ON THE POISSON KERNEL FOR THE NEUMANN PROBLEM OF SCHRÖDINGER OPERATORS

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ABSTRACT

Let D be a bounded domain in $\mathbb{R}^d$ $(d \geqslant 3)$ and let $b(t,x,y)$ be the kernel of the Feynamn–Kac semigroup associated with the reflecting Brownian motion $\{X_t : t \geqslant 0\}$ and potential V, namely

$$E^x\left[\exp\left(\int_0^t V(X_s)\,ds\right)f(X_t)\right] = \int_D b(t,x,y)f(y)\,m(dy).$$

We assume that V is in the Kato class K_d [1]. The Poisson kernel studied in this paper is

$$N_V(x,y) = \int_0^\infty b(t,x,y)\,dt.$$

In general N_V may be infinite. We show that if $N_V(x,y)$ is finite for one pair of points then it is finite for all $x \neq y$ and there exist two constants c_1, c_2 (depending on D and V) such that

$$c_1 \leqslant \|x-y\|^{d-2} N_V(x,y) \leqslant c_2.$$

This happens precisely when the spectrum of $H_V = \Delta/2 + V$ under the Neumann boundary condition lies in the negative half-axis. This result is used to discuss the Neumann boundary value problem of H_V. We prove that for any boundary function $f \in L^\alpha(\partial D)$, $\alpha \geqslant 1$, the problem has a unique weak solution

$$u_f(x) = \frac{1}{2}\int_{\partial D} N_V(x,y)f(y)\,\sigma(dy) \in C(D)$$

and its growth rate near the boundary can be estimated by $\|f\|_{\alpha,\partial D}$.

1. *Introduction*

We assume throughout the paper that D is a bounded domain in $\mathbb{R}^d$ with a C^3 boundary. We denote the Lebesgue measure of $\mathbb{R}^d$ by m and the $(d-1)$-dimensional volume measure of ∂D by σ. The outward normal derivative at ∂D will be denoted by $\partial/\partial n$. The Neumann boundary value problem of the Schrödinger operator $H_V = \Delta/2 + V$ is concerned with the existence and uniqueness of a function u such that $H_V u = 0$ and $\partial u/\partial n = f$, a function defined on the boundary ∂D. For brevity, we denote this problem by $N(D; V, f)$. We work with potentials V in the Kato class K_d. This class of functions was studied in detail in [1]. A condition equivalent to $VI_D \in K_d$ is the following:

$$\lim_{t\to 0} \alpha(t) := \lim_{t\to 0} \sup_{x\in \bar{D}} \int_0^t s^{-d/2}\,ds \int_D |V|(y)\,e^{-K\|x-y\|^2/s}\,m(dy) = 0. \tag{1.1}$$

It is this form of the condition which we shall use in this paper. The connection between the Neumann problem and reflecting Brownian motion is well known. In [5],

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we discussed a probabilistic approach to the problem with bounded measurable boundary functions. Let $X = \{X_t : t \geqslant 0\}$ be the standard reflecting Brownian motion on D, namely the diffusion process on $\bar{D}$ generated by $H_0 = \Delta/2$ with the reflecting boundary condition [6, pp. 293–295]. The transition density function of X is denoted by $p(t,x,y)$ and the expectation operator by E^x. We know that $p(t,x,y)$ is the fundamental solution of the heat equation on D with the Neumann boundary condition. We use ϕ to denote the boundary local time of the reflecting Brownian motion X. By definition, it is the unique continuous additive function of X such that

$$E^x[\phi(t)] = \int_0^t ds \int_{\partial D} p(s,x,y)\,\sigma(dy). \tag{1.2}$$

(For properties about reflecting Brownian motion see [8].) We introduce the Feynman–Kac functional

$$e_V(t) = \exp\left[\int_0^t V(X_s)\,ds\right].$$

In terms of these objects, the solution to $N(D;\, V, f)$ is given by

$$u_f(x) = \tfrac{1}{2}E^x\left[\int_0^\infty e_V(t)\,f(X_t)\,\phi(dt)\right]. \tag{1.3}$$

Since we do not assume that V and f are smooth, the term 'solution' is to be understood in the following sense. Let

$$C_\partial^2(D) = \{f \in C^2(D) \cup C^1(\bar{D}) : \partial f/\partial n = 0 \text{ on } \partial D\}.$$

DEFINITION. A function u is called a *weak solution* of $N(D, V; f)$ if $u \in L^1(D)$ and $uV \in L^1(D)$, and for any $v \in C_\partial^2(D)$ the following equality holds:

$$\int_D u(x)\,H_V v(x)\,m(dx) = -\frac{1}{2}\int_{\partial D} f(x)\,v(x)\,\sigma(dx). \tag{1.4}$$

The main result in [5] can now be stated as follows.

THEOREM 1.1. *Suppose that* (a) $VI_D \in K_d$; (b) f *is bounded and measurable on* ∂D; (c) *the gauge function*

$$G_V(x) = E^x\left[\int_0^\infty e_V(t)\,\phi(dt)\right]$$

is finite for some x in D. Then u_f in (1.3) *is continuous on $\bar{D}$ and is the unique weak solution of the Neumann problem $N(D;\, V, f)$.*

Now (1.3) can be written as

$$u_f(x) = \frac{1}{2}\int_{\partial D} N_V(x,y)\,f(y)\,\sigma(dy) \tag{1.5}$$

for some kernel defined on $\bar{D} \times \partial D$. This kernel will be called the Poisson kernel of the Neumann problem.

At this point, in order to extend (1.3) to a larger class of boundary functions, we need to study the analytic behavior of the Poisson kernel. More specifically, we want to know for what kind of V this kernel can be defined and finite, whether the kernel is continuous and how it behaves when x and y are close. These problems correspond roughly to the types of problems recently studied for the Poisson kernel of the Dirichlet problem of H_V, namely the validity of the so-called gauge theorem [**3**, **4**, **9**] and the smoothness of the Poisson kernel [**2**]. In a way, the kernel $N_V(x,y)$ behaves like the conditional gauge for the Dirichlet case studied by these authors.

