# COMPLEXITY AND DIRECTIONAL ENTROPY IN TWO DIMENSIONS

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Abstract. We study the directional entropy of a dynamical system associated to a $\mathbb{Z}^2$ configuration in a finite alphabet. We show that under local assumptions on the complexity, either every direction has zero topological entropy or some direction is periodic. In particular, we show that all nonexpansive directions in a $\mathbb{Z}^2$ system with the same local assumptions have zero directional entropy.

## 1. Introduction

A classic problem in dynamics is to deduce global properties of a system from local assumptions. A beautiful example of such a result is the Morse-Hedlund Theorem [6]: a local assumption on the *complexity* of a system is equivalent to the global property of *periodicity* of the system. Any periodic system trivially has zero topological entropy. In higher dimensions, this local to global connection is less well understood. Again, there is a natural local assumption on the system that implies zero topological entropy, but now one can study the finer notion of directional behavior and new subtleties arise: under this assumption some directions may have positive directional entropy, while others do not. We prove that there are natural local assumptions on the complexity of a $\mathbb{Z}^2$ system under which either every direction has zero topological directional entropy or some direction is periodic.

To explain the results more precisely, for a finite alphabet $\mathcal{A}$, we study functions of the form $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ which we view as colorings of $\mathbb{Z}^2$. For $\mathbf{n} \in \mathbb{Z}^2$, define the translation $T^{\mathbf{n}}\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ by $T^{\mathbf{n}}(\mathbf{x}) := \mathbf{x} + \mathbf{n}$ and for fixed $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, define $T^{\mathbf{n}}\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ by $T^{\mathbf{n}}\eta(\mathbf{x}) := \eta(T^{\mathbf{n}}\mathbf{x})$. If $\mathcal{S} \subset \mathbb{Z}^2$, then an $\eta$-coloring of $\mathcal{S}$ is any function of the form $T^{\mathbf{n}}\eta\restriction_{\mathcal{S}}$, where by $\eta\restriction_{\mathcal{S}}$ we mean the restriction of the coloring $\eta$ of $\mathbb{Z}^2$ to the set $\mathcal{S}$. To simplify the notation, we define an $\eta$-coloring of $\mathcal{S} \subset \mathbb{R}^2$ to be an $\eta$-coloring of $\mathcal{S} \cap \mathbb{Z}^2$. If $K \subset \mathbb{R}^2$ is compact, we define the complexity $P_\eta(K)$ to be the number of distinct $\eta$-colorings of $K \cap \mathbb{Z}^2$:

$$P_\eta(K) = \left|\{T^{\mathbf{n}}\eta\restriction_{K \cap \mathbb{Z}^2} \colon \mathbf{n} \in \mathbb{Z}^2\}\right|,$$

where $|\cdot|$ denotes the cardinality. This is a generalization to two dimensions of the usual one dimensional complexity $P_\alpha\colon \mathbb{N} \to \mathbb{N}$ defined for $\alpha\colon \mathbb{Z} \to \mathcal{A}$, where $P_\alpha(n)$ is defined to be the number of distinct words of length $n$ appearing in $\alpha$.

It follows immediately from the definition that $P_\eta$ is monotonic: If $A \subseteq B \subset \mathbb{R}^2$ are compact sets, then $P_\eta(A) \le P_\eta(B)$. When $K$ is the rectangle $[0, n-1] \times [0, k-1] \subset \mathbb{Z}^2$, we write $P_\eta(n, k)$ instead of $P_\eta(K)$.

There is a standard dynamical system associated to a configuration such as $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$. We endow $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{Z}^2}$ with the product

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topology. For $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, we let $X_\eta$ denote the orbit closure of $\eta$ under the $\mathbb{Z}^2$ translations $\{T^{\mathbf{n}}\colon \mathbf{n} \in \mathbb{Z}^2\}$. Then $X_\eta$, endowed with a distance $\rho$ defined by

$$\rho(x, y) = 2^{-\min\{\|\mathbf{m}\|\colon x(\mathbf{m}) \neq y(\mathbf{m})\}}$$

for $x, y \in \mathcal{A}^{\mathbb{Z}^2}$, and with the $\mathbb{Z}^2$ action by translation, is a $\mathbb{Z}^2$ topological dynamical system. We refer to an element of the system $X_\eta$ as an *$\eta$-coloring* of $\mathbb{Z}^2$.

We say that $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ is *periodic* if there exists $\mathbf{m} \neq \mathbf{0}$ such that $\eta(\mathbf{m} + \mathbf{n}) = \eta(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{Z}^2$ (note that this means that $\eta$ has a direction of periodicity, but it is not necessarily doubly periodic). Again, this is a two dimensional generalization of the usual notion of periodicity for $\alpha\colon \mathbb{Z} \to \mathcal{A}$. It was conjectured by Nivat [7] that for $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, then $\eta$ is periodic. While there are partial results toward the conjecture, the question remains open. We note that in [3], it is observed that a system with such a complexity bound has zero topological entropy.

In this work, we study the finer notion of directional entropy, which was introduced for cellular automata by Milnor [5] (see Section 2 for the definition). If the directional entropy is finite in all directions, then the system has zero topological entropy, but the converse is false: zero topological entropy does not imply anything more than the existence of a single direction with finite directional entropy. We study the directional entropy of a system under a low complexity assumption (this assumption is made precise in Theorem 1.2).

Boyle and Lind [1] further analyzed directional entropy for topological dynamical systems and related it to expansive subdynamics. We use their definition, but restricted to our two dimensional setting:

**Definition 1.1.** If $X$ is a dynamical system with a continuous $\mathbb{Z}^2$ action $(T^{\mathbf{n}}\colon \mathbf{n} \in \mathbb{Z}^2)$, we say that a line $\ell \subset \mathbb{R}^2$ is *expansive* if there exists $r > 0$ such that if $x, y \in X$ satisfy $x(\mathbf{n}) = y(\mathbf{n})$ for all $\mathbf{n} \in \{\mathbf{n} \in \mathbb{Z}^2\colon \rho(\mathbf{n}, \ell) < r\}$, then $x = y$. If $\ell$ is not an expansive line, we say that it is *nonexpansive*.

For the full shift $X = \mathcal{A}^{\mathbb{Z}^2}$ with the $\mathbb{Z}^2$ action by translations, it is easy to check that there are no expansive lines. However, restricting to a system of the form $X_\eta$ associated to some $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, there are more possibilities. If $\eta$ is periodic, then either the directional entropy $h(\mathbf{u}) = 0$ for all $\mathbf{u} \in S^1$ or there is a single direction of zero entropy. In the former case, $\eta$ has an expansive direction with zero entropy and in the latter case, the unique direction of zero entropy is nonexpansive.

Thus, assuming Nivat's conjecture, if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, then the directional entropy of $\eta$ is either zero in all directions or there is a unique direction of zero directional entropy. We show that this conclusion holds under the stronger hypothesis that the analogous complexity assumption holds for infinitely many pairs $n_i, k_i$ (as usual, $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the standard basis vectors):

**Theorem 1.2.** *Assume $\mathcal{A}$ is a finite alphabet and $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$. If there exists an infinite sequence $n_i, k_i \in \mathbb{N}$ such that $\lim_{i\to\infty} n_i k_i = \infty$ and $P_\eta(n_i, k_i) \leq n_i k_i$, then either*

*(i) $h(\mathbf{u}) = 0$ for all $\mathbf{u} \in S^1$, or*
*(ii) there is a unique nonexpansive direction for $\eta$, which is either $\mathbf{e}_1$ or $\mathbf{e}_2$, and $\eta$ is periodic in this direction.*

As an immediate consequence, we have:

**Corollary 1.3.** *Assume $\mathcal{A}$ is a finite alphabet and $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$. If there exists an infinite sequence $n_i, k_i \in \mathbb{N}$ such that $\lim_{i\to\infty} n_i k_i = \infty$ and $P_\eta(n_i, k_i) \leq n_i k_i$, then $\eta$ has zero directional entropy along each of its nonexpansive directions.*

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## 2. Sufficient conditions for zero directional entropy

We start by reviewing some definitions from [3]. If $\mathcal{S} \subset \mathbb{R}^2$, we denote the convex hull of $\mathcal{S}$ by $\mathrm{conv}(\mathcal{S})$. We say $\mathcal{S} \subset \mathbb{Z}^2$ is *convex* if $\mathcal{S} = \mathrm{conv}(\mathcal{S}) \cap \mathbb{Z}^2$. Define the *area* of a convex set $\mathcal{S} \subset \mathbb{Z}^2$ to be the area of its convex hull and define the *boundary* $\partial(\mathcal{S})$ to be the boundary of $\mathrm{conv}(\mathcal{S})$. Note that when $\mathcal{S}$ is finite, $\partial(\mathcal{S})$ is a polygon or line segment. Given a convex set in $\mathbb{Z}^2$ of positive area, we endow its boundary with the positive orientation, so that it consists of directed line segments. If $\mathcal{S} \subset \mathbb{Z}^2$ is convex and has zero area, then $\mathrm{conv}(\mathcal{S})$ is a line segment in $\mathbb{R}^2$ and in this case, we do not define an orientation on $\partial(\mathcal{S})$.

