EQUIDISTRIBUTION OF DILATED CURVES ON NILMANIFOLDS

BRYNA KRA, NIMISH A. SHAH, AND WENBO SUN

ABSTRACT. Generalizing classic results for a family of measures in the torus, for a family $(\mu_t)_{t \geq 0}$ of measures defined on a nilmanifold X, we study conditions under which the family equidistributes, meaning conditions under which the measures μ_t converge as $t \to \infty$ in the weak* topology to the Haar measure on X. We give general conditions on a family of measures defined by a dilation process, showing necessary and sufficient conditions for equidistribution as the family dilates, along with conditions such that this holds for all dilates outside some set of density zero. Furthermore, we show that these two types of equidistribution are different.

1. INTRODUCTION

1.1. Limiting distributions of measures.

A classic problem for billiards is of illumination: in a polygonal room, a light source is located at some point and the question is if there is some point not illuminated by this source. Chaika and Hubert [5] recently studied a related problem for circles of light, rather than points, showing that dilated circles around a fixed point weakly equidistribute outside a set of density zero (in their terminology, this phenomenon is called *weak illumination*). Motivated by their work, we prove weak and strong equidistribution results for a dilated family of measures on a nilmanifold.

The study of equidistribution results on a nilmanifold originates in the work of Green [1], where he showed that a flow is either equidistributed or there is a nontrivial obstruction to this flow arising from horizontal character on the nilmanifold. More precise distributional results were obtained by Shah [12], who described the limiting behavior for a polynomial series of iterates in a flow, and by Leibman [6], who proved convergence results for polynomial sequences of iterates. The asymptotic behavior of dilates of a measure supported on a curve in a nilmanifold was studied by Björklund and Fish [4], and among other

results we answer a conjecture of theirs about the general behavior of such dilates (this result is given in Theorem 1.2).

For example, we consider a family of measures that are linear expansions on the Lie algebra $\mathfrak{g}$ associated to some nilpotent Lie group G. A sample result is necessary and sufficient conditions that this family equidistributes as it dilates, and whether or not these equidistribution results for all sufficiently large dilates or only for all dilates outside a set of density zero depends on the derivative of the dilation. The simplest case of our results is for a torus, where such strong equidistribution results for a curve are implicit in the literature (though we are unaware of an explicit result like this). For a nilmanifold X, if we consider a continuous curve with the induced measure on this curve, it is easy to check that the dilates equidistribute weakly so long as the curve contains no linear segment of positive length (this corresponds to a nontrivial horizontal character picking up positive measure). More general equidistribution results on a nilmanifold, covering a broader class of dilations, require significantly more work and these characterizations are our main focus.

To give the precise formulations of our results in Sections 1.6, 1.7, and 1.8, we start by defining the objects that give us sufficient (and in some cases necessary) conditions for equidistribution.

1.2. **Equidistribution and weak equidistribution.** Let $X = G/\Gamma$ be a compact nilmanifold, meaning that G is a simply connected nilpotent Lie group and $\Gamma \subset G$ is a cocompact closed subgroup of G. The group G acts on $X = G/\Gamma$ by left translation, and there is a unique G-invariant Borel probability measure μ on X (the Haar measure).

A family of probability measures $(\mu_t)_{t \geq 0}$ on X is *equidistributed on* X if μ_t converges in the weak* topology as $t \to \infty$ to the Haar measure μ of X. Letting $C(X)$ denote the space of continuous functions on X, this means that the family of measures $(\mu_t)_{t \geq 0}$ is equidistributed if for all $f \in C(X)$,

$$\lim_{t \to \infty} \int_X f \, d\mu_t = \int_X f \, d\mu. \qquad (1.1)$$

Throughout, we let λ denote the Lebesgue measure. The family of measures $(\mu_t)_{t \geq 0}$ is said to be *weakly equidistributed on* X if the convergence in (1.1) holds for t in a set of asymptotic density 1 in $[0, \infty)$, meaning there exists $A \subset [0, \infty)$ such that

$$\lim_{T \to \infty} \frac{\lambda(A \cap [0, T])}{T} = 1$$

and for all $f \in C(X)$,

$$\lim_{\substack{t \to \infty \\ t \in A}} \int_X f \, d\mu_t = \int_X f \, d\mu.$$

If we consider discrete time, then the family $(\mu_n)_{n \in \mathbb{N}}$ of measures is weakly equidistributed on X if the convergence in (1.1) holds for a set of parameters n with asymptotic density 1, meaning along a set $A \subset \mathbb{N}$ satisfying $\lim_{N \to \infty} \frac{|A \cap \{1,\dots,N\}|}{N} = 1$, where $|\cdot|$ denotes the cardinality of the set.

Equivalently, $(\mu_t)_{t \geq 0}$ is weakly equidistributed if and only if for every $f \in C(X)$ with $\int_X f \, d\mu = 0$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \int_X f \, d\mu_t \right|^2 dt = 0, \tag{1.2}$$

and for discrete iterates $(\mu_n)_{n \in \mathbb{N}}$, this becomes

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^N \left| \int_X f \, d\mu_n \right|^2 = 0. \tag{1.3}$$

While it is clear that strong equidistribution implies weak equidistribution, for a family of measures in a nilmanifold these two notions are not equivalent. Various examples illustrating the difference are given in Section 5.

1.3. **Dilation associated to the Lie algebra.** We note that $\mathrm{Ad}(\Gamma)$ is a Zariski dense subgroup of $\mathrm{Ad}(G)$, see [10, Chapter 2]. Since Γ normalizes its connected component of the identity, say Γ^0, we have that Γ^0 is a normal Lie subgroup of G. We note that Γ^0 acts trivially on G/Γ, so G acts on $X = G/\Gamma$ via G/Γ^0, or we can say that G/Γ^0 acts on $X = G/\Gamma$ as follows:

$$(g\Gamma^0) \cdot x = gx, \text{ for all } g \in G \text{ and } x \in X.$$

Let $\mathfrak{g}$ denote the Lie algebra associated to the nilpotent Lie group G. Then $\mathfrak{g}/\mathrm{Lie}(\Gamma^0)$ is the Lie algebra associated to G/Γ^0. We define a family of *dilations* $(\rho_t)_{t \in \mathbb{R}}$ such that $\rho_t \colon \mathfrak{g} \to \mathfrak{g}/\mathrm{Lie}(\Gamma^0)$ is a linear transformation, and each matrix entry of ρ_t with respect to any bases of $\mathfrak{g}$ and $\mathfrak{g}/\mathrm{Lie}(\Gamma^0)$ is a polynomial in t. In other words, for some $m \geq 0$ and for some linear maps $B_j : \mathfrak{g} \to \mathfrak{g}/\mathrm{Lie}(\Gamma^0)$, $0 \leq j \leq m$, we have $\rho_t = \sum_{j=0}^m t^j B_j$ for all $t \in \mathbb{R}$.

1.4. **Dynamics of measures under dilations.** Let ν be a probability measure on $\mathfrak{g}$ and let $x_0 \in G/\Gamma$. For $t \geq 1$, let μ_{ν,x_0,ρ_t} denote the measure on X defined as follows: for any $f \in C(X)$,

$$\int_X f \, d\mu_{\nu,x_0,\rho_t} = \int_X f(\exp \circ \rho_t(y) \cdot x_0) \, d\nu(y), \qquad (1.4)$$

where $\exp : \mathfrak{g}/\operatorname{Lie}(\Gamma^0) \to G/\Gamma^0$ is the exponential map from Lie algebra to Lie group.

An interesting example of ν is as follows: Given a measurable map $\phi \colon (0,1) \to \mathfrak{g}$, we can take ν to be the pushforward of the Lebesgue measure λ restricted to $(0,1)$ under ϕ. Then for $t \geq 0$, taking μ_{ϕ,x_0,ρ_t} to be the measure on X defined by

$$\int_X f \, d\mu_{\phi,x_0,\rho_t} = \int_0^1 f(\exp \circ \rho_t \circ \phi(u) \cdot x_0) \, d\lambda(u) \qquad (1.5)$$

for all $f \in C(X)$, then $\mu_{\phi,x_0,\rho_t} = \mu_{\nu,x_0,\rho_t}$.

1.5. **Abelianization of a nilsystem.** Let $[G,G]$ denote the Lie subgroup of G corresponding to the commutator algebra $[\mathfrak{g},\mathfrak{g}]$. For a cocompact, closed subgroup Γ of G, we have that $[G,G]\Gamma$ is a closed subgroup of G, and $\bar{X} := G/[G,G]\Gamma$ can be identified with a compact torus $\mathbb{R}^m/\mathbb{Z}^m$ for some $m \geq 1$ (see for example [10, Corollary 1 of Theorem 2.3]). We call $\bar{X}$ the *abelianization* of X and use $\mathfrak{t} \cong \mathbb{R}^m$ to denote the associated Lie algebra of $\bar{X}$ and $q \colon X = G/\Gamma \to \bar{X} = G/[G,G]\Gamma$ to denote the natural quotient map.

Let $dq \colon \mathfrak{g}/\operatorname{Lie}(\Gamma^0) \to \mathfrak{t}$ denote the differential of the map q at the identity coset. Then

$$q(\exp(y) \cdot x_0) = \exp \circ dq(y) + q(x_0) \qquad (1.6)$$

for all $y \in \mathfrak{g}/\operatorname{Lie}(\Gamma^0)$ and all $x_0 \in X$.

Then $dq \circ \rho_t \colon \mathfrak{g} \to \mathfrak{t}$ can be expressed as

$$dq \circ \rho_t = \sum_{i=0}^{d_1} t^i A_i, \qquad (1.7)$$

where $d_1 \in \mathbb{N}$ and each $A_i \colon \mathfrak{g} \to \mathfrak{t}$ is a linear map.

When Γ is discrete; that is, $\Gamma^0 = \{e\}$, a natural choice for a family of dilations is to consider $(\rho_t)_{t \in \mathbb{R}}$, where $\rho_t v = tv$ for all $v \in \mathfrak{g}$. In this case, $d_1 = 1$, $A_1 = dq$, and $A_0 = 0$.

Let $\bar{X}^*$ denote the space of (continuous) unitary characters on the torus $\bar{X}$. Then elements of $\bar{X}^*$ are in one-to-one correspondence with unitary characters on G such that the kernel of the map contains Γ.

For any $\chi \in \bar{X}^*$, let $d\chi \colon \mathfrak{t} \to \mathbb{R}$ denote the differential of χ, meaning that for $y \in \mathfrak{t}$,

$$\chi(y + \mathbb{Z}^m) = e^{2\pi i d\chi(y)}. \tag{1.8}$$

1.6. **Weak equidistribution of dilated measures.** Our first result provides necessary and sufficient conditions for weak equidistribution of the family $(\mu_t := \mu_{\nu,x_0,\rho_t})_{t \geq 0}$.

Theorem 1.1. *Let ν be a probability measure on $\mathfrak{g}$, $x_0 \in X$, and $(\mu_t := \mu_{\nu,x_0,\rho_t})_{t \geq 0}$ be the family of measures defined in (1.4). If for all $\chi \in \bar{X}^* \setminus \{1\}$ and $(z_1, \ldots, z_{d_1}) \in \mathbb{R}^{d_1}$ we have*

$$\nu(\{v \in \mathfrak{g} \colon d\chi(A_i v) = z_i \text{ for all } 1 \leq i \leq d_1\}) = 0, \tag{1.9}$$

then the families of measure $(\mu_t)_{t \geq 0}$ and $(\mu_n)_{n \in \mathbb{N}}$ are weakly equidistributed on X.

Furthermore, if $A_0 = 0$, then condition (1.9) is necessary for weak equidistribution of $(\mu_n)_{n \in \mathbb{N}}$, and also for $(\mu_t)_{t \geq 0}$.

