb{E}_\mathbb{P} \int_0^{T_K} h(X_s)ds \ .$$

Letting $n \to \infty$ and using the facts that $\partial u_n/\partial\nu \to 0$ uniformly and that $\mathbb{E}_\mathbb{P} L_K < \infty$, we obtain

$$\lim_{n\to\infty} u_n(x_0) = \mathbb{E}_\mathbb{P} \int_0^{T_K} h(X_s)ds \ .$$

Hence the value of $\mathbb{E}_\mathbb{P} \int_0^{T_K} h(X_s)ds$ does not depend on $\mathbb{P}$.

Since $\mathbb{Q}^{x_0}$ is also in $\mathscr{M}(x_0)$, then

$$\mathbb{E}_{\mathbb{P}} \int_0^{T_K} h(X_s)ds = \mathbb{E}^{x_0} \int_0^{T_K} h(X_s)ds \ . \tag{7}$$

This is the analog of Corollary 4.6 of Bass[1].

Using Corollary 4.7 and following the proof of Proposition 4.7 of Bass[1], we see (7) holds when $x_0 \in \partial D$ as well. We now can follow the proof of Bass[1] (from Proposition 4.8 to the end of Section 4) almost exactly. (Part of that proof involves removing the restriction that D be star-like and that d be larger than 2.) □

5. Pathwise uniqueness

Theorem 5.1. *Let D be a bounded $C^{1,\alpha}$ domain and W a d-dimensional Brownian motion. Let X_0 be a $\overline{D}$-valued random variable independent of W. Any two solutions to the Skorokhod equation*

$$X_t = X_0 + W_t + \frac{1}{2}\int_0^t \nu(X_s)\,dL_s$$

agree pathwise, a.s.

Proof. By Theorem 3.5 there is a strong solution $(Y, H) = F(X_0 + W)$, so

$$Y_t = X_0 + W_t + \frac{1}{2}\int_0^t \nu(Y_s)\,dH_s \ .$$

Let X_t be another solution to the SDE. We have

$$W_t = Y_t - X_0 - \frac{1}{2}\int_0^t \nu(Y_s)\,dH_s, \qquad W_t = X_t - X_0 - \frac{1}{2}\int_0^t \nu(X_s)\,dL_s \ . \tag{8}$$

The processes Y and X have the same law because of the uniqueness in law (Theorem 4.2). By Bass and Hsu[3], Y does not spend time on the boundary, namely,

$$\mathbb{E}\int_0^\infty 1_{\partial D}(Y_s)ds = \mathbb{E}^{x_0}\int_0^\infty 1_{\partial D}(Y_s)ds = 0 \ .$$

Let ζ_n be a sequence of continuous functions with compact support mapping $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $\zeta_n(x)$ decreases boundedly and pointwise to $\nu(x)1_{\partial D}(x)$. Since W_t is a Brownian motion and Y_t spends zero time in ∂D, then $\int_0^t 1_{\partial D}(Y_s)dW_s = 0$, a.s. Hence

$$\begin{aligned}\int_0^t \zeta_n(Y_s)dY_s &= \int_0^t \zeta_n(Y_s)\cdot dW_s + \frac{1}{2}\int_0^t \zeta_n(Y_s)\cdot\nu(Y_s)dH_s \\ &\to \int_0^t \nu(Y_s)\cdot\nu(Y_s)dH_s \\ &= H_t \ .\end{aligned}$$

It now follows easily from this and (8) that there exists a measurable map $G : C(\mathbb{R}_+, \overline{D}) \to C_0(\mathbb{R}_+, \mathbb{R}^d) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$ such that $(W, H) = G(Y)$. The same proof shows that $(W, L) = G(X)$. Therefore the law of the triple (Y, H, W) is equal to the law of the triple (X, L, W). Since $(Y, H) = F(W)$, it follows that $(X, L) = F(W)$, a.s., and we then conclude that $(X, L) = F(W) = (Y, H)$, a.s. □

Corollary 5.2. *Let D be a bounded $C^{1,\alpha}$ domain. Then there is a unique strong solution $F : C_{\overline{D}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \overline{D}) \times C_0(\mathbb{R}_+, \mathbb{R}_+)$ to the Skorokhod equation*

$$X_t = X_0 + W_t + \frac{1}{2}\int_0^t \nu(X_s)dL_s \quad .$$

Furthermore F is progressively measurable, i.e., for all $t \geq 0$,

$$F(X_0 + W)_t = (X_t, L_t) \in \sigma\{X_0 + W_s, s \leq t\} \quad .$$

Proof. The corollary follows essentially from the following general fact: weak existence for each initial distribution and pathwise uniqueness together imply the existence and uniqueness of a strong solution which is *automatically* progressively measurable; see Ikeda and Watanabe[10], Theorem 1.1 on p. 163 and its proof. The two conditions are satisfied in our situation: the measure

$$\mathbb{Q}^\mu = \int_{\overline{D}} \mathbb{Q}^x \, \mu(dx)$$

is the (unique) weak solution by Theorem 4.2, and pathwise uniqueness is guaranteed by Theorem 5.1. □

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