For finite $\mathcal{S} \subset \mathbb{Z}^2$ and $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, we define $X_{\mathcal{S}}(\eta)$ to be the $\mathbb{Z}^2$ subshift of finite type generated by the $\mathcal{S}$ words of $\eta$, meaning that $X_{\mathcal{S}}(\eta)$ consists of all $f \in \mathcal{A}^{\mathbb{Z}^2}$ such that all $f$-colorings of $\mathcal{S}$ appear as $\eta$-colorings of $\mathcal{S}$.

**Definition 2.1.** Suppose $\mathcal{S} \subset \mathcal{T} \subset \mathbb{Z}^2$ are nonempty, finite sets and that $f\colon \mathcal{S} \to \mathcal{A}$ is an $\eta$-coloring of $\mathcal{S}$. We say that $f$ *extends uniquely* to an $\eta$-coloring of $\mathcal{T}$ if there is exactly one $\eta$-coloring of $\mathcal{T}$ whose restriction to $\mathcal{S}$ coincides with $f$.

**Definition 2.2.** If $\mathcal{S} \subset \mathbb{Z}^2$ is a nonempty, finite, convex set, then $x \in \mathcal{S}$ is *$\eta$-generated* by $\mathcal{S}$ if every $\eta$-coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an $\eta$-coloring of $\mathcal{S}$, and $\mathcal{S}$ is an *$\eta$-generating set* if every boundary vertex of $\mathcal{S}$ is $\eta$-generated. When $\eta$ is clear from context, we refer to an $\eta$-generating set as a *generating set*.

Generating sets give rise to zero topological entropy:

**Lemma 2.3** ([3], Lemma 2.15)**.** *If $\mathcal{S} \subset \mathbb{Z}^2$ is a generating set for $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ and $\mathcal{S}' \supset \mathcal{S}$ is finite, then the topological entropy of the $\mathbb{Z}^2$ dynamical system $(X_{\mathcal{S}'}(\eta), \{T^{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^2})$ is zero.*

We review the definition of directional entropy introduced by Milnor [5].

**Notation 2.4.** If $T$ is a continuous $\mathbb{Z}^2$ action on the compact metric space $(X, \rho)$, $E \subset \mathbb{R}^2$ is a compact set, and $\varepsilon > 0$, set $N_T(E, \varepsilon)$ to be cardinality of the smallest set $Y \subset X$ such that for each $x \in X$ there exists $y \in Y$ with $\rho(T^{\mathbf{n}}(x), T^{\mathbf{n}}(y)) < \varepsilon$ for each $\mathbf{n} \in E \cap \mathbb{Z}^2$. For a compact set $E$ and $t > 0$ , let

$$E^{(t)} = \{\mathbf{v}\colon \|\mathbf{v} - \mathbf{u}\| < t \text{ for some } \mathbf{u} \in E\}$$

denote the *$t$-neighborhood of $E$* and let $tE = \{t\mathbf{u}\colon \mathbf{u} \in E\}$ denote the *$t$-dilation* of $E$.

**Definition 2.5.** If $\Phi$ is a set of $k$ linearly independent vectors and $Q_\Phi$ is the parallelepiped spanned by $\Phi$, then the *$k$-dimensional topological directional entropy* $h_k(\Phi)$ is defined to be

$$h_k(\Phi) = \lim_{\varepsilon\to 0} \sup_{t>0} \overline{\lim_{s\to\infty}} \frac{\log N_T\big((sQ_\Phi)^{(t)}, \varepsilon\big)}{s^k}.$$

We compute the directional entropy for the $\mathbb{Z}^2$ action by translations on the space $X_\eta$, in which case it is straightforward to recast the definition in terms of complexity. As we are interested in one dimensional directional entropy, to simplify notation, when $\Phi = \{\mathbf{u}\}$ for some unit vector $\mathbf{u}$, we write $h(\mathbf{u})$, instead of $h_1(\Phi)$. Given $\mathbf{v} \in \mathbb{R}^2$, let $L_{\mathbf{v}}$ denote the line segment connecting the origin to $\mathbf{v}$, meaning that $L_{\mathbf{v}} = \{t\mathbf{v} \colon 0 \le t \le 1\}$.

**Lemma 2.6.** *Assume $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$. If $\mathbf{u}$ is a unit vector, then the (1-dimensional) topological directional entropy $h(\mathbf{u})$ of the $\mathbb{Z}^2$ action by translation on $X_\eta$ in the direction of $\mathbf{u}$ is given by*

$$h(\mathbf{u}) = \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t)})}{s}.$$

*Proof.* Let $T$ denote the $\mathbb{Z}^2$ action by translation. Fix $0 < \varepsilon < 1$ and let $M = \lfloor -\log_2 \varepsilon \rfloor$. If $x, y \in X_\eta$ satisfy

$$\rho\big(T^{\mathbf{n}}(x), T^{\mathbf{n}}(y)\big) < \varepsilon \text{ for all } \mathbf{n} \in L_{s\mathbf{u}}^{(t)}, \tag{1}$$

then $x$ and $y$ agree on $L_{s\mathbf{u}}^{(t+M)}$. Conversely, if $x$ and $y$ agree on $L_{s\mathbf{u}}^{(t+M+1)}$, they satisfy (1). Thus,

$$P_\eta(L_{s\mathbf{u}}^{(t+M)}) \le N_T(L_{s\mathbf{u}}^{(t)}, \varepsilon) \le P_\eta(L_{s\mathbf{u}}^{(t+M+1)})$$

and so

$$\lim_{M\to\infty} \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t+M)})}{s} \le h(\mathbf{u}) \le \lim_{M\to\infty} \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t+M+1)})}{s}.$$

But since $P_\eta(L_{s\mathbf{u}}^{(t)})$ is non-decreasing in $t$,

$$\begin{aligned} \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t+M)})}{s} &= \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t+M+1)})}{s} \\ &= \sup_{t>0} \limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t)})}{s}. \end{aligned}$$ □

Given a generating set $\mathcal{S} \subset \mathbb{Z}^2$ for some $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, applying Lemma 2.3 with $\mathcal{S}' = \mathcal{S}$ provides an upper bound on the entropy of the associated dynamical system $X_\eta$. We use Lemma 2.6 to strengthen this result:

**Proposition 2.7.** *Assume $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ has an $\eta$-generating set and let $X_\eta$ be the associated dynamical system endowed with the $\mathbb{Z}^2$ action by translation. Then there exists $c > 0$ such that $h(\mathbf{u}) < c$ for all unit vectors $\mathbf{u} \in \mathbb{R}^2$.*

*Proof.* Let $\mathcal{S}$ be an $\eta$-generating set. We claim that it suffices to assume that $\mathcal{S}$ is not contained in a line. If instead $\mathcal{S}$ is contained in some line $\ell$, then since $\mathcal{S}$ contains at least two (integer) points, this line has rational slope. Thus we can choose a parallel line $\ell'$ of minimal (among all lines parallel to $\ell$ that contains at least one integer point) distance to $\ell$. Let $\mathbf{v}$ be a vector of minimal length connecting a point in $\mathcal{S}$ to an integer point on $\ell'$. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are the endpoints of $\mathcal{S}$, then the vertices of $\mathcal{S} \cup (\mathcal{S} + \mathbf{v})$ are $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_1 + \mathbf{v}$, and $\mathbf{a}_2 + \mathbf{v}$. Each of these vertices is $\eta$-generated by $\mathcal{S} \cup (\mathcal{S} + \mathbf{v})$, and so this is also a generating set. Thus, replacing $\mathcal{S}$ with this larger generating set, and the claim is proven.