1.7. **Equidistribution of dilated analytic curves.** We maintain the same notation in this section. Carrying out the constructions of the family of measures in (1.5) with an analytic map $\phi \colon (0, 1) \to \mathfrak{g}$, we resolve a conjecture stated in Björklund and Fish [4] (see the discussion following Theorem 7 in their paper):

Theorem 1.2. *Assume $\phi \colon (0, 1) \to \mathfrak{g}$ is analytic, let $x_0 \in X$, and let $(\mu_t = \mu_{\phi,x_0,\rho_t})_{t \in \mathbb{R}}$ be as defined in (1.5). If for every $\chi \in \bar{X}^* \setminus \{1\}$, there exists $1 \leq i \leq d_1$ such that the map*

$$u \mapsto d\chi(A_i \phi(u)) \tag{1.10}$$

is not constant, then $(\mu_t)_{t \geq 0}$ is equidistributed on X.

Furthermore, if $A_0 = 0$, then the conditions on the maps given in (1.10) are also necessary for the equidistribution of $(\mu_t)_{t \geq 0}$ on X.

1.8. **Equidistribution of dilated differentiable curves.** Our main result of this article provides a condition on tangents to the curve for (strong) equidistribution, when ϕ is a sufficiently differentiable curve. Still maintaining the assumptions and notation stated at the beginning of this section, we have:

Theorem 1.3. *Assume that $X = G/\Gamma$ is a compact nilmanifold. There exists a natural number D, which can be expressed in terms of the degrees of polynomials in t defining ρ_t such that the following holds: suppose that $\phi^{(D)}(u)$ exists for (Lebesgue) almost all $u \in (0, 1)$ and that for every $\chi \in \bar{X}^* \setminus \{1\}$ and (Lebesgue) almost every $u \in (0, 1)$,*

$$A_i \phi^{(1)}(u) \notin \ker d\chi \tag{1.11}$$

for some $1 \leq i \leq d_1$. Then the family of measures $(\mu_{\phi,x_0,\rho_t})_{t\geq 0}$ is equidistributed on G/Γ.

Our proof is based on a stronger equidistribution result for dilates of curves that shrink (the precise statement is given in Theorem 3.1).

The following special case is of interest:

Corollary 1.4. *Let $X = \mathbb{R}^n/\mathbb{Z}^n$ and $\phi: (0,1) \to \mathbb{R}^n$ be an almost everywhere differentiable map. Suppose that for each $\mathbf{v} \in \mathbb{Z}^{\mathbf{n}} \setminus \{\mathbf{0}\}$, $\phi^{(1)}(u)$ is not orthogonal to $\mathbf{v}$ for (Lebesgue) almost every $u \in (0,1)$. Then for any $f \in C(\mathbb{R}^n/\mathbb{Z}^n)$ and $x_0 \in \mathbb{R}^n/\mathbb{Z}^n$,*

$$\lim_{t\to\infty} \int_0^1 f(t\phi(s) + x_0)\, d\lambda(s) = \int_{\mathbb{R}^m/\mathbb{Z}^m} f\, d\mu,$$

where λ denotes the Lebesgue measure on $[0,1]$ and μ the Haar measure on $\mathbb{R}^n/\mathbb{Z}^n$.

For analytic ϕ, the conclusion of Corollary 1.4 was obtained in Randol [9].

In Section 5 we provide examples where condition in (1.9) holds and so $(\mu_t)_{t\geq 0}$ is weakly equidistributed on X, but $(\mu_t)_{t\geq 0}$ is not equidistributed on X.

2. Weak equidistribution for measures

This section is devoted to proving Theorem 1.1. We give a detailed proof for the discrete case, and then indicate the modifications needed for the continuous setting.

The classic equidistribution theorem in the discrete setting is due to Weyl:

Theorem 2.1 (Weyl [13])**.** *Let $p(t) = a_d t^d + \cdots + a_1 t + a_0$ be a real polynomial. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i p(n)} = 0$$

if and only if at least one of $a_1, \ldots, a_d$ is not rational.

If G is a nilpotent Lie group, $a_1, \ldots, a_m \in G$, and $p_1, \ldots, p_m \colon \mathbb{N} \to \mathbb{N}$ are polynomials taking integer values on the integers, then a sequence $(g(n))_{n\in\mathbb{N}}$ in G of the form $a_1^{p_1(n)} a_2^{p_2(n)} \ldots a_m^{p_m(n)}$ is a *polynomial*. If $\mathfrak{g}$ denotes the Lie algebra of G, we say that $\zeta \colon \mathbb{N} \to \mathfrak{g}$ is a *polynomial map* if $\exp(\zeta(n)) = g(n)$ for some polynomial $(g(n))_{n\in\mathbb{N}}$. Generalizing Weyl's equidistribution result, Leibman [6] showed (we write his result in terms of the Lie algebra):

Theorem 2.2 (Leibman [6])**.** *Let* $\zeta \colon \mathbb{N} \to \mathfrak{g}$ *be a polynomial map such that for every nontrivial character* χ *on* $\bar{X}$,

$$d\chi \circ dq \circ \zeta(n) \not\equiv d\chi \circ dq \circ \zeta(0) \quad \mathrm{mod}\ \mathbb{Q}\ \textit{for some } n \in \mathbb{N}. \qquad (2.1)$$

Then for any $f \in C(X)$ *and any* $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\exp \circ \zeta(n) \cdot x) = \int_{X} f\, d\mu.$$

We make use of these results in the proof of the weak equidistribution result:

Proof of Theorem 1.1 (in the discrete setting). First we prove the necessity of condition (1.9) for weak equidistribution. Assume that $A_0 = 0$ and suppose that (1.9) fails to hold, and we want to prove that the families $(\mu_t)_{t \geq 0}$ and $(\mu_n)_{n \in \mathbb{N}}$ are both not weakly equidistributed. Thus there exist a nontrivial character χ on $\bar{X}$ and $z_1, \ldots, z_{d_1} \in \mathbb{R}$ such that

$$\nu(\{v \in \mathfrak{g} \colon d\chi(A_j v) = z_j \text{ for all } 1 \leq j \leq d_1\}) \qquad (2.2)$$

is positive. Since χ is a nontrivial character on the torus $\bar{X}$ and its pushforward $q_* \mu$ is the translation invariant (Haar) probability measure on the compact torus $\bar{X}$, we have

$$\int_{X} \chi \circ q\, d\mu = \int_{\bar{X}} \chi\, d(q_* \mu) = 0. \qquad (2.3)$$

Therefore in view of (1.3), in order prove that $(\mu_n)_{n \in \mathbb{N}}$ is not weakly equidistributed, we need to show that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_{X} \chi \circ q\, d\mu_n \right|^2 > 0.$$

Therefore it suffices to prove that

$$\lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_{X} \chi \circ q\, d\mu_{M!n} \right|^2 > 0, \qquad (2.4)$$

because for any sequence $\{a_n\}$ of non-negative reals, and $M \in \mathbb{N}$,

$$\frac{1}{M!N} \sum_{n=1}^{M!N} a_n \geq \frac{1}{M!} \frac{1}{N} \sum_{n=1}^{N} a_{M!n}.$$

For all $t \in \mathbb{R}$,

$$
\begin{aligned}
\int_X \chi \circ q \, d\mu_t &= \int_{\mathfrak{g}} \chi(q(\exp \circ \rho_t(y) \cdot x_0)) \, d\nu(y) \\
&= \chi(q(x_0)) \int_{\mathfrak{g}} \chi(dq \circ \rho_t(y) + \mathbb{Z}^m) \, d\nu(y) \\
&= \chi(q(x_0)) \int_{\mathfrak{g}} e^{2\pi i d\chi(dq \circ \rho_t(y))} \, d\nu(y) \\
&= \chi(q(x_0)) \int_{\mathfrak{g}} e^{2\pi i \sum_{j=0}^{d_1} t^j \cdot d\chi(A_j y)} \, d\nu(y),
\end{aligned}
$$

where we have used equations (1.6), (1.8), and (1.7). Writing $y_j := d\chi(A_j y)$ and using the assumption that $A_0 = 0$, this last quantity is the same as

$$
\chi(q(x_0)) \int_{\mathfrak{g}} e^{2\pi i \sum_{j=1}^{d_1} t^j \cdot y_j} \, d\nu(y).
$$

Combining this expression for $\int_X \chi \circ q \, d\mu_t$ with the fact that $|\chi(q(x_0))| = 1$, we have that

$$
\begin{aligned}
&\lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X \chi \circ q \, d\mu_{M!n} \right|^2 \\
&= \lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_{\mathfrak{g} \times \mathfrak{g}} e^{2\pi i (nM!)^j \sum_{j=1}^{d_1} (y_j - y_j')} \, d(\nu \times \nu)(y, y') \\
&= \int_{\mathfrak{g} \times \mathfrak{g}} \left[\lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \sum_{j=1}^{d_1} n^j ((M!)^j (y_j - y_j'))} \right] d(\nu \times \nu)(y, y').
\end{aligned}
$$

We claim that this integral over $\mathfrak{g} \times \mathfrak{g}$ is the same as integrating the indicator function of the set

$$
W = \{(y, y') \in \mathfrak{g} \times \mathfrak{g} : y_j \equiv y_j' \mod \mathbb{Q} \text{ for all } 1 \le j \le d_1\}.
$$

To see this, fix $y, y' \in \mathfrak{g}$. If $y_j \ne y_j' \mod \mathbb{Q}$ for some $1 \le j \le d_1$, by Weyl Equidistribution (Theorem 2.1) the limit equals to 0. If $y_j = y_j' \mod \mathbb{Q}$ for all $1 \le j \le d_1$, then $(M!)^j y_j = (M!)^j y_j' \mod \mathbb{Z}$ for all $1 \le j \le d_1$ and all M sufficiently large, and so the limit as $N \to \infty$ equals to 1.

Thus

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X \chi \circ q \, d\mu_n \right|^2$$

$$= \int_{\mathfrak{g} \times \mathfrak{g}} \mathbf{1}_W \, d(\nu \times \nu)(y, y')$$

$$= \int_{\mathfrak{g}} \nu(\{y \in \mathfrak{g} \colon d\chi(A_j y) \equiv y'_j \mod \mathbb{Q} \text{ for all } 1 \le j \le d_1\}) \, d\nu(y')$$

$$= \sum_{w_1, \ldots, w_{d_1} \in \mathbb{R}} \nu(\{y \in \mathfrak{g} \colon d\chi(A_j y) \equiv w_j \mod \mathbb{Q} \text{ for all } 1 \le j \le d_1\}) \times$$

$$\nu(\{y' \in \mathfrak{g} \colon d\chi(A_j y') = w_j \text{ for all } 1 \le j \le d_1\})$$

$$\ge \nu(\{v \in \mathfrak{g} \colon d\chi(A_j v) = z_j \text{ for all } 1 \le j \le d_1\})^2.$$

By (2.2), this is strictly positive and so (2.4) holds, which completes the proof that $(\mu_n)_{n \in \mathbb{N}}$ is not weakly equidistributed on X.