The purpose of this paper is to study the kernel $N_V(x,y)$ along these lines. Our results can be briefly described as follows. We first define the kernel $N_V(x,y)$ on $\bar{D}\times\bar{D}$ for any potential V in the Kato class. This kernel could be infinite everywhere. It becomes useful only when it is finite. We show that if $N_V(x,y)$ is finite at one pair (x,y), then it is finite and continuous on $\bar{D}\times\bar{D}$ minus the diagonal and blows up at the diagonal like $\|x-y\|^{-(d-2)}$. This situation happens precisely when the upper bound of the spectrum of H_V under the Neumann condition is strictly negative. Using this result, we improve Theorem 1.1 above by showing that the Neumann boundary value problem for H_V can be solved uniquely for any boundary function in $L^1(\partial D)$ and we shall be able to control the growth of the solution near the boundary by the L^α-norm of the boundary function ($\alpha \geqslant 1$).

To start with, we define the Poisson kernel $N_V(x,y)$ probabilistically as follows. Let $e_V(t)$ be the Feynman–Kac functional as before. By (1.1) and (2.10) proved below, the condition $VI_D \in K_d$ implies that $e_V(t)$ is a well-defined, almost surely finite random variable. The Feynman–Kac semigroup is

$$T_t^V f(x) = E^x[e_V(t) f(X_t)]. \tag{1.6}$$

(In the following we may omit the super- or subscript if it is meant to be V.) Now the operator T_t is given by a kernel $b(t,x,y)$

$$T_t f(x) = \int_D b(t,x,y) f(y)\, m(dy).$$

In fact, from (1.6), we have

$$b(t,x,y) = E^x[e_V(t) \mid X_t = y]\, p(t,x,y). \tag{1.7}$$

It is immediate from this expression that $b(t,x,y)$ is strictly positive. By the semigroup property, we have

$$b(t+s,x,y) = \int_D b(t,x,z)\, b(s,z,y)\, m(dz). \tag{1.8}$$

Formula (1.6) implies that $b(t,x,y)$ has the following expansion:

$$b(t,x,y) = \sum_{n=0}^{\infty} b_n(t,x,y), \tag{1.9}$$

where $b_0(t,x,y) = p(t,x,y)$ and

$$\begin{aligned} b_n(t,x,y) &= E^x\left[\int_0^t b_{n-1}(t-s, X_s, y)\, V(X_s)\, ds\right] \\ &= \int_0^t ds \int_D p(s,x,z)\, V(z)\, b_{n-1}(t-s,z,y)\, m(dz). \end{aligned} \tag{1.10}$$

The following properties of the semigroup T_t will be used repeatedly: for all $t > 0$,

$$T_t \colon L^1(D) \longmapsto C(\bar{D}) \quad \text{and} \quad \|T_t\|_{1,\infty} < \infty. \tag{1.11a}$$

For any $t_0 > 0$ there exists a constant $K = K(t_0)$ such that

$$\sup_{0 \leqslant t \leqslant t_0} \|T_t\|_{\infty,\infty} \leqslant K. \tag{1.11b}$$

These properties follow from Kash'minski's lemma and the Markov property [1, p. 213].

We define the Poisson kernel of the Neumann problem by

$$N_V(x,y) = \int_0^\infty b(t,x,y)\,dt. \tag{1.6}$$

Since $b(t,x,y) \geqslant 0$ by definition, $N_V(x,y)$ is well defined for all $VI_D \in K_d$, is positive and may be infinite somewhere or everywhere.

Next we come to $\mathrm{Spec}_N H_V$, the spectrum of H_V under the Neumann condition. Since $\|T_t\|_{1,\infty} < \infty$, we know $\{T_t: t \geqslant 0\}$ is a semigroup of bounded integral operators on $L^2(D)$. We can verify that this semigroup is continuous at $s > 0$:

$$\lim_{t \to s > 0} \|T_t - T_s\|_{2,2} = 0.$$

Using the continuity one shows easily that there is a sequence of eigenpairs $\{\lambda_n, \phi_n\}$, $n = 1, 2, \ldots$, such that (1) λ_n decreases to $-\infty$; (2) $\{\phi_n : n \geqslant 1\}$ form a complete orthornormal basis of $L^2(D)$; and (3) $T_t \phi_n = e^{\lambda_n t}\phi_n$ for all $n \geqslant 1$ (cf. the argument in [7, pp. 121–122]). Let

$$\phi = \sum_{n-1}^{\infty} a_n(\phi)\,\phi_n \in L^2(D).$$

Define

$$D(H_V) = \left\{\phi \in L^d(D): \sum_{n-1}^{\infty} \lambda_n^2 |a_n(\phi)|^2 < \infty\right\}$$

and

$$H_V \phi = \sum_{n-1}^{\infty} \lambda_n\, a_n(\phi)\,\phi_n.$$

Then $\{H_V, D(H_V)\}$ is a realization of $\Delta/2 + V$ as a self-adjoint operator on $L^2(D)$. Set

$$\mathrm{Spec}_N H_V = \{\lambda_1, \lambda_2, \ldots\}.$$

Thus $\lambda_1 = \sup \mathrm{Spec}_N (H_V)$.

In §2 we prove the following theorem (compare with [5, Theorem 2.3]).

THEOREM 1.2. *Assume that* $VI_D \in K_d$ *and define the Poisson kernel* $N_V(x,y)$ *as in* (1.12).

(1) *The following two conditions are equivalent:*

(a) $\mathrm{Spec}_N (H_V) = \lambda_1 < 0$;

(b) *there exists a pair* x, y *such that* $N_V(x,y)$ *is finite.*

(2) *If either* (a) *or* (b) *holds then* $N_V(x,y)$ *is finite and continuous on* $\bar{D} \times \bar{D}$ *minus the diagonal. In this case, if we define*

$$M_V(x,y) = N_V(x,y)/c_d\,\|x-y\|^{-(d-2)},$$

where $c_d = \Gamma(d/2-1)/2\pi^{d/2}$, *then* $M_V(x,y)$ *is bounded and the limiting set of* $M_V(x,y)$ *as* $\|x-y\| \to 0$ *is the interval* $[1,2]$.

