# CORRECTION AND ADDITION TO CHAPTER 19

## 1. Summarizing the issue

There is a gap in the proof in [1, Section 10.2.3] and the completion of this in [2, Chapter 19] does not fully address the issue and is inaccurate. This document summarizes the issue, necessary corrections, and additions.

In the notation of Section 10.2.3 of [1], we show that $Y$ is a nilmanifold $H/\Gamma$ by lifting the action of the group $G$ on $X$ to an action of the group $H$ on $Y$. We then say that this action is "transitive," writing "Choose a lift $y_1$ of $x_1$ in $Y$ and consider the map $f\colon H \to Y$ given by $f(h) = h\cdot x_1$." Next we define $\Gamma$ to be the stabilizer $\{h \in H\colon f(h) = y_1\}$ of $y_1$ and use the map $f$ to identify $Y$ and $H/\Gamma$. However, this argument does not make sense: at this stage in the proof, $Y$ is a probability space and does not have any topological structure and the action of $H$ on $Y$ is a measure preserving action that is only defined almost everywhere. In particular, the map $f$ is not defined, the notion of a transitive action does not make sense, and we can not define the stabilizer of $x_1$.

In [2, Chapter 19, Section 3] we claimed to have filled in this gap, but this proof requires correction. In Section 19.3, we first consider the general case of a measure preserving action of a group $G$ on a probability space $(X, \mu)$ and we define a convolution product $\phi * f$ for $\phi \in C_c(G)$ and $f \in L^1(X)$ (see Equation (10) on page 312). However, the comments below Equation (10) are easily seen to be false, as seen by considering a trivial action. Moreover, these comments do not even hold in our setting, meaning when $X = G/\Gamma$ is a nilmanifold.

This note corrects and completes the arguments of both [1] and [2]. The incorrect comments on page 312 are replaced by a new result (Lemma 2), and we use this endow our space $X$ with a topological structure such that the action of $G$ is continuous and transitive, avoiding any consideration of a general action.

Henceforth, we use the notation of [2], where the roles of $X$ and $Y$ are reversed with respect to [1].

## 2. Modifications of the argument

Using the set up and notation of [2, Chapter 19], we have that $Y = H/\Lambda$ is a nilmanifold with Haar measure $\nu$. We start with a preliminary lemma.

**Lemma 1.** *For every compact $M$ subset of $H$, there exists a constant $C > 0$ such that for any $f \in L^1(\nu)$ with $f \geq 0$,*

$$\sup_{y\in Y} \int_M f(h \cdot y)\, dm_H(h) \leq C\|f\|_{L^1(\nu)}. \tag{1}$$

*Proof.* Let $\pi\colon H \to Y = H/\Lambda$ be the quotient map, set $e_Y = \pi(e_H)$, and let $D$ be a compact fundamental domain of the projection $\pi$. This means that $H = \bigcup_{\lambda\in\Lambda} D\lambda$ and the sets comprising this union pairwise have $m_H$-negligible intersections. We normalize the Haar measure $m_H$ of $H$ such that $m_H(D) = 1$.

We first consider the case that $y = e_Y$. We claim that for every compact $M \subset H$, there exists a constant $c > 0$ such that for any $f \in L^1(\nu)$ with $f \geq 0$, we have that

$$\int_M f(h \cdot e_Y)\, dm_H(h) \leq c\|f\|_{L^1(\nu)}.$$

To check this, note that since $D^{-1}M$ is compact and $\Lambda$ is discrete, the set $F := D^{-1}M \cap \Lambda$ is finite. Since $M \subset \bigcup_{\lambda\in F} D\lambda$, it follows that

$$\begin{aligned}\int_M f(h \cdot e_Y)\, dm_H(h) &\leq \sum_{\lambda\in F} \int_{D\lambda} f(h \cdot e_Y)\, dm_H(h)\\ &= |F| \int_D f(h \cdot e_Y)\, dm_H(h) = |F|\, \|f\|_{L^1(\nu)}\end{aligned}$$

and the claim is proven.