Thus we assume that $\mathcal{S}$ is not contained in a line. Set $d = \operatorname{diam}(\mathcal{S})$, and note that $d \geq 1$. Fix a unit vector $\mathbf{u} \in \mathbb{R}^2$ and let $t > 0$. We claim that there exists $C > 0$ such that for sufficiently large $s > 0$, we have $P_\eta(L_{s\mathbf{u}}^{(t)}) \leq C|\mathcal{A}|^{2ds}$. Once this claim is proven, the proposition follows from Lemma 2.6. Fix $s > d$. Since $P_\eta(L_{s\mathbf{u}}^{(t)})$ is non-decreasing in $t$, we can assume that $t > d$. Then $L_{s\mathbf{u}}^{(t)}$ contains a translate of $\mathcal{S}$. Let $\mathbf{u}^\perp$ be one of the two unit vectors perpendicular to $\mathbf{u}$. Let $\ell_0$ be the line parallel to $\mathbf{u}^\perp$ passing through the origin, and for $b \in \mathbb{R}$ let $\ell_b = \ell_0 + b\mathbf{u}$. Each of the finitely many points in $\mathcal{S}$ lies on $\ell_b$ for some $b \in \mathbb{R}$. Since $\mathcal{S}$ is not contained in a line, it has nontrivial intersection with at least two different lines of the form $\ell_b$. Let $\delta = b_1 - b_2 > 0$, where $b_1$ and $b_2$ are the two largest such values of $b$. Let $\mathcal{S}'$ be a translate of $\mathcal{S}$ that satisfies

$$\mathcal{S}' \subseteq L_{(s+\delta)\mathbf{u}}^{(t)} \text{ but } \mathcal{S}' \not\subseteq L_{(s+\varepsilon)\mathbf{u}}^{(t)} \text{ for } \varepsilon < \delta,$$

and

$$\mathcal{S}' \cap (L_{(s+1)\mathbf{u}} - t\mathbf{u}^\perp) \neq \emptyset.$$

Fix a coloring of $L_{s\mathbf{u}}^{(t)} \cup \operatorname{conv}(\mathcal{S}')$. Since $\mathcal{S}$ is a generating set, this coloring extends uniquely to a coloring of $L_{s\mathbf{u}}^{(t)} \cup (\operatorname{conv}(\mathcal{S}') + L_{\varepsilon_1\mathbf{u}^\perp})$, where $\varepsilon_1 > 0$ is the smallest $\varepsilon$ such that $(\operatorname{conv}(\mathcal{S}') + L_{\varepsilon\mathbf{u}^\perp}) \cap \mathbb{Z}^2$ contains a point not in $L_{s\mathbf{u}}^{(t)} \cup \operatorname{conv}(\mathcal{S}')$. This in turn extends uniquely to a coloring of $L_{s\mathbf{u}}^{(t)} \cup (\operatorname{conv}(\mathcal{S}') + L_{\varepsilon_2\mathbf{u}^\perp})$ for some $\varepsilon_2 > \varepsilon_1$. Continuing to extend and using that $\operatorname{diam}(\mathcal{S}) = d$, each $\eta$-coloring of $L_{s\mathbf{u}}^{(t)} \cup \operatorname{conv}(\mathcal{S}')$ extends uniquely to an $\eta$-coloring of $L_{s\mathbf{u}}^{(t)} \cup (\operatorname{conv}(\mathcal{S}') + L_{(2t-2d)\mathbf{u}^\perp})$. Repeating this $\lceil 1/\delta \rceil$ times, it follows that each $\eta$-coloring of $L_{s\mathbf{u}}^{(t)} \cup (L_{(s+1)\mathbf{u}}^{(t)} \setminus L_{(s+1)\mathbf{u}}^{(t-d)})$ extends uniquely to $L_{(s+1)\mathbf{u}}^{(t)}$. Since $L_{(s+1)\mathbf{u}}^{(t)} \setminus L_{(s+1)\mathbf{u}}^{(t-d)}$ contains at most $2d$ integer points, we have that

$$P_\eta(L_{(s+1)\mathbf{u}}^{(t)}) \leq |\mathcal{A}|^{2d} P_\eta(L_{s\mathbf{u}}^{(t)}),$$

which completes the proof. □

It was shown in [9] that if a $\mathbb{Z}^2$ topological dynamical system has bounded directional entropy in all directions, then it has zero topological entropy. Thus if $X_\eta$ is the system generated by some $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ and the system $X_\eta$ also has an $\eta$-generating system (as in Proposition 2.7), then $X_\eta$ has zero topological entropy. If there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, it is proven in [3] that then there is an $\eta$-generating set. Thus for $\eta$ satisfying such a complexity bound, the $\mathbb{Z}^2$ action by translations on $X_\eta$ has bounded directional entropy in all directions. In particular, it has zero topological entropy.

However, the converse to this result fails even for a system $X_\eta$ endowed with translations, showing that Proposition 2.7 strengthens this result on the entropy of $X_\eta$:

**Example 2.8.** *Let $\alpha\colon \mathbb{Z} \to \{0,1\}$ with $P_\alpha(n) = 2^n$ and let $A = \{10^n + i^2 \colon i, n \in \mathbb{N}, 1 \leq i \leq n\}$. Define $\eta\colon \mathbb{Z}^2 \to \{0,1\}$ by*

$$\eta(i,j) = \begin{cases} \alpha(i + 2^j) & \text{if } j \in A \\ \alpha(i) & \text{otherwise.} \end{cases}$$

*Then the topological entropy of the $\mathbb{Z}^2$ action on $X_\eta$ by translations is zero and $h(\mathbf{e}_1) = \infty$.*

*Proof.* Let $\beta\colon \mathbb{Z} \to \{0,1\}$ denote the indicator function of the set $A$. We first compute a bound on the complexity function $P_\beta(k)$. Fix $k \in \mathbb{N}$. Given $m \in \mathbb{Z}$, let $I(m) = \{m, m+1, \dots, m+k-1\}$. Let $n(m)$ be the smallest $n \in \mathbb{N}$ such that $10^n + i^2 \in I(m)$ for some $1 \le i \le n$, and set $n(m) = 0$ if no such $n$ exists. If $n(m) > 0$, let $i(m)$ be the minimal $1 \le i \le n(m)$ such that $n(m) + i^2 \in I(m)$, and set $i(m) = 0$ if $n(m) = 0$. Finally, let $a(m) = \min(A \cap I(m)) - m$.

If $n = n(m) \ge k$, then $(10^n + 1) - (10^{n-1} + (n-1)^2) \ge 8 \cdot 10^{n-1} > n \ge k$ and similarly $(10^{n+1} + 1) - (10^n + n^2) > k$. Thus $10^{n'} + i^2 \notin I(m)$ for any $n' \ne n(m)$, $1 \le i \le n'$. Note also that if $i = i(m) \ge k > 1$, then $(i+1)^2 - i^2 \ge i^2 - (i-1)^2 \ge 2i - 1 > k$, and so $I(m) \cap A = \{10^{n(m)} + i(m)\}$. Thus, if $n(m)$ or $i(m)$ is strictly greater than $k$, then $\beta\restriction I(m)$ is equal to $\beta\restriction I(m')$ for some $m'$ with $n(m') \le k$ and $i(m') \le k$. Hence, $\beta\restriction I(m)$ is determined by $0 \le \min\{n(m), k\} \le k$, $0 \le \min\{i(m), k\} \le k$, and $0 \le a(m) \le k$, and so $P_\beta(k) \le (k+1)^3$. Since $I(m)$ contains at most $\sqrt{k}\log_{10} k < k^{3/4} - 1$ elements of $A$,

$$P_\eta(n,k) \le P_\beta(k)(2^n)^{k^{3/4}} = (k+1)^3 2^{nk^{3/4}}$$

and so

$$\lim_{n\to\infty} \frac{\log P_\eta(n,n)}{n^2} \le \lim_{n\to\infty} \frac{3\log(n+1) + n^{7/4}\log 2}{n^2} = 0.$$

To show $h(\mathbf{e}_1) = \infty$, we need to bound $P_\eta(n,k)$ from below for large $n$. To obtain this bound, we consider rectangles of the form $[n_0+1, n_0+n] \times [10^n+1, 10^n+k]$ for $n_0 \in \mathbb{Z}$. When $k \le n$, there are exactly $m \stackrel{\text{def}}{=} \lfloor \sqrt{k} \rfloor$ elements of $A$ in $[10^n+1, 10^n+k]$. Call these elements $a_1, \dots, a_m$. We claim that given any $m$ functions $\gamma_i\colon [1,n] \to \{0,1\}$, there exists $n_0$ such that $\eta(n_0 + j, a_i) = \gamma_i(j)$ for $1 \le j \le n$. For each $i$, $\eta(n_0 + j, a_i) = \alpha(n_0 + j + 2^{a_i})$. Since each $a_i \ge 10^n + 1$, we have that for $i \ne i'$

$$[2^{a_i} + 1, 2^{a_i} + n] \cap [2^{a_{i'}} + 1, 2^{a_{i'}} + n] = \emptyset.$$