The assertion about the behavior of the Poisson kernel near the diagonal enables us to improve Theorem 1.1 by the following Theorem 1.3, which we shall prove in §3. Let us introduce a norm $\|\cdot\|_\beta^*$ to describe the rate of growth of a function on D near the boundary:

$$\|u\|_\beta^* = \sup_{x \in D} d(x)^\beta |u(x)|.$$

Here $d(x) = d(x, \partial D)$ is the distance from x to the boundary.

THEOREM 1.3. *Assume that $VI_D \in K_d$ and the Poisson kernel $N_V(x,y)$ is finite. Assume also that $\alpha \geqslant 1$. For any $f \in L^\alpha(\partial D)$, the weak solution to the Neumann problem $N(D, V; f)$ exists, is unique and is given by* (1.3) *or* (1.5). *Moreover,*

(i) *if $1 \leqslant \alpha < d-1$, then $u_f \in C(D)$ and there exists a constant $C = C(D, V, p, \alpha)$ such that*

$$\|u_f\|^*_{(d-1-\alpha)} \leqslant C\|f\|_{\alpha, \partial D};$$

(ii) *if $\alpha = d-1$, then $u_f \in C(D) \cap L^\infty(D)$. If $\alpha > d-1$, then $u_f \in C(\bar{D})$. In both cases there exists a constant $C = C(D, V, p, \alpha)$ such that*

$$\|u_f\|_\infty \leqslant C\|f\|_{d-1, \partial D}.$$

2. *The Poisson kernel*

As a first step towards Theorem 1.2, we prove the following property of the transition density function.

PROPOSITION 2.1. *Under our assumptions on the domain, for any fixed $t_0 > 0$, the function*

$$\tilde{M}(t_0; x, y) = c_d^{-1}\|x-y\|^{d-2} \int_0^{t_0} p(t, x, y)\, dt \tag{2.1}$$

is bounded on $\bar{D} \times \bar{D}$ and has as limiting set the interval $[1, 2]$ *as* $\|x-y\| \to 0$.

To prove this result, we need to sketch the construction of the fundamental solution $p(t, x, y)$ by the method of parametrix. The reader is referred to [**8**, Appendix] for details.

Let $x \in \mathbb{R}^d$; let x_∂ denote any point on the boundary such that $\|x - x_\partial\| = d(x, \partial D)$. Let x^* be the point symmetric to x with respect to x_∂, namely, $x^* = 2x_\partial - x$. Notice that since the boundary is C^3, x_∂ and x^* are uniquely determined by x provided that x is sufficiently close to the boundary. Let now ϕ be a C^∞ function on $\mathbb{R}^d$, $0 \leqslant \phi \leqslant 1$, with $\phi(x) = 1$ if $d(x, \partial D) \leqslant \frac{1}{2}\varepsilon_0$ and $\phi(x) = 0$ if $d(x, \partial D) \geqslant \varepsilon_0$, where ε_0 is a fixed small constant. Finally, let

$$\Gamma(t, x, y) = (2\pi t)^{-d/2} e^{-\|x-y\|^2/2t}.$$

As a first approximation of $p(t, x, y)$, we take

$$p_0(t, x, y) = \Gamma(t, x, y) + \phi(x)\,\Gamma(t, x^*, y).$$

This function is defined so that it satisfies the initial condition $\lim_{t\to 0} p(t, x, y) = \delta_y(x)$ and the boundary condition $\partial p(t, x, y)/\partial n_x = 0$, and almost satisfies the heat equation $\partial p/\partial t = \Delta_x p/2$. Now $p(t, x, y)$ can be written as

$$p(t, x, y) = p_0(t, x, y) + p_1(t, x, y), \tag{2.2}$$

where $p_1(t, x, y)$ has the form

$$p_1(t, x, y) = \int_0^t ds \int_D p_0(s, x, z) f(s, z, y)\, m(dz). \tag{2.3}$$

Putting (2.2) in the heat equation, we see that f has to satisfy a differential equation which can be solved by the iteration method. We then obtain the following relations:

$$f(t, x, y) = \sum_{n=0}^{\infty} f_n(t, x, y), \tag{2.4}$$

$$f_0(t,x,y) = \left[\frac{\Delta_x}{2} - \frac{\partial}{\partial t}\right] p_0(t,x,y), \tag{2.5}$$

$$f_n(t,x,y) = \int_0^t ds \int_D f_0(t-s,x,z) f_{n-1}(s,z,y)\, m(dz). \tag{2.6}$$

We shall see below that the series converges absolutely. Now we come to the proof.

Proof of Proposition 2.1. It is elementary to verify that the proposition holds if in (2.1) we replace $p(t,x,y)$ by $p_0(t,x,y)$. Thus, in view of (2.2), we need only show that there is a constant c such that

$$\left| \int_0^{t_0} p_1(t,x,y)\, dt \right| \leqslant \frac{c}{\|x-y\|^{d-3}} \tag{2.7}$$

for all $x \neq y$. To this end, we first prove by induction that

$$|f_n(t,x,y)| \leqslant K_1 K_2^n \Gamma\left(\frac{n+1}{2}\right)^{-1} t^{(n-1-d)/2} e^{-K\|x-y\|^2/t}, \tag{2.8}$$

where K, K_1, K_2 are some constants. For $n = 0$, (2.8) follows directly from (2.5) with suitably chosen K and K_1. By the iteration formula (2.6), we have

$$\begin{aligned} |f_n(t,x,y)| &\leqslant K_1^2 K_2^{n-1} \Gamma\left(\frac{1}{2}\right)^{-1} \Gamma\left(\frac{n}{2}\right)^{-1} \int_0^t (t-s)^{-(d+1)/2} s^{(n-2-d)/2}\, ds \\ &\quad \times \int_D e^{-K(\|x-z\|^2/(t-s) + \|z-y\|^2/s)}\, m(dz) \\ &\leqslant K_1^2 K_2^{n-1} \left(\frac{\pi}{K}\right)^d \Gamma\left(\frac{1}{2}\right)^{-1} \Gamma\left(\frac{n}{2}\right)^{-1} t^{-d/2} e^{-K\|x-y\|^2/t} \\ &\quad \times \int_0^t (t-s)^{-1/2} s^{(n-2)/2}\, ds \\ &= K_1 K_2^n \Gamma\left(\frac{n+1}{2}\right)^{-1} t^{(n-1-d)/2} e^{-K\|x-y\|^2/t} \end{aligned} \tag{2.9}$$

by taking $K_2 = K_1(\pi/K)^d$. Thus (2.8) is proved.