For the general case, let $M$ be a compact subset of $H$ and let $C$ be the constant associated by to the compact set $MD$ for the case that $y = e_Y$. Let $y \in Y$ and choose $g \in D$ such that $\pi(q) = y$. In particular, $q \cdot e_Y = y$. Then we have that

$$\begin{aligned}&\int_M f(h \cdot y)\, dm_H(y) = \int_M f(hq \cdot e_Y)\, dm_H(h)\\ &\quad= \int_{Mq} f(h \cdot e_G)\, dm_H(y) \leq \int_{MD} f(h \cdot e_G)\, dm_H(y) \leq C\|f\|_{L^1(\nu)},\end{aligned}$$

and so (1) holds with the constant $C$. □

We next establish a result similar to Lemma 1, but with the group $H$ replaced by $G$ and the nilmanifold $(Y, \nu)$ replaced by $(X, \mu)$.

Recall that if $\phi$ is a bounded function with compact support on $G$ and $f \in L^1(\mu)$, the convolution function $\phi * f$ on $X$ is defined by

(Equation (10) in [2]) $$\phi * f(x) = \int_G \phi(g)\, f(S_g x)\, d\mu(g).$$

We use $\mathsf{C_c}(G)$ to denote the algebra of continuous functions on $G$ with compact support.

**Lemma 2.** *For every $\phi \in \mathsf{C_c}(G)$, there exists a constant $C > 0$ such for any $f \in L^1(\mu)$ with $f \geq 0$, we have*

$$\|\phi * f\|_{L^\infty(\mu)} \leq C\|f\|_{L^1(\mu)}.$$

*Proof.* We recall some notation and results from [2, Chapter 19]. Let $p\colon G \to H$ be the quotient map, let $L = \ker(p)$, and recall that $K$ is a finite dimensional torus. By [2, Proposition 5], we have that $K$ is an open subgroup of $L$. We normalize the Haar measure $m_L$ of $L$ such that $m_L(K) = 1$. Choosing a countable family $(z_i)_{i\in I}$ in $L$ such that the sets $z_iK$ form a partition of $L$, then for $\psi \in L^1(m_L)$ we have that

$$\int_L \psi\, dm_L = \sum_{i\in I} \int_K \psi(z_i u)\, dm_K(u).$$

The group $G$ is endowed with a right invariant distance $d_G$, and so there exists $\varepsilon > 0$ such that for all distinct $i, j \in I$,

$$d_G(z_iK, z_jK) > 4\varepsilon.$$

Let $\sigma\colon H \to G$ be a Borel cross section of the quotient map $p\colon G \to H$. We normalize the Haar measure $m_G$ of $G$ such that for all $\psi \in C_c(G)$,

$$\int_G \psi\, dm_G = \int_H \Big(\int_L \psi(z\sigma(h))\, dm_L(z)\Big)\, dm_H(h)$$

and thus we have

$$\int_G \psi\, dm_G = \int_H \Big(\sum_{i\in I} \int_K \psi(z_i u\sigma(h))\, dm_K(u)\Big)\, dm_H(h).$$

As in [2, Section 2.2], we make use of the identification $X = Y \times K$, and for $g \in G$ and $x = (y, u) \in X$, we write

$$S_g(y, u) = \big(T_{p(g)}y, u + \rho_g(y)\big),$$

where $(\rho_g)_{g\in G}$ is a cocycle on $Y$ taking values in $K$ and such that for all $g \in G$ and $u \in K$ we have

$$\rho_{gu}(y) = u + \rho_g(y) \quad \text{for } \nu\text{-almost every } y \in Y.$$

Let $\phi \in \mathsf{C_c}(G)$. We can express $\phi$ as a finite sum of functions belonging to $C_c(G)$, each of which has support with diameter bounded by

$\varepsilon$. Thus without loss of generality, we can assume that $\phi$ is a smooth function whose support $M$ has diameter bounded by $\varepsilon$. For $\chi \in \widehat{K}$, define

$$\phi_\chi(g) = \int_K \phi(gu)\overline{\chi}(u)\, dm_K(u).$$

It follows that for any $\chi \in \widehat{K}$, the function $\phi_\chi$ is smooth, supported on $MK$, and $\phi_\chi$ has vertical frequency $\chi$, meaning that for all $g \in G$ and $u \in K$,

$$\phi_\chi(gu) = \phi_\phi(g)\chi(u).$$

Thus, we can write

$$\phi = \sum_{\chi \in \widehat{K}} \phi_\chi,$$

where this series converges in the uniform norm of $\mathsf{C_c}(G)$. In particular, without loss of generality, we can restrict ourselves to the case that $\phi$ has vertical frequency $\chi$ for some $\chi \in \widehat{K}$ and $\phi$ is supported on $MK$ where $M$ is a compact set whose diameter is bounded by $\varepsilon$.