Thus, since $P_\alpha(N) = 2^N$ for all $N \in \mathbb{N}$, there exists $n_0$ such that $\alpha(n_0 + j + 2^{a_i}) = \gamma_i(j)$ for each $1 \le i \le m$ and $1 \le j \le n$. Hence, $P_\eta(n,k) \ge 2^{nm} \ge 2^{n(\sqrt{k}-1)}$, and so

$$\frac{\log P_\eta([0, n\mathbf{e}_1]^{(k/2)})}{n} \ge \frac{\log P_\eta(n,k)}{n} \ge \frac{\log 2^{n(\sqrt{k}-1)}}{n} = (\sqrt{k}-1)\log 2.$$

By Lemma 2.6, it follows that

$$h(\mathbf{e}_1) \ge \sup_{k>0} (\sqrt{k} - 1)\log 2 = \infty. \qquad \square$$

## 3. A sequence of complexity bounds

**Notation 3.1.** For a unit vector $\mathbf{u} \in \mathbb{R}^2$ and a compact set $K \subset \mathbb{R}^2$, we let $\tau_{\mathbf{u}}(K)$ denote the thickness of the compact set $K$ in the direction of $\mathbf{u}$, defined by

$$\tau_{\mathbf{u}}(K) = \sup\{\tau\colon L_{\tau\mathbf{u}}^{(1/2)} + \mathbf{n} \subset K \text{ for some } \mathbf{n} \in \mathbb{Z}^2\}.$$

We note that the choice of $1/2$ in this definition could be replaced with any $\lambda > 0$.

**Proposition 3.2.** *Assume $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$. The $\mathbb{Z}^2$ action by translation on $X_\eta$ has zero entropy in direction $\mathbf{u} \in \mathbb{R}^2$ if and only if there exist compact sets $K_i \subset \mathbb{R}^2$ such that $\lim_{i\to\infty} \frac{\log P_\eta(K_i)}{\tau_{\mathbf{u}}(K_i)} = 0$.*

*Proof.* Assume there exists such a sequence $K_i$ and assume for contradiction that $h(\mathbf{u}) = 4\delta > 0$. Since $P_\eta(L_{s\mathbf{u}}^{(t)})$ is non-decreasing in $t$ (recall Notation 2.4), using Lemma 2.6, there exists $t_0 > 1$ such that whenever $t \geq t_0$,

$$\limsup_{s\to\infty} \frac{\log P_\eta(L_{s\mathbf{u}}^{(t)})}{s} \geq 3\delta.$$

Thus there exists a sequence $(s_m)$ such that $\log P_\eta(L_{s_m\mathbf{u}}^{(t_0)}) \geq 2\delta s_m$. Set $\tau_i = \tau_{\mathbf{u}}(K_i)$. If $s_m \leq s \leq s_m + \tau_i$ and $s_m \geq \tau_i$, then

$$\log P_\eta(L_{s\mathbf{u}}^{(t_0)}) \geq \log P_\eta(L_{s_m\mathbf{u}}^{(t_0)}) \geq 2\delta s_m \geq 2\delta s \frac{s_m}{s} \geq 2\delta s \frac{s_m}{s_m + \tau_i} \geq \delta s.$$

Hence, there exist infinitely many $j \in \mathbb{N}$ such that for all $t \geq t_0$,

$$\log P_\eta(L_{j\tau_i\mathbf{u}}^{(t)}) \geq \delta j \tau_i.$$

But since $K_i$ contains a translate of $L_{\tau_i\mathbf{u}}^{(1/2)}$, it follows that

$$\log P_\eta(K_i)^{2t_0 j} \geq \log P_\eta(L_{j\tau_i\mathbf{u}}^{(t_0)}) \geq \delta j \tau_i,$$

and so

$$\frac{\log P_\eta(K_i)}{\tau_i} \geq \frac{\delta}{2t_0}$$

for all $i \in \mathbb{N}$, a contradiction.

Conversely, if no such sequence exists, then setting $K_i = L_{i\mathbf{u}}^{(1/2)}$, there exists a constant $c > 0$ such that $\log P_\eta(K_{i_j}) \geq c i_j$ for some increasing sequence $(i_j)$. By Lemma 2.6, $h(\mathbf{u}) \geq c$. □

**Corollary 3.3.** *For $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, if there exist $n_i, k_i \in \mathbb{N}$ with $\lim_{i\to\infty} n_i k_i = \infty$ and such that $P_\eta(n_i, k_i) \leq n_i k_i$ and $\lim \frac{\log(n_i)}{k_i} = \lim \frac{\log(k_i)}{n_i} = 0$, then the $\mathbb{Z}^2$ action by translation on $X_\eta$ has zero entropy in all directions.*

*Proof.* We apply Proposition 3.2 to the sets $K_i = [0, n_i - 1] \times [0, k_i - 1]$. If $\mathbf{u}$ is a unit vector, then $\tau_{\mathbf{u}}(K_i) \geq \min(n_i, k_i) - 1$. Without loss of generality, we can assume that $k_i \geq n_i$ for all $i \in \mathbb{N}$. Since $n_i$ tends to infinity, we may assume $n_i \geq 2$ for all $i \in \mathbb{N}$, so that $\min(n_i, k_i) - 1 > 0$. Then

$$\lim_{i\to\infty} \frac{\log P_\eta(K_i)}{\tau_{\mathbf{u}}(K_i)} \leq \lim_{i\to\infty} \frac{\log(n_i k_i)}{n_i - 1} \leq \lim_{i\to\infty} 2\frac{\log(k_i)}{n_i - 1} = 0. \qquad \square$$

**Remark 3.4.** By passing to a subsequence, the conclusion of Corollary 3.3 holds unless the rectangles for which we have complexity assumptions have eccentricity unbounded either above or below. In particular, it holds unless there exists $C > 1$ such that either $k_i \geq C^{n_i}$ for all $i \in \mathbb{N}$ or $n_i \geq C^{k_i}$ for all $i \in \mathbb{N}$. Applying the transformation $f(x, y) = (y, x)$ if necessary, we may assume the first of these holds, and this is the setting studied in the next section.

## 4. Proof of Theorem 1.2

In this section, we state two main lemmas used to prove Theorem 1.2. Roughly speaking, the idea of the proof is to use our assumptions to show that when the first case in the statement of the theorem fails, we can produce a single rectangle satisfying the stronger complexity bound $P_\eta(n, k) \leq nk/2$, which then allows us to apply results from [3]. To do this, we consider a fixed rectangle with given complexity bound and use it to find a larger rectangle, dependent on the first one, with the stronger complexity bound. This analysis naturally leads to the dichotomy of Theorem 1.2, with an exponential relation between the relative sides of the rectangle leading to the second case and all other situations leading to the first case.

**Lemma 4.1.** *Suppose $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ satisfies $P_\eta(n_1, k_1) \leq n_1 k_1$ for some $n_1, k_1 \in \mathbb{N}$ and set $B_{m,k}$ to be the integer points in $[0, m-1] \times [3mk_1, k-1-3mk_1]$. For any $k, m \in \mathbb{N}$ with $k > 7mk_1$, any $\eta$-coloring of $[0, n_1 - 1] \times [0, k-1]$ either*

- *(i) extends uniquely to an $\eta$-coloring of $B_{m,k}$, or*
- *(ii) extends only to vertically periodic (with period independent of $k$ and $m$) $\eta$-colorings of $B_{m,k}$.*

To bound the number of colorings of the second type, we use the following lemma:

**Lemma 4.2.** *If $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ is not vertically periodic and there exist $n_i, k_i \in \mathbb{N}$ such that $\lim_{i\to\infty} n_i k_i = \infty$ and $P_\eta(n_i, k_i) \leq n_i k_i$, then for any $p \in \mathbb{N}$ and $\lambda > 1$, there exist $w_0, h_0 \in \mathbb{N}$ such that for any $h \geq h_0$ and $w \geq w_0$, the number of $\eta$-colorings of $[0, w-1] \times [0, h-1]$ that are vertically periodic with period at most $p$ is less than $\lambda^w$.*

The proofs of these two lemmas require several auxiliary lemmas and definitions, and are deferred to the next section. We show how they can be used to deduce our main theorem:

*Proof of Theorem 1.2.* By Corollary 3.3 and Remark 3.4, without loss of generality we can assume that there exists $C > 1$ such that $k_i \geq C^{n_i}$ for all $i \in \mathbb{N}$. In particular, we may assume $n_i \leq k_i$ and so

$$\frac{\log(P_\eta(n_i, k_i))}{k_i} \leq \frac{2\log(k_i)}{k_i} \to 0.$$

Thus by Proposition 3.2, $h(\mathbf{e}_2) = 0$. Applying Theorem 6.3, Part (4) in [1], if $\mathbf{e}_2$ is an expansive direction then we have that the directional entropy is zero in all directions. Thus we can assume that $\mathbf{e}_2$ is nonexpansive, and we are left with showing that this is the unique nonexpansive direction and $\eta$ is periodic in this direction.