The converse implication. Assume that condition (1.9) of Theorem 1.1 holds. Our goal is to show that (1.3) holds for the family $(\mu_n := \mu_{\nu, x_0, \rho_n})_{n \in \mathbb{N}}$ and any given $f \in C(X)$ with $\int_X f \, d\mu = 0$. We have:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X f \, d\mu_n \right|^2$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[\int_{X \times X} f \otimes \overline{f} \, d(\mu_n \times \mu_n) \right]$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[\int_{\mathfrak{g} \times \mathfrak{g}} f(\exp \circ \rho_n(y) \cdot x_0) \overline{f(\exp \circ \rho_n(y') \cdot x_0)} \, d(\nu \times \nu)(y, y') \right]$$

$$= \int_{\mathfrak{g} \times \mathfrak{g}} \left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \otimes \overline{f} (\exp \circ \zeta_{y, y'}(n) \cdot (x_0, x_0)) \right] d(\nu \times \nu)(y, y'),$$

$$(2.5)$$

where $\zeta_{y, y'} \colon \mathbb{N} \to \mathfrak{g} / \operatorname{Lie}(\Gamma^0) \times \mathfrak{g} / \operatorname{Lie}(\Gamma^0)$ is the map given by

$$\zeta_{y, y'}(t) = (\rho_t(y), \rho_t(y')) \quad \text{for all } t \in \mathbb{N}. \qquad (2.6)$$

From the definition of $(\rho_t)_{t \in \mathbb{R}}$, it follows that $(\exp \circ \zeta_{y, y'}(n))_{n \in \mathbb{N}}$ is a polynomial on $(G / \Gamma^0) \times (G / \Gamma^0)$.

For every $(\chi, \chi') \in \bar{X}^* \times \bar{X}^* \setminus \{(1,1)\}$, we have $d(\chi \otimes \bar{\chi}') = d\chi \oplus d\bar{\chi}'$, and by (1.7),

$$
\begin{aligned}
(d\chi \oplus d\bar{\chi}') &\circ \zeta_{y,y'}(n) \\
&= d\chi(dq \circ \rho_n(y)) - d\chi'(dq \circ \rho_n(y')) \\
&= \sum_{j=1}^{d_1} [d\chi(A_j y) - d\chi'(A_j y')] n^j + d\chi(A_0 y) - d\chi'(A_0 y').
\end{aligned}
$$

Therefore in view of condition (2.1) of Theorem 2.2,

$$(d\chi \oplus d\bar{\chi}') \circ \zeta_{y,y'}(n) \equiv d\chi(A_0 y) - d\chi'(A_0 y') \mod \mathbb{Q}$$

for all $n \in \mathbb{Z}$ is equivalent to the condition that

$$d\chi(A_j y) \equiv d\chi'(A_j y') \mod \mathbb{Q}$$

for all $1 \le j \le d_1$.

In order to apply Theorem 2.2 to $\zeta_{y,y'}$ and $X \times X$, define B to be the set of $(y, y') \in \mathfrak{g} \times \mathfrak{g}$ such that for all $(\chi, \chi') \in \bar{X}^* \times \bar{X}^* \setminus \{(1,1)\}$, there exists $j \in \{1, \ldots, d_1\}$ such that $d\chi(A_j y) \not\equiv d\chi(A_j y') \mod \mathbb{Q}$. Define C to be the set of $y \in \mathfrak{g}$ such that for all $\chi \in \bar{X}^* \setminus \{1\}$, there exists $j \in \{1, \ldots, d_1\}$ such that $d\chi(A_j y) \notin \mathbb{Q}$ and define

$$E_y = \{y' \in \mathfrak{g} \colon (y, y') \notin B\}.$$

Since $\mathbb{Q}$ is countable, by the hypothesis given in (1.9), we have that $\nu(C) = 1$. Also

$$B \supset \bigcup_{y \in C} \{y\} \times (\mathfrak{g} \setminus E_y). \tag{2.7}$$

For any $y \in C$, $\chi \in \bar{X}^*$, $\chi' \in \bar{X}^* \setminus \{1\}$, define $E_{y,\chi,\chi'}$ to be the set

$$\{y' \in \mathfrak{g} \colon d\chi'(A_j y') \equiv d\chi(A_j y) \mod \mathbb{Q} \text{ for all } 1 \le j \le d_1\}.$$

Then since $\mathbb{Q}$ is countable, by the hypothesis (1.9), we have that

$$\nu(E_{y,\chi,\chi'}) = 0. \tag{2.8}$$

Suppose that $y \in C$, $y' \in E_y$, and $(\chi, \chi') \in \bar{X}^* \times \bar{X}^* \setminus \{(1,1)\}$ are such that

$$d\chi(A_j y) \equiv d\chi'(A_j y') \mod \mathbb{Q} \text{ for all } 1 \le j \le d_1. \tag{2.9}$$

By definition of the set C, we can choose j with $1 \le j \le d_1$ such that $d\chi(A_j y) \notin \mathbb{Q}$. Then by (2.9), $\chi'(A_j y') \notin \mathbb{Q}$. Therefore $\chi' \neq 1$. By definition of the set E_y,

$$E_y = \bigcup_{\chi' \in \bar{X}^* \setminus \{1\}} \bigcup_{\chi \in \bar{X}^*} E_{y,\chi,\chi'}. \tag{2.10}$$

Since $\bar{X}^*$ is countable, by (2.8), we have $\nu(E_y) = 0$. Hence by (2.7), $(\nu \times \nu)(B) = 1$.

By Theorem 2.2 applied to $X \times X$, for any $f \in C(X)$ with $\int_X f \, d\mu = 0$, for all $(y, y') \in B$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \otimes \bar{f}(\exp \circ \zeta_{y,y'}(n) \cdot (x_0, x_0)) = 0.$$

Now since $(\nu \times \nu)((\mathfrak{g} \times \mathfrak{g}) \setminus B) = 0$, it follows from (2.5) that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_{\mathfrak{g}} f(\exp \circ \rho_n(y) \cdot x_0) \, d\nu(y) \right|^2$$

$$= \int_{\mathfrak{g} \times \mathfrak{g}} \left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \otimes \bar{f}(\exp \circ \zeta_{y,y'}(n) \cdot (x_0, x_0)) \right] d(\nu \times \nu)(y, y') = 0.$$

Thus $(\mu_n)_{n \in \mathbb{N}}$ is weakly equidistributed. $\square$

The case of continuous parameter. The proof in this case is similar. Instead of using the discrete version of Weyl's Theorem to show necessity, we use:

Theorem 2.3 (Weyl [13])**.** *Let $p(t) = a_d t^d + \cdots + a_1 t + a_0$ be a polynomial with real coefficients. Then*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{2\pi i p(t)} \, dt = 0$$

if and only if at at least one of $a_1, \ldots, a_d$ is nonzero.

We also replace the use of Leibman's Theorem by the following result of Shah (which is generalizes Theorem 2.3):

Theorem 2.4 (Shah [12])**.** *Let $\zeta \colon \mathbb{R} \to \mathfrak{g}$ be a polynomial map such that for every nontrivial character χ on $\bar{X}$,*

$$d\chi \circ dq \circ \zeta(t) \neq d\chi \circ dq \circ \zeta(0) \text{ for some } t \in \mathbb{R}. \qquad (2.11)$$

Then for any $f \in C(X)$ and any $x \in X$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\exp \circ \zeta(t) \cdot x) \, dt = \int_X f \, d\mu.$$

Up to obvious changes in notation, the proof is then the same as in the discrete case; in fact it gets simplified as one can replace equivalence mod $\mathbb{Q}$ by just equality, and we omit further details.

3. Strong equidistribution

3.1. A stronger equidistribution result for dilations of curves.
The key ingredient for proving Theorem 1.3 is a stronger equidistribution of dilation of shrinking curves as stated below. First we need to recall the notation.

We continue to use λ to denote the Lebesgue measure on $(0,1)$ or $\mathbb{R}$, depending on the context.

We consider the dynamics on $X = G/\Gamma$, where G is a simply connected nilpotent Lie group and Γ a closed subgroup of G. Let $\mathfrak{g}$ denote the Lie algebra of G.

We write $\bar{\mathfrak{g}} = \mathfrak{g}/\operatorname{Lie}(\Gamma^0)$. For $t \in \mathbb{R}$, let $\rho_t : \mathfrak{g} \to \bar{\mathfrak{g}}$ denote the linear map such that ρ_t is a polynomial in t with coefficients which are linear maps from $\mathfrak{g}$ to $\bar{\mathfrak{g}}$.

Let $\mathfrak{t}$ denote the Lie algebra associated to the compact torus $\bar{X} = G/[G,G]\Gamma$. Let $dq : \bar{\mathfrak{g}} \mapsto \mathfrak{t} = \bar{\mathfrak{g}}/[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}]$ denote the natural quotient map. Then $dq \circ \rho_t : \mathfrak{g} \to \mathfrak{t}$ can be expressed as

$$dq \circ \rho_t = \sum_{i=0}^{d_1} t^i A_i,$$

where each $A_i : \mathfrak{g} \to \mathfrak{t}$ is a linear map.

Consider the lower central series $\bar{\mathfrak{g}}^{(k)} = [\bar{\mathfrak{g}}, \bar{\mathfrak{g}}^{(k-1)}]$ for $k \geq 1$. Put $\bar{\mathfrak{g}}^{(0)} = \bar{\mathfrak{g}}$. Let $\kappa \geq 1$ be an integer such that $\bar{\mathfrak{g}}^{(\kappa)} = 0$.

For $1 \leq k \leq \kappa$, let $Q_k \colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^{(k)}$ denote the natural quotient map and choose an integer $D_k \geq 0$ such that the map $Q_k \circ \rho_t \colon \mathfrak{g} \to \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^{(k)}$ is a polynomial of degree at most D_k in t and $D_1 \leq D_2 \leq \cdots \leq D_\kappa$. Set

$$D = \max\left\{ \sum_{m=1}^{n} D_{k_m} : \sum_{m=1}^{n} k_m \leq \kappa, \ 1 \leq k_m \leq \kappa, \ 1 \leq n \leq \kappa \right\}. \quad (3.1)$$

The purpose of such a definition of D becomes apparent later.

Theorem 3.1. *Suppose that $\phi \colon (0,1) \to \mathfrak{g}$ is a function such that $\phi^{(D)}(u)$ exists for (Lebesgue) almost all $u \in (0,1)$. Assume further that for every nontrivial character χ on $\bar{X}$ and (Lebesgue) almost every $u \in (0,1)$,*

$$A_i \phi^{(1)}(u) \notin \ker d\chi \text{ for some } 1 \leq i \leq d_1. \quad (3.2)$$

Then there exists $W \subset (0,1)$ with $\lambda(W) = 1$ such that for all $s_0 \in W$ the following holds: Let $(\ell_t)_{t \geq 1}$ be a family such that $\ell_t > 0$, and $\ell_t \to \infty$ and $\ell_t t^{-1} \to 0$ as $t \to \infty$. Then

$$\lim_{t \to \infty} \frac{1}{\ell_t t^{-1}} \int_{s_0}^{s_0 + \ell_t t^{-1}} f(\exp \circ \rho_t \circ \phi(\xi) \cdot x_0) \, d\xi = \int_X f \, d\mu. \quad (3.3)$$

We defer the proof of this result to the next two sections. Before this, we explain its use in completing the proof Theorem 1.3 and obtain some of its immediate consequences.

Proof of Theorem 1.3. Let $f \in C(G/\Gamma)$ be such that $\int_{G/\Gamma} f \, d\mu = 0$, and without loss of generality we can assume that $\|f\|_\infty \leq 1$. Given $\varepsilon > 0$, choose $W \subset (0,1)$ with $\lambda(W) = 1$ as in Theorem 3.1. The conclusion of Theorem 3.1 can be interpreted as follows: for every $s \in W$, there exists $\ell_s \geq 1$ such that for any $\ell \geq \ell_s$, there exists $t_{s,\ell} > \ell(1-s)^{-1}$ such that for all $t \geq t_{s,\ell}$,

$$\left| \int_s^{s+\ell t^{-1}} f(\exp \circ \rho_t \circ \phi(\xi) \cdot x_0) \, d\xi \right| \leq \ell t^{-1} \varepsilon. \qquad (3.4)$$

For $L > 0$, let

$$W(L) = \{s \in W : \ell_s \leq L\}.$$

By our choice of ℓ_s, there exists $L > 0$ such that $\lambda(W \setminus W(L)) \leq \varepsilon$.
For $t \geq 1$, let

$$W(L,t) = \{s \in W(L) : t_{s,L} \leq t\}.$$

By our choice of $t_{s,L}$, there exists $T_L \geq 1$ such that $\lambda(W(L) \setminus W(L,T_L)) < \varepsilon$.