Now by (2.3), (2.4), (2.8) and the same reasoning as in (2.9), we conclude that for any $t_0 > 0$, there exists a constant K_1 such that

$$|p_1(t,x,y)| \leqslant K_1 t^{-(d-1)/2} e^{-K\|x-y\|^2/t}$$

for all $0 < t \leqslant t_0$. This implies (2.7) by integrating out t. The proof is complete.

The following inequalities follow directly from the above proof. First of all, (2.2) and the last inequality imply that

$$p(t,x,y) \leqslant K_1 t^{-d/2} e^{-K\|x-y\|^2/t} \tag{2.10}$$

for $t \leqslant t_0$ and some constant K depending on t_0. Let

$$D_\varepsilon = \{x \in D \colon d(x, \partial D) \leqslant \varepsilon\}.$$

Now (2.10) implies there exist positive ε_0 and K_2 such that for any $\varepsilon \leqslant \varepsilon_0$,

$$\frac{1}{\varepsilon}\int_{D_\varepsilon} p(t,x,y)\,m(dy) \leqslant \frac{K_2}{\sqrt{t}}. \tag{2.11a}$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{\partial D} p(t,x,y)\,\sigma(dy) \leqslant \frac{K_2}{\sqrt{t}}. \tag{2.11b}$$

Hence we have, by (1.2),

$$E^x\phi(t) \leqslant K_3\sqrt{t}, \tag{2.11c}$$

and by the Markov property of the process X and the additivity of ϕ, we see that

$$E^x\,\phi(t)^2 = 2E^x\left[\int_0^t E^{X_s}\phi(t-s)\,\phi(ds)\right] \leqslant K_4\,t. \tag{2.11d}$$

LEMMA 2.2. *The function $b(t,x,y)$ is strictly positive and continuous on $(0,\infty)\times\bar{D}\times\bar{D}$.*

Proof. The strict positivity is clear from (1.7). By the semigroup property, it suffices to show continuity for small t. Using (1.10), we can obtain the following estimates by induction:

$$|b_n|(t,x,y) \leqslant K^{n+1}\,t^{-d/2}\,M(t)^n \tag{2.12a}$$

for some constant K, and

$$\int_0^t ds \int_D |V|(x)\,|b_n|(s,x,y)\,m(dx) \leqslant M(t)^n, \tag{2.12b}$$

where

$$M(t) = \sup_{x\in\bar{D}} E^x\left[\int_0^t |V|(X_s)\,ds\right].$$

By (1.1) and (2.10), $\lim_{t\to 0} M(t) = 0$. Thus (2.12a) implies that there exists a constant t_1 such that the series in (1.4) converges uniformly and absolutely for any $0 < t < t_1$. It remains to show that each $b_n(t,x,y)$ is continuous. Clearly $b_0(t,x,y) = p(t,x,y)$ is continuous. Let us assume inductively that $b_{n-1}(t,x,y)$ is continuous on $(0,\infty)\times\bar{D}\times\bar{D}$. Split the integral (1.5) into three parts I_1, I_2, I_3, namely from 0 to ε, from ε to $t-\varepsilon$, and from $t-\varepsilon$ to t. Now I_2 is continuous by the induction hypothesis (note that $VI_D \in K_d \subset L^1_{\mathrm{loc}}(D)$). As ε tends to zero, I_1 tends to zero uniformly by $\lim_{t\to 0} M(t) = 0$; so does I_3 by (2.12b). Therefore $b_n(t,x,y)$ is continuous. The lemma is proved.

For the proof of Theorem 1.2 we also need the following estimate.

LEMMA 2.3. *For any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $(x,y)\in\bar{D}\times\bar{D}$, $x \neq y$,*

$$\sum_{n=1}^{\infty}\int_0^{t_0} |b_n(t,x,y)|\,dt \leqslant \frac{\varepsilon}{\|x-y\|^{d-2}}.$$

Proof. From the iteration formula (1.5), we have

$$b_n(t,x,y) = \int_{0\leqslant t_1+\dots+t_n\leqslant t} dt_1\dots dt_n \int_{D\times\dots\times D} \prod_{i=1}^{n} p(t_i, x_{i-1}, x_i)\, V(x_i) \times p\left(t-\sum_{i=1}^{n} t_i, x_n, x_{n+1}\right) m(dx_1)\dots m(dx_n). \tag{2.13}$$

Here we have set $x_0 = x$ and $x_{n+1} = y$. Now one of $\|x_i - x_{i+1}\|$, $i = 0,\dots,n$, has to be greater than or equal to $\|x-y\|/(n+1)$. Recall the definition of $\alpha(t)$ in (1.1). We have

$$\begin{aligned}\int_0^{t_0} |b_n(t,x,y)|\,dt &\leqslant K_1^{n+1} \int_{[0,t_0]^{n+1}} dt_0\dots dt_n \int_{D\times\dots\times D} \prod_{i=0}^{n} t_i^{-d/2} e^{-K\|x_i-x_{i+1}\|^2/t_i} \times \prod_{i=1}^{n} |V|(x_i)\, m(dx_i)\\ &\leqslant K_1^{n+1}(n+1) \int_0^\infty t^{-d/2} e^{-K\|x-y\|^2/(n+1)t}\, dt\cdot \alpha(t_0)^n\\ &\leqslant K_2 \frac{K_3^n\, \alpha(t_0)^n}{\|x-y\|^{d-2}}.\end{aligned}$$

Summing from $n = 1$ to ∞, we obtain

$$\sum_{n=1}^{\infty} \int_0^{t_0} |b_n(t,x,y)|\,dt \leqslant \frac{K_2 K_3\, \alpha(t_0)}{1-K_3\,\alpha(t_0)} \|x-y\|^{-(d-2)}.$$

Since $\alpha(t_0) \to 0$ as $t_0 \to 0$, the lemma follows.