For any $m, k \in \mathbb{N}$, the complexity of $B_{m,k}$ can bounded by the sum of the number of colorings of $[0, n_1 - 1] \times [0, k-1]$ that extend uniquely to $B_{m,k}$ plus the number of colorings of $B_{m,k}$ that do not arise as the unique extension of a coloring of the rectangle $[0, n_1 - 1] \times [0, k-1]$. The number of colorings of the first type is clearly bounded above by $P_\eta(n_1, k)$. By Lemma 4.1, each of the colorings of $B_{m,k}$ of the latter type is vertically periodic with period independent of $m$ and $k$. Applying Lemma 4.2, we see that for sufficiently large $m$ and $k$, the number of such colorings is at most $(C^{1/8})^m$. Set $k = k_i$ and $m = 8n_i$ for $i$ large enough such that this bound holds and also sufficiently large such that $48 n_i k_1 \leq k_i/2$. Then the number of such colorings is at most

$$(C^{1/8})^m \leq n_i C^{n_i} \leq n_i k_i.$$

Thus

$$P_\eta(B_{m,k}) \leq P_\eta(n_1, k_i) + (C^{1/8})^m \leq P_\eta(n_i, k_i) + n_i k_i \leq 2n_i k_i.$$

But by the choice of $i$,

$$|B_{m,k}| = 8n_i(k_i - 6(8n_i)k_1) \geq 8n_i(k_i/2) = 4n_i k_i,$$

and so $P_\eta(B_{m,k}) \leq \dfrac{|B_{m,k}|}{2}$. Hence, by Theorems 1.4 and 1.5 in [3], vertical is the unique nonexpansive direction for $\eta$, and $\eta$ is vertically periodic. $\square$

## 5. Proofs of Lemmas 4.2 and 4.1

We say two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2 \setminus \{0\}$ are *parallel* if $\mathbf{v} = c\mathbf{w}$ for some $c > 0$, and we say they are *antiparallel* if $\mathbf{v} = c\mathbf{w}$ for some $c < 0$. We define these terms analogously for directed lines and line segments.

Recall that we endow the boundary of a convex set $S \subset \mathbb{Z}^2$ with positive orientation. Given $\mathbf{v} \in \mathbb{R}^2 \setminus \{0\}$, a $\mathbf{v}$-plane is a closed half-plane whose boundary is parallel to $\mathbf{v}$. For example, $\{(x,y)\colon x \leq 2\}$ is an $\mathbf{e}_2$-plane, while $\{(x,y)\colon x \geq 2\}$ is a $(-\mathbf{e}_2)$-plane.

**Notation 5.1.** If $\mathcal{S} \subset \mathbb{Z}^2$ is convex and $\ell$ is a directed line, we write $E(\ell, \mathcal{S}) = \ell' \cap \mathcal{S}$, where $\ell'$ is the boundary of the intersection of all $\ell$-planes containing $\mathcal{S}$.

Note that $E(\ell, \mathcal{S})$ is the set of integer points on some edge of $\partial(\mathcal{S})$, and it may reduce to a single vertex.

We recall a definition from [3]:

**Definition 5.2.** Suppose $\ell \subset \mathbb{R}^2$ is a directed line. A finite, convex set $\mathcal{S} \subset \mathbb{Z}^2$ is *$\ell$-balanced for $\eta$* if

(i) The endpoints of $E(\ell, \mathcal{S})$ are $\eta$-generated by $\mathcal{S}$;
(ii) The set $\mathcal{S}$ satisfies $P_\eta(\mathcal{S} \setminus E(\ell, \mathcal{S})) > P_\eta(\mathcal{S}) - |E(\ell, \mathcal{S})|$;
(iii) Every line parallel to $\ell$ that has nonempty intersection with $\mathcal{S}$ intersects $\mathcal{S}$ in at least $|E(\ell, S)| - 1$ integer points.

We also call such a set $\mathbf{v}$-*balanced for $\eta$* whenever $\mathbf{v}$ is a vector $\mathbf{v}$ parallel to $\ell$.

We note that the endpoints of $E(\ell, \mathcal{S})$ could consist of a single endpoint. The following lemma allows us to avoid the degenerate case where our balanced set is a line segment.

**Lemma 5.3.** *Suppose $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ and $\mathcal{S} \subset \mathbb{Z}^2$ is a discrete line segment (the intersection of a line segment with $\mathbb{Z}^2$, which contains at least two integer points) whose endpoints are $\eta$-generated by $\mathcal{S}$. Then $\eta$ is periodic with period vector parallel to the line determined by $\mathcal{S}$.*

*Proof.* It suffices to show this when $\mathcal{S}$ is a vertical line segment, as otherwise we can change coordinates by composing $\eta$ with an appropriate element of $SL_2(\mathbb{Z})$. Thus suppose $\mathcal{S} = \{(0,y) \in \mathbb{Z}^2\colon 0 \leq y < h\}$ and that $(0,0)$ and $(0,h)$ are both $\eta$-generated by $\mathcal{S}$. There are at most $|\mathcal{A}|^h$ distinct colorings of $\mathcal{S}$ that arise in $\eta$. So for all $x \in \mathbb{Z}$, there exist $y_1(x), y_2(x) \in \{0, 1, \ldots, |\mathcal{A}|^h\}$ such that the coloring of $\mathcal{S}$ that arises from the restriction of $\eta$ to $\mathcal{S} + (x, y_1(x))$ is that same as that from the restriction of $\eta$ to $\mathcal{S} + (x, y_2(x))$. Since the endpoints of $\mathcal{S}$ are $\eta$-generated by $\mathcal{S}$, the restriction of $\eta$ to the line $x = x_0$ must be periodic of (not necessarily minimal) period $|y_1(x_0) - y_2(x_0)| \leq |\mathcal{A}|^h$. Therefore $(0, (|\mathcal{A}|^h)!)$ is a period vector for $\eta$. $\square$

**Remark 5.4.** In light of Lemma 5.3, our main theorem is easily deduced if there is a discrete line segment whose endpoints are $\eta$-generated. Thus, it suffices to assume for the remainder of this work that any balanced set for $\eta$ (Definition 5.2) has positive area (in the sense that its convex hull has positive area).

**Lemma 5.5.** *Suppose $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ satisfies $P_\eta(n_1,k_1) \leq n_1k_1$ for some $n_1,k_1 \in \mathbb{N}$. Then there exist $n_1',k_1' \in \mathbb{N}$ with $n_1' \leq n_1$ and $k_1' \leq k_1$ and there exist $a_i,b_i \in \mathbb{Z}$ with $0 \leq a_i \leq b_i < k_1$ for $i=1,2$ and $0 \leq a_j \leq b_j < n_1$ for $j=3,4$ such that if*

$$\begin{aligned}
\mathcal{S}_1 &:= \big([0,n_1'-2]\times[0,k_1-1] \cup \{n_1'-1\}\times[a_1,b_1]\big)\cap\mathbb{Z}^2\\
\mathcal{S}_2 &:= \big([1,n_1'-1]\times[0,k_1-1] \cup \{0\}\times[a_2,b_2]\big)\cap\mathbb{Z}^2\\
\mathcal{S}_3 &:= \big([0,n_1-1]\times[0,k_1'-2] \cup [a_3,b_3]\times\{k_1'-1\}\big)\cap\mathbb{Z}^2\\
\mathcal{S}_4 &:= \big([0,n_1-1]\times[1,k_1'-1] \cup [a_4,b_4]\times\{0\}\big)\cap\mathbb{Z}^2
\end{aligned}$$

*then $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced, $\mathcal{S}_2$ is $-\mathbf{e}_2$-balanced, $\mathcal{S}_3$ is $-\mathbf{e}_1$-balanced, and $\mathcal{S}_4$ is $\mathbf{e}_1$-balanced.*

Each of the sets $\mathcal{S}_i$ is a rectangle with an interval of points added either on the line one to the right or one to the left (for $i=1,2$) or on the line above or the line below (for $i=3,4$). This is essentially proved in Lemma 4.7 in [3]. There, the assumption that there exist $n,k$ with $P_\eta(n,k)\leq nk$ is replaced by the stronger assumption that we can find $n,k$ with $P_\eta(n,k) \leq \frac{nk}{2}$, but this hypothesis is not necessary in the case of a horizontal or vertical line, which are the only ones needed here. We include a proof for completeness.