Pick a compact set $W_2 \subset W(L,T_L)$ such that $\lambda(W(L,T_L) \setminus W_2) \leq \varepsilon$.
Let $t \geq T_L$. We cover W_2 by disjoint intervals $[s_i, s_i + Lt^{-1}]$ for $0 \leq i \leq n$, where $s_0 = \min W_2$, and if $s_i \in W_2$ is chosen for some $i \geq 0$, then choose

$$s_{i+1} = \min\bigl(W_2 \setminus (0, s_i + Lt^{-1})\bigr) \in W_2, \text{ if } s_i + Lt^{-1} \leq \max(W_2),$$

otherwise put $n = i$ and stop the induction. Since $s_n \in W(L,T_L)$, and $t \geq T_L$, we have $s_n + Lt^{-1} < 1$. Therefore the intervals $[s_i, s_i + Lt^{-1})$ for $0 \leq i \leq n$ are disjoint and contained in $(0,1)$. Hence

$$(n+1)(Lt^{-1}) \leq 1.$$

Let $W_2' = \bigcup_{i=0}^n [s_i, s_i + Lt^{-1})$. Then $W_2 \subseteq W_2'$ and by (3.4),

$$\left| \int_{W_2'} f(\exp \circ \rho_t \circ \phi(s) \cdot x_0) \, ds \right| \leq \sum_{i=0}^n \left| \int_{s_i}^{s_i + Lt^{-1}} f(\exp \circ \rho_t \circ \phi(s) \cdot x_0) \, ds \right|$$

$$\leq (n+1)(Lt^{-1})\varepsilon \leq \varepsilon.$$

Since $\lambda(W_2') \geq \lambda(W_2) \geq 1 - 3\varepsilon$, for all $t \geq T_L$,

$$\left| \int_0^1 f(\exp \circ \rho_t \circ \phi(s) \cdot x_0) \, ds \right| \leq \varepsilon + 3\varepsilon \|f\|_\infty \leq 4\varepsilon. \qquad \square$$

The following special case of Theorem 1.3 is of interest:

Corollary 3.2. *Let $\kappa \geq 1$ be such that $\bar{\mathfrak{g}}^{(\kappa)} = 0$. Suppose that that for $1 \leq k \leq \kappa$,*

$$Q_k \circ \rho_t \colon \mathfrak{g} \to \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^{(k)} \text{ is a polynomial of degree at most } k \text{ in } t. \quad (3.5)$$

Let $\phi\colon (0,1) \to \mathfrak{g}$ be such that $\phi^{(\kappa)}(u)$ exists for almost all $u \in (0,1)$. Suppose that the condition on $\chi \in \bar{X}^* \setminus \{1\}$ given in (1.11) holds. Then the family of measures $(\mu_{\phi,x_0,\rho_t})_{t\geq 0}$ is equidistributed on X as $t \to \infty$.

Note that the example of ρ_t being multiplication by t on $\mathfrak{g}$ is an example satisfying (3.5) and generalizing Corollary 1.4 to nilpotent groups. Other natural examples of family of dilations $(\rho_t)_{t\in\mathbb{R}}$ satisfying the condition given in (3.5) are given in [2, 3, 8]. In particular, when ρ_t is a Lie algebra automorphism of $\mathfrak{g}$, then ρ_t satisfies (3.5).

Proof of Corollary 3.2. Given the assumption in (3.5), the bound D_k on the degree of the quotient map $Q_k \circ \rho_t$ satisfies $D_k = k$ for all $1 \leq k \leq \kappa$. Therefore in (3.1), we have

$$\sum_{m=1}^{n} D_{k_m} = \sum_{m=1}^{n} k_m \leq \kappa,$$

and hence $D \leq \kappa$. Thus the result is a special case of Theorem 1.3. $\quad\square$

We also complete the proof of Theorem 1.2:

Proof of Theorem 1.2. Since ϕ is an analytic curve, the condition on the χ given in (1.10) implies the condition given in (1.11). Therefore the equidistribution follows from Theorem 1.3.

For the converse, suppose that the condition on the map given in (1.10) fails to hold and $A_0 = 0$. Then there exists a nontrivial character χ on $\bar{X}$ such that $d\chi(A_i\phi^{(1)}(u)) = 0$ for all $u \in (0,1)$ and all $1 \leq i \leq d_1$. But then by analyticity of ϕ, for every $1 \leq i \leq d_1$, there exists $z_i \in \mathbb{R}$, such that $d\chi(A_i\phi(u)) = z_i$ for all $u \in (0,1)$. Hence, since $A_0 = 0$,

$$d\chi(dq \circ \rho_t \circ \phi(u)) = \sum_{i=1}^{d_1} t^i d\chi(A_i\phi(u)) = \sum_{i=1}^{d_1} t^i z_i. \quad (3.6)$$

Let $f = \chi \circ q \in C(X)$. Then

$$\int_X f \, d\mu = \int_{\bar{X}} \chi \, d(q_*\mu) = 0,$$

because χ is a nontrivial character on the torus $\bar{X}$ and $q_*\mu$ is the invariant probability measure on $\bar{X}$.

For any $x_0 \in X$, by (1.8),

$$f(\exp \circ \rho_t \circ \phi(u) \cdot x_0) = e^{2\pi i d\chi(dq\circ\rho_t\circ\phi(u))} \cdot \chi(q(x_0)). \quad (3.7)$$

If $\lim_{t\to\infty} \int f \, d\mu_t$ exists, then by (3.6) and (3.7) $\lim_{t\to\infty} e^{2\pi i \sum_{i=1}^{d_1} t^i z_i}$ exists. We conclude that $z_i = 0$ for all i. It follows that the limit is equal to $\chi(q(x_0)) \neq 0$. Therefore the sequence μ_t does not converge to μ as $t \to \infty$ with respect to the weak* topology. $\square$

3.2. **Equidistribution of expanding translates of shrinking curves.**

First we describe a quantitative condition that we want for $s_0 \in (0,1)$ to hold, so that (3.3) holds.

Assume that $\phi \colon (0,1) \to \mathfrak{g}$ is a function satisfying the hypotheses of Theorem 3.1. That is, $\phi^{(D)}(u)$ exists for almost all $u \in (0,1)$, where D is defined as in (3.1), and for every $\chi \in \bar{X}^* \setminus \{1\}$, for almost all $u \in (0,1)$, there exists $1 \leq i \leq d_1$, such that $A_i \phi^{(1)}(u) \notin \ker d\chi$.

Define a map $\psi_t \colon \{(u,\xi) \colon u + \xi, u \in (0,1)\} \to \mathfrak{g}/\operatorname{Lie}(\Gamma^0)$ by

$$\psi_t(u,\xi) = \log[\exp(\rho_t \circ \phi(u + \xi)) \cdot \exp(-\rho_t \circ \phi(u))], \qquad (3.8)$$

where $\exp : \mathfrak{g}/\operatorname{Lie}(\Gamma^0) \to G/\Gamma^0$ is the exponential map, and $\log$ is its inverse.

Definition 3.3 (Property $\mathcal{W}(\Gamma^0)$). We say that $s_0 \in (0,1)$ has property $\mathcal{W}(\Gamma^0)$ if the following holds: there exists $\alpha \geq 1$ such that for any sequences $t_n \to \infty$ and $\ell_n \to \infty$ with $\ell_n t_n^{-1} \to 0$ and for any $\epsilon > 0$, there exists a compact set $I_\epsilon \subset [0,1]$ with $\lambda([0,1] \setminus I_\epsilon) < \epsilon$ such that the following holds: for any $s \in I_\epsilon$ and $\zeta \in \mathbb{R}$, for each $n \in \mathbb{N}$, if we put $u_n = s_0 + s\ell_n t_n^{-1}$ and $\xi_n = \zeta t_n^{-\alpha}$, then

$$\lim_{n\to\infty} \psi_{t_n}(u_n, \xi_n) = \eta(\zeta), \qquad (3.9)$$

where $\eta : \mathbb{R} \to \mathfrak{g}/\operatorname{Lie}(\Gamma^0)$ is a non-constant polynomial map.

If Λ is any closed subgroup of G containing Γ, then let $dp : \bar{\mathfrak{g}} = \mathfrak{g}/\operatorname{Lie}(\Gamma^0) \to \mathfrak{g}/\operatorname{Lie}(\Lambda^0)$ be the natural quotient map. Then we replace the dilations ρ_t with $dp \circ \rho_t : \mathfrak{g} \to \mathfrak{g}/\operatorname{Lie}(\Lambda^0)$. With these modification, property $\mathcal{W}(\Lambda^0)$ is also defined.

Proposition 3.4. *Suppose that $s_0 \in (0,1)$ has property $\mathcal{W}(\Lambda^0)$ for every closed subgroup Λ of G containing Γ. For all $t \geq 1$, let $\ell_t > 0$ be such that as $t \to \infty$, $\ell_t \to \infty$ and $\ell_t t^{-1} \to 0$. For all $t \geq 1$ such that $s_0 + \ell_t t^{-1} < 1$, let ν_t be the probability measure on X such that all $f \in C(X)$,*

$$\int_X f \, d\nu_t = \frac{1}{\ell_t t^{-1}} \int_{s_0}^{s_0 + \ell_t t^{-1}} f(\exp(\rho_t \circ \phi(\xi) \cdot x_0) \, d\xi$$

$$= \int_0^1 f(\exp \circ \rho_t \circ \phi(s_0 + s\ell_t t^{-1}) \cdot x_0) \, ds.$$

Then $\nu_t \to \mu$ with respect to the weak topology as $t \to \infty$.*

Proof. The result is trivial if $\dim(G/\Gamma^0) = 0$. We intend to prove the result by induction on $\dim(G/\Gamma^0)$. In particular, we can assume that the result is valid for all closed subgroups Λ of G containing Γ such that $\dim(\Lambda^0) > \dim(\Gamma^0)$. In other words, if $p\colon G/\Gamma \to G/\Lambda$ is the natural quotient map, and p_* denotes the corresponding pushforward of measures then as $t \to \infty$, $p_*(\nu_t)$ converges to the G-invariant probability measure on G/Λ with respect to the weak* topology.

Let $t_n \to \infty$ be any given sequence. It suffices to show that after passing to a subsequence of $\{t_n\}$, we have that $\nu_{t_n} \to \mu$ in the weak* topology as $n \to \infty$, where μ denotes the G-invariant probability measure on G/Γ.

Since G/Γ is compact, after passing to a subsequence, we may assume that $\nu_{t_n} \to \nu$ in the space of probability measures on G/Γ with respect to the weak* topology as $n \to \infty$.