LEMMA 2.4. *If $N(x,y)$ is finite for some (x_0, y_0), then for any $t_0 > 0$, there are positive constants K and β such that, for all $t \geqslant 0$,*

$$\sup_{(x,y)\in\bar D\times\bar D} b(t,x,y) \leqslant K e^{-\beta t}.$$

Proof. The proof is similar to that of [5, Theorem 2.2]. It is enough to prove this for large t_0. Fix $t_1 > 0$. From Lemma 2.2 it is clear that there exists a constant c depending on t_1 such that for any $f \geqslant 0$,

$$c^{-1}\|f\|_1 \leqslant T_{t_1} f(x) = E^x[e(t_1) f(X_{t_1})] \leqslant c\|f\|_1. \tag{2.14}$$

By the semigroup property we have, for $t > t_1$,

$$b(t,x,y) = T_{t_1}[b(t-t_0, \cdot, y)](x).$$

Hence, using (2.15), we get a Harnack type inequality

$$b(t,x,y) \leqslant c^2 b(t,z,y) \tag{2.15}$$

for all x,y,z in $\bar D$ and $t \geqslant t_0$. Now the condition $N(x_0, y_0) = \int_0^\infty b(t, x_0, y_0)\,dt < \infty$ implies that there exists $t_0 > t_1$ such that $b(t_0, x_0, y_0) \leqslant c^{-4}/2$. Therefore, by (2.15),

$$\sup_{x\in\bar D} b(t_0, x, y_0) \leqslant \frac{c^{-2}}{2}.$$

Since $b(t,x,y)$ is symmetric in (x,y) (the symmetry follows from that of $p(t,x,y)$ and (2.13)) we have, by (2.15) again,

$$\sup_{(x,y)\in \bar{D}\times\bar{D}} b(t_0,x,y) \leqslant c^2 \sup_{x\in\bar{D}} b(t_0,x,y_0) \leqslant \tfrac{1}{2}. \tag{2.16}$$

Thus $\|T_{t_0}1\|_\infty \leqslant \frac{1}{2}$. Now the semigroup property also implies that $\|T_{nt_0}1\|_\infty \leqslant \|T_{t_0}1\|_\infty^n$. Therefore, for any $t \geqslant 0$, letting $n = [t/t_0]$, we have by (1.11b) that

$$T_t 1(x) = T_{t-nt_0} T_{nt_0} 1(x) \leqslant K_1 \|T_{nt_0}1\|_\infty \leqslant K_1 2^{-nt_0} \leqslant K_2 e^{-\beta t} \tag{2.17}$$

for some K and $\beta > 0$. Finally, by (2.16) and (2.17), for $t \geqslant t_0$,

$$b(t,x,y) = T_{t-t_0} b(t_0,\cdot,y)(x) \leqslant \frac{K_2 e^{\beta t_0}}{2} e^{-\beta t}.$$

The lemma is proved.

We can now give the main result of this section.

Proof of Theorem 1.2. Let us first prove part (2). By the foregoing two lemmas,

$$\left| N(x,y) - \int_0^{t_0} p(t,x,y)\,dt \right| \leqslant \int_{t_0}^\infty |b(t,x,y)|\,dt + \int_0^{t_0} |b(t,x,y) - b_0(t,x,y)|\,dt$$

$$\leqslant K + \varepsilon \|x-y\|^{-(d-2)}.$$

Therefore $N(x,y)$ is finite for all $x \neq y$. Since ε can be made arbitrarily small, the assertion about the limiting set follows from Proposition 2.1.

Now by Lemma 2.3, the integral $\int_0^{t_0} b(t,x,y)\,dt$ converges uniformly to zero as $t_0 \to 0$ on the region $\|x-y\| \geqslant \delta > 0$. This fact together with Lemmas 2.2 and 2.4 implies that $N(x,y)$ is continuous on $\bar{D}\times\bar{D}$ minus the diagonal.

We now show part (1) of Theorem 1.2. Assume again that $N(x_0,y_0) < \infty$. For the eigenfunction ϕ_1 we have, by Lemma 2.4,

$$e^{\lambda_1 t}\phi_1 = T_t\phi_1 = \int_D b(t,x,y)\,\phi_1(y)\,m(dy) \leqslant Km(D)\,\|\phi_1\|_\infty e^{-\beta t}.$$

Hence $\lambda_1 \leqslant -\beta$. Conversely, if $\lambda_1 < 0$, then using $\|T_t 1\|_1 \leqslant (m(D))^{\frac{1}{2}} \|T_t 1\|_2 \leqslant m(D)\,e^{\lambda_1 t}$, we have

$$\int_{D\times D} N(x,y)\,m(dx)\,m(dy) = \int_0^\infty \|T_t 1\|_1\,dt \leqslant -\frac{m(D)}{\lambda_1}.$$

Thus $N(x,y)$ must be finite. Theorem 1.2 is proved.

3. *The Neumann problem*

The aim of this section is to prove Theorem 1.3. As before, let ϕ be the boundary local time of the reflecting Brownian motion X. For any f defined on ∂D, we set

$$\phi_f(t) = \int_0^t f(X_s)\,\phi(ds).$$

By (2.11c) and (2.11d),

$$E^x|\phi_f(t)| \leqslant K\|f\|_\infty \sqrt{t},$$

$$E^x \phi_f(t)^2 \leqslant K\|f\|_\infty^2\, t.$$

For g defined on D, we set

$$\phi_g^\varepsilon(t) = \int_0^t \frac{I_{D_\varepsilon}(X_s)}{\varepsilon} g(X_s)\, ds.$$

LEMMA 3.1. *Assume that f is bounded measurable on ∂D. There exists a uniformly bounded sequence $\{g_n : n \geqslant 1\}$ of continuous functions on $\bar{D}$ and $\{\varepsilon_n : n \geqslant 1\}$ tending to zero such that*

$$\lim_{n\to\infty} \sup_{\substack{0\leqslant t\leqslant T\\ x\in\bar{D}}} |E^x[\phi_{g_n}^{\varepsilon_n}(t) - \phi_f(t)]| = 0.$$