On the other hand, for $f \in L^1(\mu)$, it is easy to check that

$$\phi * f(x) = \phi * f_{\overline{\chi}}, \qquad \text{where } f_{\overline{\chi}}(x) = \int_K f(S_z x)\chi(u)\, dm_k(u).$$

The function $f_{\overline{\chi}}$ satisfies $\|f_{\overline{\chi}}\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$ and has vertical frequency $\overline{\chi}$, meaning that for any $u \in K$,

$$f_\chi(S_u x) = \overline{\chi}(u) f(x).$$

Therefore, without loss of generality, we can restrict ourselves to the case that $f$ has vertical frequency $\overline{\chi}$.

Using the identification $X = Y \times K$, we write the function $f$ on $X$ as

$$f(y,u) = F(y)\overline{\chi}(u)$$

for some function $F$ on $Y$ satisfying

$$\|F\|_{L^1(\nu)} = \|f\|_{L^1(\mu)}$$

and where $\overline{\chi}$ is the vertical frequency of $f$. For $g \in G$, we have

$$f(S_g(y,u)) = F(T_{p(g)}y)\, \overline{\chi}(u)\, \overline{\chi}(\rho_g(y)).$$

Computing the convolution, we have that

$$\begin{aligned}\phi * f(y,u) &= \overline{\chi}(u) \int_G \phi(g)\, F(T_{p(g)}y)\overline{\chi}(\rho_g(y))\, dm_G(g) \\ &= \overline{\chi}(u) \int_H F(T_h y) \sum_{i\in I} \Big( \int_K \phi(z_i u\sigma(h))\overline{\chi}(\rho_{z_i u\sigma(h)}(y))\, dm_K(u)\Big)\, dm_H(h) \\ &= \overline{\chi}(u) \int_H F(T_h y) \Big( \sum_{i\in I} \Psi_i(y,h)\Big)\, dm_H(h),\end{aligned}$$

where

$$\Psi_i(y,h) = \int_K \phi(z_i u\sigma(h))\overline{\chi}(\rho_{z_i u\sigma(h)}(y))\, dm_K(u).$$

By hypothesis, for distinct $i,j \in I$ and $u,u' \in K$, we have

$$d_G(z_i u\sigma(h), z_j u'\sigma(h)) > 4\varepsilon.$$

Since $\phi$ is supported on $MK$ and the diameter of $M$ is bounded by $\varepsilon$, it follows that for a given $h \in H$, there is at most one value $i = i(h) \in I$ such that $\psi_i(y,h) \neq 0$ for some $y \in Y$. Therefore,

$$\Big| \sum_{i\in I} \Psi_i(y,h)\Big| \leq \sup_{i\in I} |\Psi_i(y,h)|$$

Since $\phi$ vanishes outside $MK$, for every $i \in I$ we have that $\Psi_i(y,h) = 0$ except when $z_i\sigma(h) \in MK$. Since $z_i \in L = \ker(p)$, it follows that $h \in p(M)$. Combining these, we obtain that

$$\sup_{i\in I} |\Psi_i(y,h)| \leq \|\phi\|_\infty\, \mathbf{1}_{p(M)}(h).$$

Summarizing, we have that

$$\begin{aligned}|\phi * f(y,u)| &\leq \|\phi\|_\infty \int_{p(M)} |F(T_h y)|\, dm_H(h) \\ &\leq C\|\phi\|_\infty \|F\|_{L^1(\nu)} = C\|\phi\|_\infty \|f\|_{L^1(\mu)},\end{aligned}$$

where $C$ is the constant from Lemma 1 that is associated to the compact subset $p(M)$ of $H$. □

**Corollary 3.** *For a compact subset $M$ of $G$, there exists a constant $C$ such that*

$$\Big| \int_M f(S_g x)\, dm_G(g)\Big| \leq C\|f\|_{L^1(\mu)} \quad \text{for } \mu\text{-almost every } x \in X.$$

Note that this bound can be rewritten as

$$\|\mathbf{1}_M * f\|_{L^\infty(\mu)} \leq C\|f\|_{L^1(\mu)}.$$

We quote the following lemma from [2], whose proof is correct as written there.