Claim 3.4.1. *The measure ν is invariant under a connected subgroup U of G properly containing Γ^0.*

Proof. Let $f \in C(G/\Gamma)$. Since s_0 has property $\mathcal{W}(\Gamma^0)$, let $\alpha \geq 1$ be such that (3.9) holds. For the remainder of the proof, we make use of some shorthand to simplify formulas. For all $n \in \mathbb{N}$, we write $t = t_n$, for all $s \in [0,1]$, we write $u = s_0 + s\ell_t t^{-1}$, and for all $\zeta \in \mathbb{R}$, we write $\xi = \zeta t^{-\alpha}$. Then, since $\nu_{t_n} \to \nu$ as $n \to \infty$, by (3.3) we have

$$\int_{G/\Gamma} f(x)\, d\nu(x) = \lim_{n\to\infty} \int_0^1 f(\exp \circ \rho_{t_n} \circ \phi(u) \cdot x_0)\, ds. \qquad (3.10)$$

Now $\xi \ell_t^{-1} t = \zeta \ell_t^{-1} t^{-(\alpha-1)} \to 0$, because $\alpha \geq 1$ and $\ell_t \to \infty$ as $n \to \infty$. Therefore (3.10) can be rewritten as

$$\lim_{n\to\infty} \int_{I_\varepsilon + \xi\ell_t^{-1}t} f(\exp \circ \rho_t \circ \phi(u) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon)$$

$$= \lim_{n\to\infty} \int_{I_\varepsilon + \xi\ell_t^{-1}t} f(\exp \circ \rho_t \circ \phi(s_0 + s\ell_t t^{-1}) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon)$$

$$= \lim_{n\to\infty} \int_{I_\varepsilon} f(\exp \circ \rho_t \circ \phi(s_0 + (s + \xi\ell_t^{-1}t)\ell_t t^{-1}) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon)$$

$$= \lim_{n\to\infty} \int_{I_\varepsilon} f(\exp \circ \rho_t \circ \phi(u + \xi) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon).$$

Applying (3.8), this becomes

$$\lim_{n\to\infty} \int_{I_\varepsilon} f(\exp(\psi_t(u,\xi)) \exp(\rho_t \circ \phi(u)) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon)$$

$$= \lim_{n\to\infty} \int_{I_\varepsilon} f(\exp(\eta(\zeta)) \exp(\rho_t \circ \phi(u)) \cdot x_0)\, ds + O(\|f\|_\infty \varepsilon)$$

$$= \lim_{n\to\infty} \int_{G/\Gamma} f(\exp(\eta(\zeta)) \cdot x)\, d\nu_{t_n}(x) + O(2\|f\|_\infty \varepsilon),$$

where we have used (3.9) in the second to last step. Taking the limit, in view of (3.10),

$$\left| \int_{G/\Gamma} f(x)\, d\nu(x) - \int_{G/\Gamma} f(\exp(\eta(\zeta))x)\, d\nu(x) \right| \le O(2\|f\|_\infty \varepsilon).$$

Thus ν is invariant under the action of U_1, where U_1 is the closure of the subgroup of G/Γ^0 generated by $\{\exp(\eta(\zeta))\colon \zeta \in \mathbb{R}\}$. But $\zeta \mapsto \eta(\zeta)$ is a non-constant polynomial map and so U_1 is a nontrivial connected subgroup of G/Γ^0. Let $U = \pi^{-1}(U_1)$. Then ν is U-invariant, U is connected, and properly contains Γ^0. $\qquad\square$

Claim 3.4.2. *The measure ν is invariant under a closed connected normal subgroup F of G containing U such that $F\Gamma/\Gamma$ is compact.*

Proof. Let $\mathcal{H}$ be the collection of all closed connected subgroups H of G containing Γ^0 with $H\Gamma/\Gamma$ is compact. Then $\mathcal{H}$ is countable. By Lesigne [7, Theorem 2], or more generally Ratner's Theorem [11], any U-ergodic invariant probability measure on G/Γ is of the form $g\mu_H$ for some $H \in \mathcal{H}$, where μ_H denotes the H-invariant probability measure on $H\Gamma/\Gamma$, and $g \in G$ is such that $Ug\Gamma \subset gH\Gamma$, or equivalently, $g \in X(U,H)$, where

$$X(U,H) = \{g \in G\colon g^{-1}Ug \subset H\}. \tag{3.11}$$

Note that $X(U,H)$ is an algebraic subvariety of G with respect to any algebraic group structure on G.

Let $F \in \mathcal{H}$ be a subgroup of smallest possible dimension such that

$$\nu(X(U,F)\Gamma/\Gamma) > 0. \tag{3.12}$$

We note that

$$X(U,H)Z = X(U,H), \tag{3.13}$$

where $Z \subset G$ is the inverse image of the center of G/Γ^0. Now $\Lambda := Z\Gamma$ is closed and G/Λ is compact, see [10, Chapter II]. Consider the natural quotient map $p\colon G/\Gamma \to G/\Lambda$. Let p_* denote corresponding the pushforward map of measures. Then $p_*(\nu_{t_n}) \to p_*(\nu)$ as $t_n \to \infty$.

If $G = \Gamma^0$, the lemma is trivial. So we assume that G/Γ^0 is a nontrivial connected nilpotent group. Therefore its center is of strictly positive dimension. Therefore $\dim(G/Z) < \dim(G/\Gamma^0)$. So by our induction hypothesis stated on the beginning of the proof of Proposition 3.4, we conclude that $p_*(\nu)$ is the G-invariant probability measure on $G/Z\Gamma$.

By (3.12) and (3.13),

$$p_*(\nu)(X(U,F)Z\Gamma/Z\Gamma) = \nu(X(U,F)\Gamma/\Gamma) > 0.$$

Since $X(U,F)Z = X(U,F)$ and $p_*(\nu)$ is the G-invariant measure on $G/Z\Gamma$, the Haar measure of $X(U,F)$ is strictly positive. As we noted before, $X(U,F)$ is an algebraic subvariety of G. Therefore $X(U,F) = G$; that is, $Ug \subset gF$ for all $g \in G$.

Now we show that that F is normal in G, and ν is F-invariant.

By minimality of F,

$$\nu(S(U,F)\Gamma/\Gamma) = 0,$$

where

$$S(U,F) := \bigcup\{X(U,H)\colon H \in \mathcal{H},\ \dim H < \dim F\}.$$

Therefore

$$\nu(X(U,F)\Gamma/\Gamma \setminus S(U,F)\Gamma/\Gamma) = 1.$$

As a consequence, almost every U-ergodic component of ν is of the form $g\mu_F$ for some $g \in G$.

Now $G \setminus S(U,F) \neq \emptyset$. Let $g \in G \setminus S(U,F)$. We claim that $\overline{Ug\Gamma} = gF\Gamma$. To see this note that $Ug \subset gF$ and $F\Gamma$ is closed, so $\overline{Ug\Gamma} \subset gF\Gamma$, and by the orbit closure theorem (cf. Lesigne [7, Theorem 2]), $\overline{Ug\Gamma} = Lg\Gamma$ and $g^{-1}Lg \in \mathcal{H}$. Therefore $g^{-1}Lg \subset F$. Since $g \in X(U, g^{-1}Lg)$ and $g \notin S(U,F)$, we conclude that $\dim(g^{-1}Lg) \geq \dim F$, and hence $g^{-1}Lg = F$, and so $\overline{Ug\Gamma} = gF\Gamma$.

Now let $\gamma \in \Gamma$. Since $g, g\gamma \in G = X(F,U)$, we have that $Ug\gamma \subset g\gamma F$, and hence $gF\Gamma = \overline{Ug\Gamma} \subset g\gamma F\Gamma$. Therefore $gFg^{-1} \subset g\gamma F\gamma^{-1}g^{-1}$, and hence $F = \gamma F\gamma^{-1}$. This proves that F is normalized by Γ. But $\mathrm{Ad}\,\Gamma$ is Zariski dense in $\mathrm{Ad}\,G$; see [10, Chapter 2]. Hence F is normalized by G.

As we noted above, almost every U-ergodic component of ν is of the form $g\mu_F$ for some $g \in G$, so it is $gFg^{-1} = F$-invariant. Therefore ν is F-invariant. This completes the proof of the claim. □

Now it remains to prove that ν is G-invariant. Let $p\colon G/\Gamma \to G/F\Gamma$ be the natural quotient map, and let p_* denote the corresponding push-forward map of measures. Then $p_*(\nu_{t_n}) \to p_*(\nu)$ as $n \to \infty$. Since $U \subset F$, we have that $\dim((F\Gamma)^0) > \dim(\Gamma^0)$. Therefore by the induction hypothesis stated in the begining of the proof of Proposition 3.4,

$p_*(\nu)$ is the G-invariant probability measure on $G/F\Gamma$. Since ν is F-invariant, it follows that ν is G-invariant. Since the Haar measure is the unique G-invariant probability measure on X, we have that $\nu = \mu$, completing the proof of Proposition 3.4. $\qquad\square$

4. POINTS WITH PROPERTY $\mathcal{W}(\Gamma^0)$

Now we determine which points $s_0 \in (0,1)$ satisfy the property $\mathcal{W}(\Gamma^0)$. For this purpose, we study the following function using Taylor's expansion and Baker-Campbell-Hausdorff formula: for $u \in (0,1)$ and $\xi \in (-1,1)$ such that $u + \xi \in (0,1)$, we have

$$\psi_t(u,\xi) = \log[\exp(\rho_t \circ \phi(u+\xi)) \cdot \exp(-\rho_t \circ \phi(u))] \in \bar{\mathfrak{g}}. \qquad (4.1)$$

Let $u \in (0,1)$ be such that $\phi^{(D)}(u)$ exists. By Taylor's formula, for any $\xi \in \mathbb{R}$ such that $u + \xi \in (0,1)$, we have

$$\phi(u+\xi) = \sum_{l=0}^{D} \frac{1}{l!} \phi^{(l)}(u)\xi^l + \varepsilon(u,\xi)\xi^D, \qquad (4.2)$$

where $\varepsilon(u,\xi) \to 0$ as $\xi \to 0$.

Since $\bar{\mathfrak{g}}^{(k)} \subset \bar{\mathfrak{g}}^{(k-1)}$ for all $1 \le k \le \kappa$, we can choose a subspace $V_k \subset \bar{\mathfrak{g}}^{(k-1)}$ such that $\bar{\mathfrak{g}}^{(k-1)} = V_k \oplus \bar{\mathfrak{g}}^{(k)}$. Then

$$\bar{\mathfrak{g}} = V_1 \oplus \cdots \oplus V_\kappa \text{ and } \bar{\mathfrak{g}}^{(k)} = V_{k+1} \oplus \cdots \oplus V_\kappa.$$

Let $P_k \colon \bar{\mathfrak{g}} \to V_k$ denote the corresponding projection.

By the choice of the integers D_k (the polynomial $Q_k \circ \rho_t$ is of degree at most D_k), for all $1 \le k \le \kappa$ and for each $1 \le i \le D_k$, there exists a linear map $A_{i,k} \colon \mathfrak{g} \to V_k$, such that

$$P_k \circ \rho_t = \sum_{i=0}^{D_k} t^i A_{i,k}. \qquad (4.3)$$

Then

$$\rho_t \circ \phi(u+\xi) = \sum_{k=1}^{\kappa} (P_k \circ \rho_t) \circ \phi(u+\xi)$$

$$= \sum_{k=1}^{\kappa}\sum_{i=0}^{D_k} \left(\sum_{l=0}^{D} t^i \xi^l \frac{1}{l!} A_{i,k} \phi^{(l)}(u) + t^i \xi^D A_{i,k} \varepsilon(u,\xi) \right). \qquad (4.4)$$

By the Baker-Campbell-Hausdorff formula, since $\bar{\mathfrak{g}}^{(\kappa)} = 0$, using the definition of ψ_t in (4.1), we have

$$\psi_t(u,\xi) = \sum_{i=0}^{\kappa D}\sum_{l=0}^{\kappa D} t^i \xi^l Z_{(i,l)}(u,\xi), \qquad (4.5)$$

where the functions $Z_{(i,l)}$ can be expressed as

$$Z_{(i,l)}(u,\xi) = Y_{(i,l)}(u) + \varepsilon_{(i,l)}(u,\xi), \tag{4.6}$$

where $Y_{(i,l)}(u)$ is a fixed linear combination of nested commutators of the form

$$[X_1, [X_2, [\cdots, [X_{n-1}, X_n]]]], \quad \text{where } 1 \le n \le \kappa. \tag{4.7}$$

and

$$X_m = A_{i_m, k_m} \phi^{(l_m)}(u) \in V_{k_m}, \quad \text{for } 1 \le m \le n, \tag{4.8}$$

and $\varepsilon_{(i,l)}(u,\xi)$ is a fixed linear combination of similar nested commutators where one or more of the X_m's with $l_m = D$ are replaced by

$$\tilde{X}_m = A_{i_m, k_m} \varepsilon(u, \xi) \in V_{k_m}. \tag{4.9}$$

Claim 4.1. *We have the following:*

(1) $\lim_{\xi \to 0} \varepsilon_{(i,l)}(u, \xi) = 0$.
(2) *For all i,* $Z_{(i,0)} = 0$.
(3) *If $l < D$, then* $\varepsilon_{(i,l)} = 0$.
(4) *If $i > D$, then* $Z_{(i,l)} = 0$.