Proof. We have

$$E^x[\phi_g^\varepsilon(t) - \phi_f(t)] = \int_0^t ds \left(\frac{1}{\varepsilon}\int_{D_\varepsilon} p(s,x,y)\, g(y)\, m(dy) - \int_{\partial D} p(s,x,y) f(y)\, \sigma(dy)\right)$$

$$= I_1 + I_2,$$

where

$$I_1 = \int_0^t ds \left(\frac{1}{\varepsilon}\int_{D_\varepsilon} p(s,x,y)\, g(y)\, m(dy) - \int_{\partial D} p(s,x,y)\, g(y)\, \sigma(dy)\right)$$

$$= \left(\int_0^{\delta\wedge t} + \int_{\delta\wedge t}^t\right)(\ldots)\, ds,$$

$$I_2 = \int_0^t ds \int_{\partial D} p(s,x,y)\, [g(y) - f(y)]\, \sigma(dy).$$

By (2.11a) and (2.11b),

$$\left|\int_0^{\delta\wedge t} (\ldots)\, ds\right| \leqslant K\|g\|_\infty \sqrt{\delta}.$$

For fixed δ, as $\varepsilon \to 0$, the integral $\int_{\delta\wedge t}^t (\ldots)\, ds$ clearly tends to zero uniformly in $x \in \bar{D}$ and $0 \leqslant t \leqslant T$. Hence we get $\lim_{\varepsilon\to 0} I_1 = 0$. Next, by (2.10) and the Schwarz inequality,

$$|I_2| \leqslant \int_0^t ds \int_{\partial D} p(s,x,y)\, |g(y) - f(y)|\, \sigma(dy)$$

$$\leqslant K \int_{\partial D} \frac{|g(y) - f(y)|}{\|x-y\|^{d-2}}\, \delta(dy)$$

$$\leqslant K_1 \|g - f\|_{d-1,\partial D}$$

for some constants K and K_1 (see the proof of (3.6) below). Since we can choose $g \in C(\bar{D})$ such that $\|g\|_{\infty,\bar{D}} \leqslant \|f\|_{\infty,\partial D}$ and $\|g - f\|_{d-1,\partial D}$ is arbitrarily small, the lemma now is clear.

LEMMA 3.2. *For any $VI_D \in K_d$ and non-negative measurable f defined on ∂D, we have*

$$E^x\left[\int_0^\infty e(s) f(X_s)\, \phi(ds)\right] = \int_{\partial D} N(x,y) f(y)\, \sigma(dy). \tag{3.1}$$

Proof. By the monotone convergence theorem and the linearity of (3.1) in f, it is enough to show that (3.1) is true for bounded measurable $f \geq 0$. We have

$$E^x\left[\int_0^T e(s) f(X_s)\,\phi(ds)\right] = E^x[\phi_f(T)] + E^x\left[\int_0^T (e(s)-1)\,\phi_f(s)\,ds\right].$$

Recall that ϕ is a continuous additive functional. The second term on the right-hand side is equal to

$$E^x\left[\int_0^T \phi_f(ds)\int_0^t e(t)\,V(X_t)\,dt\right] = E^x\left[\int_0^T e(t)\,V(X_t)\,(\phi_f(T)-\phi_f(t))\,dt\right]$$

$$= E^x\left[\int_0^T e(t)\,V(X_t)\,E^{X_t}(\phi_f(T-t))\,dt\right].$$

Hence,

$$E^x\left[\int_0^T e(t)\,\phi_f(dt)\right] = E^x[\phi_f(T)] + E^x\left[\int_0^T e(t)\,V(X_t)\,E^{X_t}(\phi_f(t-t))\,dt\right]. \quad (3.2)$$

Choose ε_n and g_n so that Lemma 3.1 holds and let

$$\Delta_n(t) = \phi_{g_n}^{\varepsilon_n}(t) - \phi_f(t).$$

By the same argument, we see that (3.2) also holds if ϕ_f there is replaced by $\phi_{g_n}^{\varepsilon_n}$. Subtracting the resulting two equations, we obtain

$$\left|E^x\left[\int_0^T e(t)\,d\Delta_n(t)\right]\right| \leq E^x|\Delta_n(T)| + E^x\left[\int_0^T e(t)\,|V|\,(X_t)\,E^{X_t}|\Delta_n(T-t)|\,dt\right].$$

Lemma 3.1 and the dominated convergence theorem show that the right-hand side tends to zero. Hence,

$$\lim_{n\to\infty} E^x\left[\int_0^T e(t)\,\phi_{g_n}^{\varepsilon_n}(dt)\right] = E^x\left[\int_0^T e(t)\,f(X_t)\,\phi(dt)\right].$$

The left-hand side is equal to

$$E^x\left[\int_0^T e(t)\,\frac{I_{D_{\varepsilon_n}}(X_t)}{\varepsilon_n}\,g_n(X_t)\,dt\right] = \int_0^T dt\,\frac{1}{\varepsilon_n}\int_{D_{\varepsilon_n}} b(t,x,y)\,g_n(y)\,m(dy).$$

We now let $n \to \infty$. An argument similar to the one used in Lemma 3.1 shows that

$$E^x\left[\int_0^T e(t)\,f(X_t)\,\phi(dt)\right] = \int_{\partial D}\left(\int_0^T b(t,x,y)\,dt\right) f(y)\,\sigma(dy).$$

Now letting $T \to \infty$ we have (3.1). The proof is complete.

Let

$$u_f(x) = \frac{1}{2}\int_{\partial D} N(x,y)\,f(y)\,\sigma(dy)$$

as before. From now on, we assume that $N(x,y)$ is finite. The only result we need from the last section is the estimate

$$N(x,y) \leqslant \frac{K}{\|x-y\|^{d-2}} \tag{3.3}$$

for some constant K. From (3.3) we obtain the following estimate.