**Lemma 4** ([2, Chapter 19, Lemma 7]). *For $1 \leq p < \infty$, the action of $G$ on $L^p(\mu)$ defined by $(g, f) \mapsto S_g f$ is strongly continuous. In particular, for $f \in L^p(\mu)$, $\Phi_j * f \to f$ in $L^p(\mu)$ as $j \to \infty$.*

Let $\mathcal{A}$ denote the family of regular functions, meaning that collection of $f \in L^\infty(\mu)$ such that the map $g \mapsto S_g f$ is a continuous map from $G$ to $L^\infty(\mu)$.

**Lemma 5** ([2, Chapter 19, Lemma 8]).

(i) *If $\phi \in \mathsf{C_c}(G)$ and $f \in L^1(\mu)$, then $\phi * f \in \mathcal{A}$.*
(ii) *For $1 \leq p < \infty$, the algebra $\mathcal{A}$ is dense in $L^p(\mu)$.*
(iii) *If $f \in \mathcal{A}$, then $\Phi_j * f \to f$ in $L^\infty(\mu)$ as $j \to \infty$.*
(iv) *The algebra $\mathcal{A}$ is the closure in $L^\infty(\mu)$ of*

$$\{\phi * f \colon \phi \in \mathsf{C_c}(X),\ f \in L^1(\mu)\}.$$

(v) *The algebra $\mathcal{A}$ is separable with respect to the norm of $L^\infty(\mu)$.*

*Proof. Part* (i): Let $M$ be the support of $\phi$ and let $W$ be a compact neighborhood of $e_G$ in $G$. Let $C$ be the constant associated to the compact set $MW$ in Corollary 3.

Let $\varepsilon > 0$. Since $\phi$ is uniformly continuous, there exists a neighborhood $V$ of $e_G$ in $G$, contained in $W$, and such that $|\phi(hg) - \phi(g)| < \varepsilon/C$ for all $g \in G$.

Fix $h \in V$. Then for every $x \in X$, we have

$$\phi * f(S_h x) = \int_G \phi(gh^{-1}) f(S_g x)\, dm_G(g).$$

It thus follows that for every $x \in X$, we have

$$\big|\phi * f(S_h x) - \phi * f(x)\big| \leq \int_G |\phi(gh^{-1}) - \phi(g)|\, |f(S_g x)|\, dm_G(g).$$

Since $|\phi(gh^{-1}) - \phi(g)| \leq \varepsilon/C$ for every $g \in G$ and since $\phi$ is supported on $M$ and $h \in V \subset W$, we have that $\phi(gh^{-1}) - \phi(g) = 0$ and for all $g$ not contained in $MW$. Therefore, $|\phi(gh^{-1}) - \phi(g)| \leq \mathbf{1}_{MW}(g)\varepsilon/C$. By the definition of $C$, we conclude that for every $h \in V$,

$$|\phi * f(S_h x) - \phi * f(x)| < \varepsilon \quad \text{for } \mu\text{-almost every } x \in X.$$

*Part* (ii): This follows by combining Lemma 4 and Part (i).

*Part* (iii): Let $\varepsilon > 0$. Since $f \in \mathcal{A}$, there exists $j \in \mathbb{N}$ such that $\|S_g f - f\|_{L^\infty(\mu)} < \varepsilon$ for every $g$ in the ball $B$ of radius $1/j$ centered at $e_G$. For any $g$ in this ball and $x \in X$, we have

$$\phi_j * f(x) - f(x) = \int_G \Phi_j(g)\big(f(S_g x) - f(x)\big)\, dm_G(g).$$

It then follows that

$$\|\Phi_j * f - f\|_{L^\infty(\mu)} \le \|\Phi_j\|_{L^1(\mu)} \|S_g f - f\|_{L^\infty(\mu)} \le \varepsilon.$$

*Part* (iv): By Part (i), the set $\{\phi * f \colon \phi \in C_c(X),\ f \in L^1(\mu)\}$ is contained in $\mathcal{A}$ and by Part (iii) its closure with respect to the norm of $L^\infty(\mu)$ contains $\mathcal{A}$.