In particular,

$$\psi_t(u,\xi) = \sum_{(i,l) \in \mathcal{P}} t^i \xi^l Y_{(i,l)}(u) + \sum_{i=0}^{D} \sum_{l=D}^{\kappa D} t^i \xi^l \varepsilon_{(i,l)}(u, \xi), \tag{4.10}$$

where

$$\mathcal{P} = \{(i,l) \colon 0 \le i \le D, 1 \le l \le \kappa D\}. \tag{4.11}$$

Proof. From the description of $\varepsilon_{(i,l)}(u,\xi)$ and (4.9), due to (4.2) we conclude that (1) holds.

Next (2) holds, because $\psi_t(u, 0) = 0$.

In view of (4.4), the following relations hold between various indices in (4.8) and (4.9):

$$1 \le k_m \le \kappa, \ 0 \le i_m \le D_{k_m}, \text{ and } 0 \le l_m \le D. \tag{4.12}$$

In view of equations (4.5)-(4.9), we get

$$\sum_{i=1}^{n} i_m = i \text{ and } \sum_{m=1}^{n} l_m = l. \tag{4.13}$$

For every commutator appearing in the expression for $\varepsilon_{(i,l)}(u,\xi)$, we have $l_m = D$ for at least one m, and hence $l \ge D$ by (4.13). Therefore if $l < D$ then $\varepsilon_{(i,l)} = 0$, which proves (3).

By (4.8) $X_m \in V_{k_m}$, and by (4.9) $\tilde{X}_m \in V_{k_m}$. Also

$$V_{k_m} \subset \bar{\mathfrak{g}}^{(k_m - 1)} \text{ and } [\bar{\mathfrak{g}}^{(k-1)}, \bar{\mathfrak{g}}^{(k'-1)}] \subset \bar{\mathfrak{g}}^{(k+k')-1} \quad \text{for all } k, k'.$$

Therefore the nested commutator as in (4.7) and its analogue involving $\tilde{X}_m$'s belong to $\bar{\mathfrak{g}}^{((\sum_{m=1}^{n} k_m)-1)}$. Since $\bar{\mathfrak{g}}^{(\kappa)} = 0$, if the nested commutator is nonzero then

$$\sum_{m=1}^{n} k_m \leq \kappa. \tag{4.14}$$

By (4.12) and (4.13), we have that if the nested commutator as in (4.7) or its analogue involving $\tilde{X}_m$'s is nonzero then

$$i = \sum_{m=1}^{n} i_m \leq \sum_{m=1}^{n} D_{k_m}. \tag{4.15}$$

Now in view of (4.7), (4.14) and (4.15), we recall (3.1):

$$D = \max \left\{ \sum_{m=1}^{n} D_{k_m} : \sum_{m=1}^{n} k_m \leq \kappa, \ 1 \leq k_m \leq \kappa, \ 1 \leq n \leq \kappa \right\}.$$

Therefore by (4.15), if $i > D$ then $Z_{(i,l)} = 0$. This proves (4) $\qquad \square$

Claim 4.2. *For $0 \leq i \leq d_1$,*

$$dq(Y_{(i,1)}(u)) = A_i \phi^{(1)}(u).$$

Proof. First note that $dq = dq \circ P_1$. Now

$$P_1 \circ Y_{(i,1)}(u) = A_{i,1} \phi^{(1)}(u), \tag{4.16}$$

because $l = 1$ and by (4.7) and (4.8) we have $n = 1$, and hence by (4.7) $l_1 = l = 1$.

So by (4.3),

$$dq \circ \rho_t = dq \circ (P_1 \circ \rho_t) = dq \circ \left(\sum_{i=0}^{D_1} t^i A_{i,1} \right) = \sum_{i=0}^{D_1} t^i dq \circ A_{i,1}. \tag{4.17}$$

Therefore by (1.7), $d_1 \leq D_1$ and

$$dq \circ A_{i,1} = \begin{cases} A_i & \text{if } 0 \leq i \leq d_1 \\ 0 & \text{if } d_1 < i \leq D_1. \end{cases} \tag{4.18}$$

Hence the claim follows from (4.16). $\qquad \square$

Proposition 4.3. *Given $\varepsilon_0 > 0$, there exists a Borel measurable set $J_1 \subset (0,1)$ such that $\lambda(J_1) \geq 1 - \varepsilon_0$ and the following conditions hold:*

(1) *The derivative $\phi^{(\kappa)}(u)$ exists and is uniformly continuous for $u \in J_1$ and it is bounded on J_1. In particular, for each $(i,l) \in \mathcal{P}$, $Y_{(i,l)}(u)$ is uniformly continuous and bounded for $u \in J_1$.*

(2) *For every $u \in J_1$ and every nontrivial unitary character χ on the torus $G/[G,G]\Gamma = \bar{X}$, we have*

$$A_i \phi^{(1)}(u) \notin \ker d\chi \quad \text{for some } 0 \le i \le d_1.$$

(3) *For each $(i,l) \in \mathcal{P}$, $\varepsilon_{(i,l)}(u,\xi) \to 0$ as $\xi \to 0$, and this convergence is uniform for $u \in J_1$.*

Proof. By our assumption, $\phi^{(\kappa)}(u)$ exists for almost all $u \in (0,1)$, hence by Lusin's Theorem there exists a compact set $J_2 \subset (0,1)$ such that $\phi^{(\kappa)}$ is uniformly continuous on J_2 and $\lambda(J_2) \ge 1 - \varepsilon_0/2$.

By the condition given in (1.11), since $X^* \setminus \{1\}$ is countable, there exists a Borel set $J_3 \subset J_2$ such that $\lambda(J_2 \setminus J_3) = 0$ and for every $\chi \in \bar{X}^* \setminus \{1\}$ and every $u \in J_3$, there exists $1 \le i \le d_1$ such that $A_i \phi^{(1)}(u) \notin \ker d\chi$.

Given $p, q \ge 1$, define the set $J_{p,q}$ to be

$$\{u \in J_3 \colon \text{ for all } (i,l) \in \mathcal{P}, \ |\varepsilon_{(i,l)}(u,\xi)| \le 1/p \text{ for all } |\xi| \le 1/q\}.$$

Then by Claim 4.1((1)), the sets $J_{p,q}$ form a nested sequence of sets growing to J_3 as $q \to \infty$. Choose, $q_p \ge 1$ such that

$$\lambda(J_3 \setminus J_{p,q_p}) < 2^{-p}(\varepsilon_0/2)$$

and set $J_1 = \bigcap_{p=1}^{\infty} J_{p,q_p}$. Then $\lambda(J_3 \setminus J_1) \le \varepsilon_0/2$, and for every $(i,l) \in \mathcal{P}$, $\varepsilon_{i,l}(u,\xi) \to 0$ as $\xi \to 0$ uniformly for $u \in J_1$. Also $\lambda(J_1) \ge 1 - \varepsilon_0$. $\square$

Corollary 4.4. *If J_1 satisfies conditions (1),(2) and (3) of Proposition 4.3 for Γ, then J_1 also satisfies the same three conditions when we replace Γ by any closed subgroup Λ of G containing Γ and ρ_t by $dp \circ \rho_t$, where $dp : \mathfrak{g}/\operatorname{Lie}(\Gamma^0) \to \mathfrak{g}/\operatorname{Lie}(\Lambda^0)$ it the natural quotient map.*

Proof. When Λ replaces Γ, each $Y_{(i,l)}$ gets replaced by $dp(Y_{(i,l)})$ and $\varepsilon_{(i,l)}$ gets replaced by $dp(\varepsilon_{(i,l)})$. Therefore it directly follows that Conditions (1) and (3) hold.

Now $\bar{X}_1 = G/[G,G]\Lambda$ is the abelianization of G/Λ with $\mathfrak{t}_1 = \operatorname{Lie}(G/[G,G]\Lambda^0)$ being its Lie algebra. Let $dr_1 : \mathfrak{t} \to \mathfrak{t}_1$ the natural quotient map. Then dq gets replaced by $dq_1 : \operatorname{Lie}(G/\Lambda^0) \to \mathfrak{t}_1$ such that $dq_1 \circ dp = dr_1 \circ dq$. Then in view of (1.7),

$$dq_1 \circ (dp \circ \rho_t) = dr_1 \circ (dq \circ \rho_t) = \sum_{i=0}^{d_1} t^i (dr_1 \circ A_i).$$

Thus in case of Λ replacing Γ, we have that A_i gets replaced by $dr_1 \circ A_i$ in Condition (2). To verify this condition, let $\chi_1 \in \bar{X}_1^* \setminus \{1\}$ be given. Put $\chi = \chi_1 \circ r_1 \in \bar{X}^* \setminus \{1\}$. Then

$$d\chi_1 \circ (dr_1 \circ A_i) = d\chi \circ A_i.$$

For any $u \in J_1$, by Condition (2) for Γ, pick $0 \le i \le d_1$ such that $d\chi(A_i\phi^{(1)}(u)) \ne 0$. Then $d\chi_1((dr_1 \circ A_i)\phi^{(1)}(u)) \ne 0$. $\qquad\square$

Let $\mathcal{F}_\Gamma$ denote the collection of all normal connected normal subgroups F of G such that $F \supset \Gamma^0$ and $F\Gamma$ is closed. In particular, $F/F \cap \Gamma \cong F\Gamma/\Gamma$ is compact. By [10, Chapter II], there exists a $\mathbb{Q}$-structure on G such that Γ^0 is a $\mathbb{Q}$-subgroup of G and the image of Γ on G/Γ^0 consists of integral points with respect to the $\mathbb{Q}$-structure on the quotient algebraic group G/Γ^0. Moreover for any $F \in \mathcal{F}_\Gamma$, we have that F/Γ^0 must be an algebraic $\mathbb{Q}$-subgroup of G/Γ^0. Therefore $\mathcal{F}_\Gamma$ is countable.

For any $F \in \mathcal{F}_\Gamma$, let $\pi_F\colon G \to G/F$ be the natural quotient map and let $d\pi_F\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{f}$ denote its differential, where $\mathfrak{f}$ denotes the Lie algebra of F.

For all $(i,l) \in \mathcal{P}$ and $F \in \mathcal{F}_\Gamma$, define

$$K_{(i,l),F} = \{u \in (0,1) \colon \phi^{(D)}(u) \text{ exists and } d\pi_F(Y_{(i,l)}(u)) = 0\}. \quad (4.19)$$

Let J_1 be a Borel measurable set which satisfies the conditions (1),(2) and (3) of Proposition 4.3.