*Proof.* We only prove the case that $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced, as the other three cases are analogous.

Let $n_1' \leq n_1$ be the minimal positive integer such that $P_\eta(n_1',k_1) \leq n_1'k_1$. First suppose $n_1'=1$ and let $k_1'$ be the minimal positive integer such that $P_\eta(1,k_1') \leq n_1'k_1'$. Note that $P_\eta(1,1)=|\mathcal{A}|>1$ and so $k_1'>1$. Since

$$k_1'-1 < P_\eta(1,k_1'-1) \leq P_\eta(1,k_1') \leq k_1',$$

it follows that $P_\eta(1,k_1'-1)=P_\eta(1,k_1')$ and so the points $(0,0)$ and $(0,k_1'-1)$ are both $\eta$-generated by the set $\mathcal{S}=\big(\{0\}\times[0,k_1'-1]\big)\cap\mathbb{Z}^2$. Our assumption, as justified in Remark 5.4, is that we do not need to consider the case of a discrete line segment whose endpoints are $\eta$-generated. Thus for the remainder of this proof we assume that $n_1'>1$.

Let $R := [0,n_1'-1]\times[0,k_1-1]$ and $\tilde{R} := [0,n_1'-2]\times[0,k_1-1]$ denote the integer points in these rectangles . By choice of $n_1'$, we have that $P_\eta(\tilde{R}) > |\tilde{R}|$ and $P_\eta(R) \leq |R|$. Therefore there exist $a$ and $b$ with $0\leq a\leq b<k_1$ such that the set (of integer points)

$$R_{a,b} := \tilde{R} \cup \{n_1'-1\}\times[a,b]$$

satisfies $P_\eta(R_{a,b}) \leq |R_{a,b}|$, but there is no set $S$ such that $\tilde{R}\subseteq S \subset R_{a,b}$ and $P_\eta(S)\leq|S|$. In particular, the points $(n_1'-1,b)$ and $(n_1'-1,a)$ are both $\eta$-generated by $R_{a,b}$. We claim that $\mathcal{S}_1 := R_{a,b}$ is an $\mathbf{e}_2$-balanced set. We have just shown that $\mathcal{S}_1$ satisfies Property (i) of Definition 5.2. Next observe that

$$P_\eta(\mathcal{S}_1) - |E(\mathbf{e}_2,\mathcal{S}_1)| \leq |\mathcal{S}_1| - |E(\mathbf{e}_2,\mathcal{S}_1)| = |\tilde{R}| < P_\eta(\tilde{R}) = P_\eta(\mathcal{S}_1 \setminus E(\mathbf{e}_2,\mathcal{S}))$$

and so $\mathcal{S}_1$ satisfies Property (ii) of Definition 5.2. Finally observe that every line parallel to $\mathbf{e}_2$ that has nonempty intersection with $\mathcal{S}_1$ has intersection in either $k_1$ places or in $|E(\mathbf{e}_2,\mathcal{S}_1)|$ places (and the only line that intersects in $|E(\mathbf{e}_2,\mathcal{S}_1)|$ places is the line containing $E(\mathbf{e}_2,\mathcal{S}_1)$). Thus $\mathcal{S}_1$ satisfies Property (iii) of Definition 5.2. □

We define ways to extend a rectangle in each of the four cardinal directions, starting with a description. For example, the extension of a rectangle $R$ over its boundary edge parallel to $\mathbf{e}_2$ is obtained by adding the integer points on a line segment adjacent to this edge with length decreased by $2p + k_1 - |E(\mathbf{e}_2, \mathcal{S}_1)|$ for some choice of the integer parameter $p$. Then for successively higher choices of the integer parameter $m$, we add another line segment with length decreased by another $2p + k_1 - |E(\mathbf{e}_2, \mathcal{S}_1)|$. Overall, $\mathrm{ext}_1^m(R, p) \setminus R$ is the set of integer points in a trapezoid whose nonvertical edges have slope $a_1 + p$ and $-(k_1 - 1 - b_1 + p)$, respectively. Extensions across the other three cardinal directions can be described analogously. In the case $p = 0$, we write $\mathrm{ext}_i^m(R, 0) = \mathrm{ext}_i^m(R)$.

We now give explicit formulas for these extensions. Let $n_1'$, $k_1'$, $\mathcal{S}_i$, $a_i$, $b_i$ for $1 \le i \le 4$ be as in Lemma 5.5. Let

$$R_1 = ([0, n' - 1] \times [0, k' - 1]) \cap \mathbb{Z}^2$$

and

$$R_2 = ([0, n'' - 1] \times [0, k'' - 1]) \cap \mathbb{Z}^2,$$

where $n' \ge n_1' - 1$, $k' \ge k_1$, $n'' \ge n_1$, and $k'' \ge k_1' - 1$. Let $m, p \ge 0$ and define the sets

$$\begin{aligned}
A_1^m(R_1, p) &:= \{(x, y) \in \mathbb{Z}^2 \colon n' \le x \le n' - 1 + m, \\
&\qquad (x - n' + 1)(a_1 + p) \le y \le k' - 1 - (x - n' + 1)(k_1 - 1 - b_1 + p)\} \\
A_2^m(R_1, p) &:= \{(x, y) \in \mathbb{Z}^2 \colon -m \le x \le -1, \\
&\qquad -x(a_2 + p) \le y \le k' - 1 + x(k_1 - 1 - b_2 + p)\} \\
A_3^m(R_2, p) &:= \{(x, y) \in \mathbb{Z}^2 \colon k'' \le y \le k'' - 1 + m, \\
&\qquad (y - k'' + 1)(a_3 + p) \le x \le n'' - 1 - (y - k'' + 1)(n_1 - 1 - b_3 + p)\} \\
A_4^m(R_2, p) &:= \{(x, y) \in \mathbb{Z}^2 \colon -m \le y \le -1, \\
&\qquad -y(a_4 + p) \le x \le n'' - 1 + y(n_1 - 1 - b_4 + p)\}
\end{aligned}$$

and set

$$\begin{aligned}
\mathrm{ext}_1^m(R_1, p) &:= R_1 \cup A_1^m(R_1, p) \\
\mathrm{ext}_2^m(R_1, p) &:= R_1 \cup A_2^m(R_1, p) \\
\mathrm{ext}_3^m(R_2, p) &:= R_2 \cup A_3^m(R_2, p) \\
\mathrm{ext}_4^m(R_2, p) &:= R_2 \cup A_4^m(R_2, p).
\end{aligned}$$

We also define the border of the $m^{th}$ extension of a rectangle. Maintaining the notation for rectangles $R_1$ and $R_2$ and $m, p, q \ge 0$, define the boundaries

$$\begin{aligned}
&\partial_1^m(R_1, p, q) := \\
&\quad [n' - n_1' + 1 + m, n' - 1 + m] \times [m(a_1 + p) + q, k' - 1 - m(k_1 - 1 - b_1 + p) - q], \\
&\partial_2^m(R_1, p, q) := \\
&\quad [-m, n_1' - 2 - m] \times [m(a_2 + p) + q, k' - 1 - m(k_1 - 1 - b_2 + p) - q], \\
&\partial_3^m(R_2, p, q) := \\
&\quad [m(a_3 + p) + q, n'' - m(n_1 - 1 - b_3 + p) - q] \times [k'' - k_1' + 1 + m, k'' - 1 + m], \\
&\partial_4^m(R_2, p, q) := \\
&\quad [m(a_4 + p) + q, n'' - m(n_1 - 1 - b_4 + p) - q] \times [-m, k_1' - 2 - m].
\end{aligned}$$

where in each case we take the convention that the boundary only consists of integer points (strictly speaking, each of these sets should be intersected with $\mathbb{Z}^2$).