*Part* (v): Since $L^1(\mu)$ is separable, it follows from Lemma 2 that for every $j \in \mathbb{N}$ the set $\{\Phi_j * f \colon f \in L^1(\mu)\}$ is separable with respect to the norm of $L^\infty(\mu)$. Thus the union of these sets is separable. By Part (iii), the closure of this union is equal to $\mathcal{A}$. □

## 3. An alternate approach

Another possible method to endow $X$ with a topological structure is by using the dual functions studied in [2, Chapter 8]. We outline this approach, using the notation of that chapter.

Define

$$\mathcal{D} = \left\{ \mathrm{D}_{k+2}(f_{\underline{\epsilon}} \colon \underline{\epsilon} \in [\![k+2]\!]^*) \colon f_{\underline{\epsilon}} \in L^{2^{k+2}-1}(\mu) \text{ for every } \underline{\epsilon} \in [\![k+2]\!]^* \right\}.$$

By [2, Chapter 8, Proposition 21], $\mathcal{D} \subset L^\infty(\mu)$, and when $f_{\underline{\epsilon}} \in L^{2^{k+2}-1}(\mu)$ for $\underline{\epsilon} \in [\![k+2]\!]^*$, the dual function $\mathrm{D}_{k+2}(f_{\underline{\epsilon}} \colon \underline{\epsilon} \in [\![k+2]\!]^*)$ satisfies

$$\|\mathrm{D}_{k+2}(f_{\underline{\epsilon}} \colon \underline{\epsilon} \in [\![k+2]\!]^*)\|_{L^\infty(\mu)} \le \prod_{\underline{\epsilon}\epsilon \in [\![k+2]\!]^*} \|f_{\underline{\epsilon}}\|_{L^{2^{k+2}-1}(\mu)}. \tag{2}$$

It follows that $\mathcal{D}$ is separable with respect to the norm of $L^\infty(\mu)$. Furthermore, by definition of $G$, for every $g \in G$ the measure $\mu^{[\![k+2]\!]}$ is invariant under $S_g^{[\![k+2]\!]}$ and it follows that

$$S_g \mathrm{D}_{k+2}(f_{\underline{\epsilon}} \colon \underline{\epsilon} \in [\![k+2]\!]) = \mathrm{D}_{k+2}(S_g^{-1} f_{\underline{\epsilon}} \colon \underline{\epsilon} \in [\![k+2]\!]^*).$$

Therefore, $\mathcal{D}$ is invariant under $S_g$. By using the inequality (2) again and Lemma 4, we deduce that for $h \in \mathcal{D}$, the map $g \mapsto S_g h$ from $G$ to $L^\infty(\mu)$ is continuous. Thus $\mathcal{D}$ is contained in $\mathcal{A}$. Furthermore, since $(X, \mu, T)$ is an ergodic $(k+1)$-step nilsystem, the linear span of $\mathcal{D}$ is dense in $L^1(\mu)$ by [2, Chapter 9, Proposition 17].

Therefore, instead of using the algebra $\mathcal{A}$, we can make use of the closed subalgebra of $L^\infty(\mu)$ spanned by $\mathcal{D}$. This method eliminates the need of Lemmas 1 and 2, Corollary 3, and Lemma 5.

We note that we can not, instead, use the closed linear span of $\mathcal{D}$, as we do not have a method of of showing that this space in an algebra; while it can be checked that the product of two elements of this space is a limit in $L^1(\mu)$ of elements of this space, this is not sufficient.

## References

[1] B. Host & B. Kra. Non conventional ergodic averages and nilmanifolds. *Ann. of Math. (2)*, **161** (2005), 397–488.

[2] B. Host & B. Kra. *Nilpotent Structures in Ergodic Theory.* Mathematical Surveys and Monographs **236**. American Mathematical Society, Providence, RI, 2018.