PROPOSITION 3.3. (i) *If* $1 \leqslant \alpha < d-1$, *then* $u_f \in C(D)$. *There exists a constant* $C = C(D, V, \alpha)$ *such that*

$$\|u_f\|^*_{(d-1-\alpha)/\alpha} \leqslant C\|f\|_{\alpha,\partial D}. \tag{3.4}$$

(ii) *If* $\alpha = d-1$, *then* $u_f \in L^\infty(D) \cap C(D)$. *If* $\alpha > d-1$, *then* $u_f \in C(\bar{D})$. *In both cases, there exists a constant* $C = C(D, V)$ *such that*

$$\|u_f\|_\infty \leqslant C\|f\|_{d-1,\partial D}. \tag{3.5}$$

Proof. Let us show first that, for any $f \in L^\alpha(\partial D)$,

$$\int_{\partial D} \frac{|f|(y)}{\|x-y\|^{d-2}}\,\sigma(dy) \leqslant K\|f\|_{\alpha,\partial D}(d(x)^{-(d-1-\alpha)/\alpha}+1). \tag{3.6}$$

Let $1/\alpha + 1/\beta = 1$. Using Schwarz's inequality, we have

$$\int_{\partial D} \frac{|f|(y)}{\|x-y\|^{d-2}}\,\sigma(dy) \leqslant K\|f\|_{\alpha,\partial D}\left(\int_{\partial D} \|x-y\|^{-(d-2)\beta}\,\sigma(dy)\right)^{1/\beta}.$$

By a local computation we can prove easily that, for any $s \geqslant 0$,

$$\int_{\partial D} \|x-y\|^{-s}\,\sigma(dy) \leqslant K_1(d(x)^{d-1-s}+1),$$

where $d(x) = d(x, \partial D)$. The inequality (3.6) is now clear; (3.4) and (3.5) follow from (3.6) and (3.3).

The interior continuity can be proved as follows. Let B be a ball contained in D and let

$$\tau_B = \inf\{t > 0 : X_t \in B^c\}.$$

Since $\phi(t) = 0$ for $0 \leqslant t \leqslant \tau_B$, we have

$$u_f(x) = E^x[e(\tau_B)\,u(X_{\tau_B})].$$

Now applying [1, Theorem A.4.6] (see also [1, A.4.9]), we conclude that u_f is continuous in B.

It remains to prove that u_f is continuous on $\bar{D}$ if $f \in L^\alpha(\partial D)$ for some $\alpha > d-1$. We have by (3.1) that

$$u_f(x) = I(s,x) + T_s u_f(x),$$

where

$$I(s,x) = \tfrac{1}{2}E^x\left[\int_0^s e(t)\,f(X_t)\,\phi(dt)\right].$$

Since $u_f \in L^\infty(D)$, we have by (1.11a) that $T_s u_f \in C(\bar{D})$. We assert that $I(s,x)$ tends to zero uniformly on $\bar{D}$ as $s \to 0$. To prove this assertion, let $\mu = \alpha/(d-1) > 1$ and $1/\mu + 1/\lambda = 1$. We have

$$2|I(s,x)| \leqslant \left(E^x\left[\int_0^s e_{\lambda V}(t)\,\phi(dt)\right]\right)^{1/\lambda}\left(E^x\left[\int_0^s |f|^\mu(X_t)\,\phi(dt)\right]\right)^{1/\mu}. \tag{3.7}$$

By (1.11b) and (2.11d), the first factor raised to λth power is bounded by

$$E^x[e_{\lambda|V|}(s)\,\phi(s)] \leqslant (E^x[e_{2\lambda|V|}(s)])^{\frac{1}{2}}(E^x\,\phi(s)^2)^{\frac{1}{2}} \leqslant K\sqrt{s}.$$

Hence it tends to zero uniformly as $s \to 0$. For the second factor in (3.7), by (2.10) and (3.6),

$$E^x\left[\int_0^s |f|^\mu(x_t)\,ds\right] = \int_{\partial D} |f|^\mu(y)\,\sigma(dy)\int_0^s p(t,x,y)\,dt$$

$$\leqslant K\int_{\partial D}\frac{|f|^\mu(y)}{\|x-y\|^{d-2}}\,\sigma(dy)$$

$$\leqslant K_1\|f\|^\mu_{\alpha,\partial D}.$$

Therefore the second factor in (3.7) is bounded and we have shown that $I(s,x)$ tends to zero uniformly as $s\to 0$. Consequently $u_f\in C(\bar{D})$, and the poof is complete.

REMARK. The mapping $f\mapsto u_f$ is continuous from $L^\alpha(\partial D)$ to $L^\beta(D)$ for any α,β such that $1\leqslant\alpha<d-1$ and $1\leqslant\beta<1+\alpha/(d-1-\alpha)$. In fact, if $f\in L^\alpha(\partial D)$,

$$\|u_f\|^\beta_\beta = \int_D |u_f|(x)\,|u_f|^{\beta-1}(x)\,m(dx)$$

$$\leqslant K\|f\|^{\beta-1}_{\alpha,\partial D}\int_{\partial D}|f|(y)\,\sigma(dy)\int_D\frac{d(x)^{-\gamma}}{\|x-y\|^{d-2}}\,m(dx)$$

$$\leqslant K_1K_2\|f\|^{\beta-1}_{\alpha,\partial D}\,\|f\|_{1,\partial D}$$

$$\leqslant K_3\|f\|^\beta_{\alpha,\partial D}.$$

Here $\gamma=(d-1-\alpha)(\beta-1)/\alpha<1$, and we have also used the fact that for any $\gamma<1$ there is a constant K_2 such that, for any $y\in\partial D$,

$$\int_D\frac{d(x)^{-\gamma}}{\|x-y\|^{d-2}}\,m(dx)\leqslant K_2.$$

Thus we obtain $u_f\in L^\beta(D)$ and $\|u_f\|_{\beta,D}\leqslant K_4\|f\|_{\alpha,\partial D}$.

Finally, we complete the following.

Proof of Theorem 1.3. First of all, we verify that the integrability condition of the definition (given in the introduction) is satisfied by u_f. By the remark above, we have $u_f\in L^1(D)\cap C(D)$ for all $f\in L^1(\partial D)$. Since $\int_D|V|(y)\,\|x-y\|^{-(d-2)}\,m(dy)$ is bounded on $\bar{D}$ (this follows from (1.1) by integrating out s), we have

$$\|u_f V\|_{1,D}\leqslant K\int_{\partial D}|f|(y)\,\sigma(dy)\int_D\frac{|V|(x)}{\|x-y\|^{d-2}}\,m(dx)\leqslant K_1\|f\|_{1,\partial D}. \tag{3.8}$$

Therefore $u_f V\in L^1(D)$.