For any $(i,l) \in \mathcal{P}$ and $F \in \mathcal{F}_\Gamma$, let

$$S_{(i,l),F} = \{s \in J_1 \cap K_{(i,l),F} \colon$$
$$s \text{ is not a Lebesgue Density point of } J_1 \cap K_{(i,l),F}\}. \quad (4.20)$$

Then by the Lebesgue Density theorem, $\lambda(S_{(i,l),F}) = 0$. Let

$$S_\Gamma = \bigcup_{(i,l)\in\mathcal{P},\, F\in\mathcal{F}_\Gamma} S_{(i,l),F}. \quad (4.21)$$

Since $\mathcal{P}$ is finite and $\mathcal{F}_\Gamma$ is countable,

$$\lambda(S_\Gamma) = 0. \quad (4.22)$$

If Λ is a closed subgroup of G containing Γ, then $\Lambda^0 \in \mathcal{F}_\Gamma$. Hence $\mathcal{F}_\Lambda \subset \mathcal{F}_\Gamma$. Therefore using Corollary 4.4 and (4.21), we have that

$$S_\Lambda \subset S_\Gamma. \quad (4.23)$$

Proposition 4.5. *Let J_1 be a Borel set which satisfies all the three conditions of Proposition 4.3. Let $s_0 \in J_1 \setminus S_\Gamma$. Then s_0 has property $\mathcal{W}(\Gamma^0)$.*

Proof. . Let

$$\mathcal{P}' = \{(i,l) \in \mathcal{P} \colon Y_{(i,l)}(s_0) \ne 0\}. \quad (4.24)$$

In view of (4.19),

$$s_0 \in J_1 \cap K, \text{ where } K := \bigcap_{(i,l)\in\mathcal{P}\setminus\mathcal{P}'} K_{(i,l),\Gamma^0}. \quad (4.25)$$

Set
$$\alpha = \max\{i/l \colon (i,l) \in \mathcal{P}'\}. \tag{4.26}$$
We first show that $\alpha \geq 1$. It suffices to show that
$$Y_{(i,1)}(s_0) \neq 0 \quad \text{for some } 1 \leq i \leq d_1. \tag{4.27}$$
Let χ be a nontrivial character on $G/[G,G]\Gamma = \bar{X}$. Since $s_0 \in J_1$, by condition (2) of Proposition 4.3 pick $1 \leq i \leq d_1$ such that $d\chi(A_i\phi^{(1)}(s_0)) \neq 0$. Therefore $A_i\phi^{(1)}(s_0) \neq 0$. Therefore by Claim 4.2,
$$dq(Y_{(i,1)}(s_0)) = A_i\phi^{(1)}(s_0) \neq 0.$$
Therefore by (4.24) and (4.27), there exists $1 \leq i \leq d_1$ such that $(i,1) \in \mathcal{P}'$. Hence by (4.26)
$$\alpha \geq i/1 \geq 1. \tag{4.28}$$
Fix $\varepsilon > 0$. For every $(i,l) \in \mathcal{P} \setminus \mathcal{P}'$, $s_0 \in J_1 \cap K_{(i,l),\Gamma^0}$ by (4.25), and since $s_0 \notin S_\Gamma$, s_0 is a Lebesgue density point of $J_1 \cap K_{(i,l),\Lambda^0}$ by (4.20). For every $n \geq 1$, we can choose $k_n \geq n$ and a compact set $I_n \subset [0,1]$ such that if we put $t = t_{k_n}$, then
$$s_0 + s\ell_t t^{-1} \in J_1 \cap K_{(i,l),\Gamma^0} \text{ for all } s \in I_n \text{ and } (i,l) \in \mathcal{P} \setminus \mathcal{P}',$$
and
$$\lambda([0,1] \setminus I_n) \leq 2^{-n}\varepsilon.$$
Let $I_\varepsilon = \bigcap_{i=1}^\infty I_n$. Then $\lambda((0,1) \setminus I_\varepsilon) \leq \varepsilon$. For any $s \in I_\varepsilon$, $\zeta \in \mathbb{R}$, and $n \geq 1$, put
$$t = t_{k_n}, \quad u = s_0 + s\ell_t t^{-1} \in J_1 \cap K, \text{ and } \xi = \zeta t^{-\alpha}. \tag{4.29}$$
Then the following statements hold:
(1) For $l > i$, since $\alpha \geq 1$, by (1) of Proposition 4.3,
$$t^i \xi^l Y_{(i,l)}(u) \to 0 \quad \text{as } n \to \infty. \tag{4.30}$$
(2) For $l \geq D \geq i$, since $\alpha \geq 1$ and $u \in J_1$, by (3) of Proposition 4.3,
$$t^i \xi^l \varepsilon_{(i,l)}(u,\xi) = \zeta^l t^{-(\alpha l - i)} \varepsilon_{(i,l)}(u,\xi) \to 0 \quad \text{as } n \to \infty. \tag{4.31}$$
(3) For all $(i,l) \in \mathcal{P} \setminus \mathcal{P}'$, $u \in K \subset K_{(i,l),\Gamma^0}$, so by (4.19),
$$Y_{(i,l)}(u) = 0. \tag{4.32}$$
(4) Since $s_0, u \in J_1$, for each $(i,l) \in \mathcal{P}'$ we have that
$$Y_{(i,l)}(u) \to Y_{(i,l)}(s_0) \quad \text{as } n \to \infty, \tag{4.33}$$
by continuity of $Y_{(i,l)}$ on J_1.
(5) For each $(i,l) \in \mathcal{P}'$,
$$\lim_{n \to \infty} t^i \xi^l = \lim_{n \to \infty} \zeta^l t^{i - \alpha l} = \begin{cases} \zeta^l, & \text{if } i/l = \alpha \\ 0, & \text{if } i/l < \alpha. \end{cases} \tag{4.34}$$

In view of (4.10), by the above list of observations

$$\lim_{n \to \infty} \psi_t(u, \xi) = \sum_{l \in \mathcal{L}} \zeta^l Y_{(\alpha l, l)}(s_0) := \eta(\zeta), \qquad (4.35)$$

where $\mathcal{L} = \{l : 1 \leq l \leq \kappa D, (\alpha l, l) \in \mathcal{P}'\}$, which is nonempty by the definition of α. By (4.24), $Y_{(\alpha l, l)} \neq 0$ for all $l \in \mathcal{L}$. Therefore s_0 has property $\mathcal{W}(\Gamma^0)$. □

Proof of Theorem 3.1. Let $\epsilon > 0$. Obtain J_1 as in Proposition 4.3 such that $\lambda(J_1) \geq 1 - \epsilon$. Let S_Γ be as defined in (4.21). Then by (4.22) we have $\lambda(S_\Gamma) = 0$. Let $W_\epsilon = J_1 \setminus S_\Gamma$. Then $\lambda(W_\epsilon) \geq 1 - \epsilon$. Let $s_0 \in W_\epsilon$. By Corollary 4.4 and (4.23), and Proposition 4.5, s_0 has property $\mathcal{W}(\Lambda^0)$ for every closed subgroup Λ of G containing Γ. Therefore (3.3) holds by Proposition 3.4. Now let $W = \cup_{n \in \mathbb{N}} W_{1/n}$. Then $\lambda(W) = 1$ and (3.3) is satisfied for every $s_0 \in W$. □

5. WEAK EQUIDISTRIBUTION DOES NOT IMPLY EQUIDISTRIBUTION

In this section we provide different instances where weak equidistribution of sequence of measures hold, but (strong) equidistribution does not. As before let λ denote the Lebesgue measure on $\mathbb{R}$ restricted to $(0, 1)$.

Proposition 5.1. *Let $X = \mathbb{T} = G/\Gamma$, where $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$. Let $\rho_t(v) = tv$ for all $v \in \mathbb{R} = \mathrm{Lie}(G)$ and $t \in \mathbb{R}$, and let $x_0 \in X$. There exists a measure ν on $\mathbb{R}$ such that the family of measures $(\mu_t := \mu_{\nu, x_0, \rho_t})_{t \geq 0}$ is weakly equidistributed but not equidistributed on (X, μ), where μ is the Haar measure on X.*

Proof. We first construct a non-atomic probability measure ν on $\mathbb{R}$ such $\mu_{3^m} = \mu_1$ for all $m \in \mathbb{N}$. To construct such a measure, let $\psi \colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\psi(u) = \sum_{\{n \geq 1 : a_n(u) = 1\}} \frac{1}{3^n} \qquad (5.1)$$

for

$$u = \sum_{n=1}^{\infty} \frac{a_n(u)}{3^n} \mod 1, \quad \text{where } a_n(u) \in \{0, 1, 2\},$$

and then take ν be the pushforward of λ under the map ψ to $\mathbb{R} = \mathrm{Lie}(G)$. We now check that it has the stated properties.

Note that if $0 < u_1 < u_2 < 1$ and $\psi(u_1) = \psi(u_2)$ then $x_2 - x_1 = \sum_{n=1}^{\infty} a_n/3^n$, where $a_n \in \{0, 2\}$. So $u_2 - u_1$ belongs to the standard Cantor middle third set, which has zero Lebesgue measure. Therefore $\lambda(\psi^{-1}(\{y\})) = 0$ for all $y \in \mathbb{R}$. Hence ν is non-atomic.

Let $f \in C(\mathbb{T})$. Then for all $t \in \mathbb{R}$,

$$\int_X f \, d\mu_t = \int_0^1 f(t\psi(u) + x_0) \, du.$$

For any $u \in [0, 1/3)$ and $b \in \{0, 1, 2\}$,

$$3\psi(b/3 + u) \equiv \sum_{\{n \geq 1 \colon a_n(u)=1\}} \frac{1}{3^{n-1}} \equiv \psi(3u) \mod \mathbb{Z}. \qquad (5.2)$$

Thus it follows that for any $x_0 \in \mathbb{T}$,

$$\begin{aligned}
\int_0^1 f(3\psi(u) + x_0) \, du &= \sum_{b=0}^2 \int_{b/3}^{b/3+1/3} f(3\psi(u) + x_0) \, du \\
&= \sum_{b=0}^2 \int_0^{1/3} f(\psi(3u) + x_0) \, du \\
&= \sum_{b=0}^2 (1/3) \int_0^1 f(\psi(u) + x_0) \, du \\
&= \int_0^1 f(\psi(u) + x_0) \, du.
\end{aligned}$$

From this we conclude that for any $m \in \mathbb{N}$,

$$\int_0^1 f(3^m \psi(u) + x_0) \, du = \int_0^1 f(3^{m-1}\psi(u) + x_0) \, du.$$

It follows that $\mu_{3^m} = \mu_1$ for all $m \geq 0$. Thus the measures ν and μ_t satisfy the stated conditions.

Since μ_1 differs from the Haar measure μ, the family of measures $(\mu_t)_{t \geq 0}$ is not equidistributed with respect to μ. On the other hand, since $\ker(d\chi)$ is countable set for all nontrivial unitary characters χ, we have that $\nu(\ker(d\chi) + v) = 0$ for all $v \in \mathrm{Lie}(G) = \mathbb{R}$, since ν is non-atomic. By Theorem 1.1, $(\mu_t)_{t \geq 0}$ is weakly equidistributed. $\square$

This example can be generalized to higher dimensional tori:

Corollary 5.2. *For every torus $X = \mathbb{T}^d = G/\Gamma$, where $G = \mathbb{R}^d$, $\Gamma = \mathbb{Z}^d$, let $\rho_t(v) = tv$ for all $v \in \mathrm{Lie}(G) = \mathbb{R}^d$ and $t \in \mathbb{R}^n$. There exists a measure σ on $\mathbb{R}^d$ such that for any $x_0 \in G/\Gamma$, the family of measures $(\mu_t := \mu_{\sigma,x_0,\rho_t})_{t \geq 0}$ is weakly equidistributed but not equidistributed on (X, μ), where μ is the Haar measure on X.*

Proof. Let ν be the measure of $\mathbb{R}$ as defined in Proposition 5.1 and define $\sigma = \nu \times \cdots \times \nu$. We claim that σ satisfies these conditions.