Thus $\partial_1^m(R,p,0)$ is the rectangle of width $n_1'-2$ that shares an edge parallel to $\mathbf{e}_2$ with $\mathrm{ext}_1^m(R,p)$, while $\partial_1^m(R,p,q)$ is a subset of this rectangle with vertical length decreased by $2q$. For the case $p=q=0$, we write $\partial_i^m(R,0,0)=\partial_i^m(R)$ and for the case $m=0$, we write $\partial_i^0(R,p,q)=\partial_i(R,p,q)$. Note that if we translate the balanced set $\mathcal{S}_1$ such that its edge parallel to $\mathbf{e}_2$ is placed on the line closest to the edge of $R$ parallel to $\mathbf{e}_2$ but outside of $R$, then the translated edge is contained in $\mathrm{ext}_1^1(R)\setminus R$ and the rest of the translated balanced set is contained in $\partial_1(R_1)$. More generally, setting $\tilde{\mathcal{S}}_1=\mathcal{S}_1\setminus E(\mathbf{e}_2,\mathcal{S}_1)$ and

$$J_1^{m-1}(R,p,q)=\{\mathbf{j}\in\mathbb{Z}^2\colon \tilde{\mathcal{S}}_1\subset\partial_1^{m-1}(R,p,q)\},$$

then it follows directly from the definitions that

$$\mathrm{ext}_1^m(R,p)\setminus\mathrm{ext}_1^{m-1}(R,p)=\partial_1^m(R,p,0)\setminus\partial_1^{m-1}(R,p,0)\subset\bigcup_{\mathbf{j}\in J_1^{m-1}(R,p,p)}(\mathcal{S}_1+\mathbf{j}).\tag{2}$$

The analogous statements hold for $i=2,3,4$. If $R'=R+\mathbf{t}$ is a translate of $R=[0,n-1]\times[0,k-1]$, we define $\mathrm{ext}_i^m(R',p)=\mathrm{ext}_i^m(R,p)+\mathbf{t}$, $\partial_i^m(R',p,q)=\partial^m(R,p,q)+\mathbf{t}$, and $J_i^m(R',p,q)=J_i^m(R,p,q)+\mathbf{t}$.

Given two sets $\mathcal{S},R\subset\mathbb{Z}^2$, we say that $f\colon R\to\mathcal{A}$ is an $(\mathcal{S},\eta)$-*coloring* of $R$ if $f=g\restriction_R$ for some $g\colon\mathbb{Z}^2\to\mathcal{A}$ such that $g\restriction_{\mathcal{S}+\mathbf{j}}$ is an $\eta$-coloring of $\mathcal{S}$ for each $\mathbf{j}\in\mathbb{Z}^2$. Note that every $\eta$-coloring of $R$ is also an $(\mathcal{S},\eta)$-coloring of $R$, but the converse does not always hold.

To prove the next lemma, we recall a finite version of the Morse-Hedlund Theorem.

**Definition 5.6.** If $a\in\mathbb{Z}$ and $f\colon\{a,a+1,\dots,a+i-1\}\to\mathcal{A}$, define $Tf\colon\{a-1,a,\dots,a+i-2\}\to\mathcal{A}$ by $(Tf)(n):=f(n+1)$ and define $P_f(n)$ to be the number of distinct functions of the form $(T^mf)\restriction_{\{a,a+1,\dots,a+n-1\}}$, where $0\le m\le i-n$ and $0\le n\le i$.

The following is essentially due to Morse and Hedlund [6], and appears with this formulation in [4]:

**Theorem 5.7.** *Let $a\in\mathbb{Z}$ and $f\colon\{a,a+1,\dots,a+i-1\}\to\mathcal{A}$ and suppose there exists $n_0\in\mathbb{N}$ such that $P_f(n_0)\le n_0$. If $i>3n_0$, then the restriction of $f$ to the set $\{a+n_0,a+n_0+1,\dots,a+i-n_0\}$ is periodic of period at most $n_0$.*

**Lemma 5.8.** *Suppose $\eta\colon\mathbb{Z}^2\to\mathcal{A}$ and $P_\eta(n_1,k_1)\le n_1k_1$ for some $n_1,k_1\in\mathbb{N}$. Let $\mathcal{S}_1$ and $n_1'$ be as in Lemma 5.5. Fix $m\in\mathbb{N}$ and suppose the rectangle $R$ of integer points in $[0,n-1]\times[0,k-1]$ is such that $k\ge 4mk_1$ and $n\ge n_1'-1$. If $f$ is an $\eta$-coloring of $\mathrm{ext}_1^m(R)$ such that $f\restriction_{\mathrm{ext}_1^{m-1}(R)}$ does not extend uniquely to an $(\mathcal{S}_1,\eta)$-coloring of $\mathrm{ext}_1^m(R)$, then $f$ is vertically periodic on $\partial_1^{m-1}(R,0,k_1)$, with period at most $|E(\mathbf{e}_2,\mathcal{S}_1)|-1$.*

*Analogous statements hold for $-\mathbf{e}_2,\mathbf{e}_1,-\mathbf{e}_1$ and their corresponding balanced sets, with the roles of $n$ and $k$ interchanged in the latter two cases.*

*Proof.* We prove the first statement; the analogs for the other three cardinal directions are proved similarly. By definition, $\partial_1^{m-1}(R)$ is a rectangle $I\times J\subset\mathbb{Z}^2$ with $|I|=n_1'-1$ and $L\stackrel{\mathrm{def}}{=}|J|=k-m(k_1-|E(\mathbf{e}_2,\mathcal{S}_1)|)$. For convenience, write $J=[j_0,j_0+L-1]\cap\mathbb{Z}$ and $J'=[j_0,j_0+L-k_1]\cap\mathbb{Z}$. Let $\tilde{\mathcal{S}}_1=\mathcal{S}_1\setminus E(\mathbf{e}_2,\mathcal{S}_1)$. Since

$\tilde{\mathcal{S}}_1 = ([0, n_1'-2] \times [0, k_1-1]) \cap \mathbb{Z}^2$, by replacing it, if needed, by one of its translates, we can assume without loss of generality that $\partial_1^{m-1}(R) = \bigcup_{j \in J'} \big(\tilde{S}_1 + (0,j)\big)$. The assumptions imply that for each $j \in J'$, $f\restriction_{\tilde{\mathcal{S}}_1 + (0,j)}$ does not extend uniquely to an $\eta$-coloring of $\mathcal{S}_1 + (0,j)$. Indeed, if it did extend uniquely for some $j$, then since the endpoints of $E(\mathbf{e}_2, \mathcal{S}_1)$ are $\eta$-generated, an induction argument (on the position of the balanced set) shows that $f\restriction_{\mathrm{ext}_{\mathbf{e}_2}^{m-1}(R)}$ extends uniquely to an $(\mathcal{S}_1, \eta)$-coloring of $\mathrm{ext}_{\mathbf{e}_2}^{m}(R)$.

Since $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced, $P_\eta(\tilde{\mathcal{S}}_1) > P_\eta(\mathcal{S}_1) - |E(\mathbf{e}_2, \mathcal{S}_1)|$. Thus there are at most $|E(\mathbf{e}_2, \mathcal{S}_1)|-1$ distinct $\eta$-colorings of $\tilde{\mathcal{S}}_1$ that do not extend uniquely to an $\eta$-coloring of $\mathcal{S}_1$. Thus, at most $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1$ such colorings appear as a coloring of the form $f\restriction_{\tilde{\mathcal{S}}_1 + (0,j)}$ for $j \in J'$.