Next we show that u_f satisfies (1.7). We shall first outline the proof of Theorem 1.1 as given in [5] and then show how to obtain Theorem 1.3 from Theorem 1.1 by a limiting procedure. Let $\{\mathscr{F}_t: t\geqslant 0\}$ be the natural filtration of σ-fields associated with the reflecting Brownian motion. Let f be a bounded measurable function defined on the boundary ∂D and let u be a continuous function on $\bar{D}$. Define

$$M^u_f(t) = u(X_t)-u(X_0)+\int_0^t V(X_s)\,u(X_s)\,ds+\frac{1}{2}\int_0^t f(X_s)\,\phi(ds).$$

The proof that u_f satisfies (1.7) is accomplished in two steps. The first step is to show that for u_f defined by (1.3), the process $\{M_f^{u_f}(t): t \geqslant 0\}$ is a continuous $(P^x, \mathscr{F}_t)$-martingale for any $x \in \bar{D}$. In fact, a calculation shows that the following equality holds:

$$M_f^{u_f}(t) = \int_0^t e(s)^{-1}\, dM_s,$$

where $\{M_s: s \geqslant 0\}$ is the martingale defined by

$$M_s = E^x\left[\int_0^\infty e(t) f(X_t)\, \phi(dt) \mid \mathscr{F}_s\right].$$

The second step is to show that u is *continuous on* $\bar{D}$; then u is a weak solution of the Neumann problem $N(D;\, V, f)$ as defined if and only if $\{M_f^u(t): t \geqslant 0\}$ is a continuous $(P^x, \mathscr{F}_t)$-martingale for all $x \in \bar{D}$.

To derive Theorem 1.3 from Theorem 1.1, let $f \in L^1(\partial D)$ and let f_n be a sequence of bounded measurable functions on ∂D converging to f in $L^1(\partial D)$. By Theorem 1.1, (1.7) holds for u_{f_n}, namely,

$$\int_D u_{f_n} H_V v(x)\, m(dx) = -\frac{1}{2}\int_{\partial D} f_n(x)\, v(x)\, \sigma(dx).$$

By the remark after Proposition 3.3 and (3.8), we have $u_{f_n} \to u_f$ and $u_{f_n} V \to u_f V$ in $L^1(D)$. Thus we can take the limit as $n \to \infty$ to obtain the desired equality (1.7). We have therefore shown that for any $f \in L^1(\partial D)$, u_f is a weak solution of the Neumann problem $N(D, V; f)$. The estimates claimed in the theorem follow from Proposition 3.3.

It remains to prove the uniqueness of the weak solution. Let $f \in L^1(\partial D)$ and suppose that u is a weak solution of $N(D, V; f)$, namely $u \in L^1(D)$, $uV \in L^1(D)$ and u satisfies (1.7). We show that $u(x)$ must be given by

$$u(x) = \tfrac{1}{2}E^x\left[\int_0^\infty e(t) f(X_t)\, \phi(dt)\right], \quad m\text{-a.e.} \tag{3.9}$$

Let

$$A_t = \int_0^t u(X_s)\, V(X_s)\, ds + \frac{1}{2}\int_0^t f(X_s)\, \phi(ds)$$

and

$$M_t \equiv M_f^u(t) = u(X_t) - u(X_0) + A_t. \tag{3.10}$$

Denoting the shifting operator of the process X by θ we have, for any $t \geqslant s \geqslant 0$,

$$M_t = M_s + M_{t-s} \circ \theta_s.$$

Since $dp(t,x,y)/dt = \Delta_y p(t,x,y)/2$, for $t \geqslant s > 0$ we have

$$E^x[M_t - M_s] = \int_D [p(t,x,y) - p(s,x,y)]\, u(y)\, m(dy) + \int_s^t dl \int_D u(y)\, V(y)\, p(l,x,y)\, m(dy)$$

$$+\frac{1}{2}\int_s^t dl \int_{\partial D} f(y)\, p(l,x,y)\, \sigma(dy)$$

$$= \int_s^t \left[\int_D u(y)\left(\frac{\Delta_y}{2} + V(y)\right) p(l,x,y)\, m(dy) + \frac{1}{2}\int_{\partial D} f(y)\, p(l,x,y)\, \sigma(dy)\right] dl.$$

The last term is zero by (1.7) with $v = p(l, x, \cdot)$, because $p(t, x, \cdot) \in C_\partial^2(D)$. Therefore

$$E^x M_t = E^x M_s \quad \text{for } t > s > 0. \tag{3.11}$$

Let us show now that, for all $t > 0$,

$$E^x M_t = 0, \quad m\text{–a.e. } x. \tag{3.12}$$

To prove this, assume that μ is a probability measure on D with bounded density (with respect to the Lebesgue measure) $d\mu/dm \in C(\bar{D})$. As usual, P^μ denotes the law of X with initial distribution μ. From the definition of M_s and our choice of μ, we prove easily that $\lim_{s\to 0} E^\mu M_s = 0$. Hence by (3.11) $E^\mu M_t = 0$ for any $t \geqslant 0$. Then (3.12) follows. Now integrating by parts, we have

$$A_t e(t) - \int_0^t A_s\, de(s) = \int_0^t e(s)\, dA_s$$

$$= \int_0^t u(X_s)\, de(s) + \frac{1}{2}\int_0^t e(s) f(X_s)\, \phi(ds). \tag{3.13}$$

Using (3.10), we find that (3.13) is equivalent to

$$u(X_0) = e(t)\, u(X_t) - M_t - \int_0^t (M_t - M_s)\, de(s) + \frac{1}{2}\int_0^t e(s) f(X_s)\, \phi(ds). \tag{3.14}$$

Now take the expectation E^x on both sides of (3.14). On the left-hand side we have $E^x u(X_0) = u(x)$. By Lemma 2.4, the first term on the right-hand side, $E^x[e(t)\, u(X_t)]$, tends to zero exponentially as $t \to \infty$. By (3.12), the second term is equal to zero almost everywhere in x. The third term is equal to

$$E^x\left[\int_0^t M_{t-s} \circ \theta_s\, de(s)\right] = E^x\left[\int_0^t E^{X_s} M_{t-s}\, de(s)\right] = 0.$$

Thus after taking E^x in (3.14) and letting $t \to \infty$, we obtain (3.9). The proof of Theorem 1.3 is complete.

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