Suppose that $x_0 = (y_1, \ldots, y_d)$. Then

$$\mu_{3^n} = \mu_{\nu, y_1, \rho_{3^n}} \times \cdots \times \mu_{\nu, y_d, \rho_{3^n}} = \mu_{\nu, y_1, \rho_3} \times \cdots \times \mu_{\nu, y_d, \rho_3},$$

and so $(\mu_t)_{t \geq 0}$ is not equidistributed.

On the other hand, for every nontrivial unitary character χ on $\mathbb{T}^d$ and every $v \in \mathrm{Lie}(G) = \mathbb{R}^d$, there exist finitely many linear functionals $g_1, \ldots, g_k \colon \mathbb{R}^{d-1} \to \mathbb{R}$ and some $1 \leq i \leq d$ such that

$$v + \ker(d\chi) = \bigcup_{m=1}^{k} \{(y_1, \ldots, y_d) \in \mathbb{R}^d \colon g_m(y_1, \ldots, \hat{y}_i, \ldots, y_d) = y_i\}.$$

Assuming without loss of generality that $i = d$, for all $1 \leq m \leq k$,

$$\sigma(\{(y_1, \ldots, y_d) \colon g_m(y_1, \ldots, y_{d-1}) = y_d\})$$
$$= \int_{\mathbb{R}^{d-1}} \nu(\{y_d \in \mathfrak{g} \colon y_d = g_m(y_1, \ldots, y_{d-1})\}) \, d\nu(y_1) \ldots d\nu(y_{d-1})$$
$$= 0,$$

because ν is non-atomic. Thus $\sigma(v + \ker(d\chi)) = 0$ and by Theorem 1.1, $(\mu_t)_{t \geq 0}$ is weakly equidistributed. $\qquad\square$

Theorem 5.3. *Let $X = \mathbb{T}^2 = G/\Gamma$, where $G = \mathbb{R}^2$ and $\Gamma = \mathbb{Z}^2$ and let $\rho_t(v) = tv$ for all $v \in \mathrm{Lie}(G) = \mathbb{R}^2$ and $t \in \mathbb{R}$. There is a function $\phi \colon (0, 1) \to \mathfrak{g}$ such that for any $x_0 \in \mathbb{T}^2$, the family of measures $(\mu_t := \mu_{\phi, x_0, \rho_t})_{t \geq 0}$ is weakly equidistributed but not equidistributed on X.*

Proof. Let $\phi \colon (0, 1) \to \mathrm{Lie}(G) = \mathbb{R}^2$ be defined by

$$\phi(u) = (u, \psi(u)), \ \forall u \in (0, 1),$$

where $\psi \colon (0, 1) \to \mathbb{R}$ is defined by (5.1). Let $\mu_t = \mu_{\phi, x_0, \rho_t}$.

We first show that $(\mu_t)_{t \geq 0}$ is weakly equidistributed.

Let ν be the probability measure on $\mathbb{R}^2$ which is the pushforward of the Lebesgue measure on $(0, 1)$ under the map ϕ. Let

$$C = \{(u, \psi(u)) \in \mathbb{R}^2 : u \in (0, 1)\}.$$

Let $P_1 : \mathbb{R}^2 \to \mathbb{R}$ denote the projection on the first factor. Then for any $E \subset \mathbb{R}^2$ we have $\nu(E) = \lambda(P_1(C \cap E))$. Then $\mu_{\phi, x_0, \rho_t} = \mu_{\nu, x_0, \rho_t}$ for all $t \in \mathbb{R}$. Therefore by Theorem 1.1, to prove the weak equidistribution of $(\mu_{\nu, x_0, \rho_t})_{t \geq 0}$, it suffices to show that for all $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and $z \in \mathbb{R}$, if we put

$$L(p, q, z) = \{(x, y) : px + qy = z\}$$

then

$$\nu(L(p, q, z)) = \lambda(P_1(L(p, q, z) \cap C) = 0.$$

First suppose that $p = 0$. Then $q \neq 0$ and $P_1(L(p, q, z) \cap C) = \psi^{-1}(z/q)$, which is contained in a translate of a Cantor middle set. Therefore $\lambda(P_1(L(p, q, z) \cap C)) = 0$.

Now assume that $p \neq 0$. For $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in \{0, 1, 2\}$, set

$$B_{a_1,\ldots,a_N} = \left[\sum_{n=1}^N \frac{a_n}{3^n}, \sum_{n=1}^N \frac{a_n}{3^n} + \frac{1}{3^N} \right) \times \left[\sum_{a_n=1} \frac{1}{3^n}, \sum_{a_n=1} \frac{1}{3^n} + \frac{1}{2 \cdot 3^N} \right].$$

Note that for $b_n \in \{0, 1, 2\}$ for all $n \geq N + 1$,

$$u = \sum_{n=1}^N \frac{a_n}{3^n} + \sum_{n=N+1}^\infty \frac{b_n}{3^N} \implies \sum_{a_n=1} \frac{1}{3^n} \leq \psi(u) \leq \sum_{a_n=1} \frac{1}{3^n} + \sum_{n=N+1}^\infty \frac{1}{3^n}.$$

Also

$$(0, 1) \subset \bigcup_{a_1,\ldots,a_N \in \{0,1,2\}} \left[\sum_{n=1}^N \frac{a_n}{3^n}, \sum_{n=1}^N \frac{a_n}{3^n} + \frac{1}{3^N} \right).$$

Therefore

$$C \subset B_N = \bigcup_{a_1,\ldots,a_N \in \{0,1,2\}} B_{a_1,\ldots,a_N}.$$

Therefore it suffices to prove that

$$\lim_{N \to \infty} \lambda(P_1(L(p, q, z) \cap B_N)) = 0.$$

It is easy to see that there exist $s_1, \ldots, s_{2^N} \in (0, 1)$ such that for all $a_1, \ldots, a_N \in \{0, 1, 2\}$, $\sum_{a_n=1} \frac{1}{3^n} = s_i$ for some $1 \leq i \leq 2^N$. Let

$$C_{j,N} = \bigcup_{a_1,\ldots,a_N \in \{0,1,2\}, \sum_{a_n=1} \frac{1}{3^n} = s_j} B_{a_1,\ldots,a_N}.$$

Then

$$B_N = \bigcup_{j=1}^{2^N} C_{j,N}.$$

Now

$$P_1(L(p, q, z) \cap C_{j,N}) \subset \{(z/p - (q/p)y : y \in [s_j, s_j + 1/(2 \cdot 3^N)]\}.$$

Therefore

$$\lambda(P_1(L(p, q, z) \cap C_{j,N})) \leq |q/p|/(2 \cdot 3^N).$$

Hence

$$\lambda(P_1((L(p, q, z) \cap B_N)) \leq 2^N \cdot |q/2p|/3^N \to 0 \text{ as } N \to \infty.$$

Thus we can now conclude that the family of measures $(\mu_t)_{t \geq 0}$ is weakly equidistributed.

We now prove that $(\mu_t)_{t \geq 0}$ is not equidistributed. By definition, for any continuous function f on $\mathbb{T}^2$, we have

$$\int_{\mathbb{T}^2} f \, d\mu_t = \int_0^1 f((tu, t\psi(u)) + x_0) \, du.$$

Recall that by (5.2), For any $u \in [0, 1/3)$ and $b \in \{0, 1, 2\}$,

$$3\psi(u + b/3) \equiv \sum_{\{n \geq 1 \,:\, a_n(u) = 1\}} \frac{1}{3^{n-1}} \equiv \psi(3u) \mod \mathbb{Z}. \qquad (5.3)$$

Therefore, for any $f \in C(\mathbb{T}^2)$, and $b \in \{0, 1, 2\}$

$$\int_{\frac{b}{3}}^{\frac{1}{3} + \frac{b}{3}} f((3u, 3\psi(u)) + x_0) \, du = \int_0^{1/3} f((3u, \psi(3u)) + x_0) \, du.$$

So

$$\int_0^1 f((3u, 3\psi(u)) + x_0) \, du = 3 \int_0^{\frac{1}{3}} f((3u, \psi(3u)) + x_0) \, du$$

$$= \int_0^1 f((u, \psi(u)) + x_0) \, du.$$

From this we can deduce that for any $m \geq 1$,

$$\int_0^1 f((3^m u, 3^m \psi(u)) + x_0) \, du = \int_0^1 f((3^{m-1} u, 3^{m-1} \psi(u)) + x_0) \, du.$$

Therefore, $\mu_{3^m} = \mu_{3^{m-1}} = \mu_1$. Since μ_1 is not a Haar measure on $\mathbb{T}^2$, $(\mu_t)_{t \geq 0}$ is not equidistributed. $\qquad \square$

References

[1] L. AUSLANDER, L. GREEN, & F. HAHN. Flows on homogeneous spaces. With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg. Annals of Mathematics Studies, **53** Princeton University Press, Princeton, N.J. 1963.

[2] E. BREUILLARD, & E. LE DONNE. On the rate of convergence to the asymptotic cone for nilpotent groups and subFinsler geometry. *Proc. Natl. Acad. Sci. USA* **110** (2013), no. 48, 19220–19226.

[3] E. BREUILLARD, & E. LE DONNE. Nilpotent groups, asymptotic cones, and subFinsler geometry. `www.math.u-psud.fr/~breuilla/Balls.pdf`

[4] M. BJÖRKLUND & A. FISH. Equidistribution of dilations of polynomial curves in nilmanifolds. *Proc. Amer. Math. Soc.* **137** (2009), no. 6, 2111–2123.

[5] J. CHAIKA & P. HUBERT. Circle averages and disjointness in typical flat surfaces on every Teichmüller disc. *Bull. Lond. Math. Soc.*, to appear.

[6] A. LEIBMAN. Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold. *Ergodic Theory Dynam. Systems* **25** (2005), no. 1, 201–213.

[7] E. Lesigne. Sur une nil-variété, les parties minimales associés á une translation sont uniquement ergodiques. *Ergodic Theory Dynam. Systems* **11** (1991), no. 2, 379–391.

[8] P. Pansu. Croissance des boules et des géodésiques fermées dans les nil-variétés. *Ergodic Theory Dynam. Systems* **3** (1983), no. 3, 415–445.

[9] B. Randol. The behavior under projection of dilating sets in a covering space. *Trans. Amer. Math. Soc.* **285** (1984), no. 2, 855–859.

[10] M.S. Raghunathan. *Discrete Subgroups of Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972. ix+227 pp.

[11] M. Ratner. Strict measure rigidity for unipotent subgroups of solvable groups. *Invent. Math.* **101** (1990), no. 2, 449–482.

[12] N. A. Shah. Limit distributions of polynomial trajectories on homogeneous spaces. *Duke Math. J.* **75** (1994), no. 3, 711–732.

[13] H. Weyl. Über die Gleichverteilung von Zahlen mod Eins. *Math. Ann.* **77** (1916), 313–352.

Northwestern University, Evanston, IL 60208 USA
E-mail address: kra@math.northwestern.edu

The Ohio State University, Columbus, Ohio 43210 USA
E-mail address: shah@math.osu.edu

The Ohio State University, Columbus, Ohio 43210 USA
E-mail address: sun.1991@osu.edu