Set $\mathcal{C} = \{(i,j) \in \tilde{\mathcal{S}}_1 \colon (i, j-1) \notin \tilde{\mathcal{S}}_1\}$. Again, since $\tilde{\mathcal{S}}_1$ is a translate of $([0, n_1' - 2] \times [0, k_1-1]) \cap \mathbb{Z}^2$, we have that $\mathcal{C}$ is a horizontal line segment and $\mathcal{C} + (0,j) \subset \tilde{\mathcal{S}}_1$ for each $0 \le j < k_1$. Let $\mathcal{B}$ be the set of $\mathcal{A}$-colorings of $\mathcal{C}$ and define $g \colon J \to \mathcal{B}$ by $g(j) = f\restriction_{\mathcal{C} + (0,j)}$. Then since $J = J' + \{0, 1, \dots, k_1 - 1\}$, the one-dimensional complexity $P_g(|E(\mathbf{e}_2, \mathcal{S}_1)| - 1)$ is bounded above by the number of colorings of $\tilde{\mathcal{S}}_1 \supset \bigcup_{0 \le j < k_1} (\mathcal{C} + (0,j))$ that arise as a coloring of the form $f\restriction_{\tilde{\mathcal{S}}_1 + (0,j)}$ with $j \in J'$, and so $P_g(|E(\mathbf{e}_2, \mathcal{S}_1)| - 1) \le |E(\mathbf{e}_2, \mathcal{S}_1)| - 1$. Since

$$L \ge k - mk_1 \ge 4mk_1 - mk_1 > 3(|E(\mathbf{e}_2, \mathcal{S}_1)| - 1),$$

we can apply Theorem 5.7. Thus $g$ is periodic on

$$\{j_0 + |E(\mathbf{e}_2, \mathcal{S}_1)| - 1, j_0 + |E(\mathbf{e}_2, \mathcal{S}_1)|, \dots, j_0 + L - (|E(\mathbf{e}_2, \mathcal{S}_1)| - 1)\}$$

with period at most $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1$. Since $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1 < k_1$, this implies that $f\restriction_{\partial_1^{m-1}(R, 0, k_1)}$ is vertically periodic with period at most $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1$. $\square$

In fact, the proof gives a slightly stronger result that we note for use in the sequel:

**Lemma 5.9.** *Let $\mathcal{S}_1$ be as in Lemma 5.5, let $\tilde{\mathcal{S}}_1 = \mathcal{S}_1 \setminus E(\mathbf{e}_2, \mathcal{S}_1)$, and let $J$ be the set of integers in an interval $[j_0, j_0 + L - k_1]$, with $L > 3(|E(\mathbf{e}_2, \mathcal{S}_1)| - 1)$. If for some $\mathbf{t} \in \mathbb{Z}^2$, $f$ is an $\eta$-coloring of $\mathcal{S}_1 + (\{0\} \times J) + \mathbf{t}$ such that for all $j \in J$, the restriction of $f$ to $\tilde{\mathcal{S}}_1 + (0,j) + \mathbf{t}$ does not extend uniquely to an $\eta$-coloring of $\mathcal{S}_1 + (0,j) + \mathbf{t}$, then $f$ is vertically periodic on $\tilde{\mathcal{S}}_1 + \mathbf{t} + \{(0,j) \in \mathbb{Z}^2 \colon j_0 + k_1 \le j \le j_0 + L - 2k_1\}$ with period at most $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1$.*

*Analogous statements hold for $-\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_1$ and their corresponding balanced sets, with the roles of $n$ and $k$ interchanged in the latter two cases.*

**Lemma 5.10.** *Let $\eta, R, m$ be as in Lemma 5.8. Suppose $f$ is an $\eta$-coloring of $R$ which is vertically periodic on $R$ of period $p$ and set $P := \max\{p, 2|E(\mathbf{e}_2, \mathcal{S}_1)|\}$. Then any extension of $f$ to an $\eta$-coloring of $\mathrm{ext}_1^m(R, k_1)$ must be vertically periodic on $\partial_1^{m'}(R, k_1, 0)$ with period at most $P$ for each $0 \le m' \le m$, and therefore vertically periodic on $\mathrm{ext}_1^m(R, k_1)$ with period at most $P!$.*

*Analogous statements hold for $-\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_1$ and their corresponding balanced sets, with the roles of $n$ and $k$ interchanged in the latter two cases.*

We note that the proof is similar in spirit to the proof of Proposition 4.8 in [3].

*Proof.* We prove the stated version, by induction on $m'$. Again, the other three cases are proved similarly. Let $\tilde{f}$ be an extension of $f$ to an $\eta$-coloring of $\mathrm{ext}_1^m(R, k_1)$

and suppose that the restriction of $\tilde{f}$ to $\partial_1^{m''}(R,k_1,0)$ is periodic of period $p' \leq P$ for each $0 \leq m'' \leq m' < m$. Then there are two possibilities: either there exists $\mathbf{j} \in J_1^{m'}(R,k_1)$ such that the restriction of $\tilde{f}$ to $\tilde{\mathcal{S}}_1 + \mathbf{j}$ extends uniquely to an $\eta$-coloring of $\mathcal{S}_1$, or no such $\mathbf{j}$ exists. We claim that in either case, $f\restriction\partial_1^{m'+1}(R,k_1,0)$ is vertically periodic of period at most $p$. For convenience, set $J_{m''} = J_1^{m''}(R,k_1,0)$ for each $0 \leq m'' < m$.

**Case 1:** *There exists* $\mathbf{j} \in J_{m'}$ *such that the restriction of* $\tilde{f}$ *to* $\tilde{\mathcal{S}}_1 + \mathbf{j}$ *extends uniquely to an* $\eta$*-coloring of* $\mathcal{S}_1$. By the periodicity of $\tilde{f}$ on $\partial_1^{m'}(R,k_1,0)$, we have that the restrictions of $\tilde{f}$ to $\tilde{\mathcal{S}}_1 + \mathbf{j} + (0,ip')$ and $\tilde{\mathcal{S}}_1 + \mathbf{j}$ coincide for any $i \in \mathbb{Z}$ such that $\mathbf{j} + (0,ip') \in J_{m'}$. Since $\mathbf{j}$ has the property that this $\eta$-coloring of $\tilde{\mathcal{S}}_1$ extends uniquely to an $\eta$-coloring of $\mathcal{S}_1$, we have that the restrictions of $\tilde{f}$ to $\mathcal{S}_1 + \mathbf{j}$ and $\mathcal{S}_1 + \mathbf{j} + (0,ip')$ coincide whenever $\mathbf{j} + (0,ip') \in J_{m'}$. As $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced, we have (using Definition 5.2 (i)) that the endpoints of $E(\mathcal{S}_1,\mathbf{e}_2)$ are $\eta$-generated by $\mathcal{S}_1$. A simple induction argument then shows that for any $0 < t < p'$, the restrictions of $\tilde{f}$ to $\mathcal{S}_1 + \mathbf{j} + (0,t)$ and $\mathcal{S}_1 + \mathbf{j} + (0,t+ip')$ coincide whenever both $\mathbf{j} + (0,t)$ and $\mathbf{j} + (0,t+ip')$ are elements of $J_{m'}$. By (2), it follows that $\tilde{f}\restriction\partial_1^{m'+1}(R,k_1,0)$ is periodic of period $p' \leq P$.

**Case 2:** *There does not exist* $\mathbf{j} \in J_{m'}$ *such that the restriction of* $\tilde{f}$ *to* $\tilde{\mathcal{S}}_1 + \mathbf{j}$ *extends uniquely to an* $\eta$*-coloring of* $\mathcal{S}_1$. Let $\tilde{\mathcal{S}}_1 := \mathcal{S}_1 \setminus E(\mathbf{e}_2,\mathcal{S}_1)$. By Lemma 5.9, $\tilde{f}$ is vertically periodic on $\partial_1^{m'}(R,k_1,k_1)$ of period at most $h := |E(\mathbf{e}_2,\mathcal{S}_1)| - 1$.

Let $N$ be the number of $\eta$-colorings $\alpha$ of $\mathcal{S}_1$ such that $\alpha\restriction_{\tilde{\mathcal{S}}_1}$ extends in more than one way to an $\eta$-coloring of $\mathcal{S}_1$. We claim that $N \leq 2h$. For each such $\alpha$, let $C_\alpha$ be the $\eta$-colorings $\alpha'$ of $\mathcal{S}_1$ such that $\alpha\restriction_{\tilde{\mathcal{S}}_1} = \alpha'\restriction_{\tilde{\mathcal{S}}_1}$. Since $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced for $\eta$,

$$P_\eta(\tilde{\mathcal{S}}_1) + h \geq P_\eta(\mathcal{S}_1) = P_\eta(\tilde{\mathcal{S}}_1) + \sum_{C_\alpha}(|C_\alpha| - 1).$$

In particular, $\alpha\restriction_{\tilde{\mathcal{S}}_1}$ extends in more than one way exactly when $|C_\alpha| > 1$.

Enumerating the colorings of $\tilde{\mathcal{S}}_1$ that extend in more than one way to a coloring of $\mathcal{S}_1$ as $\alpha_1, \ldots, \alpha_r$ (where $r \leq h$), we have that

$$N = \sum_{i=1}^{r} |C_{\alpha_i'}| = \sum_{i=1}^{r}(|C_{\alpha_i'}| - 1) + \sum_{i=1}^{r} 1 \leq h + r \leq 2h,$$

where $\alpha_i'$ is a choice of a coloring of $\mathcal{S}_1$ that restricts to $\alpha_i$ on $\tilde{\mathcal{S}}_1$. By the pigeonhole principle, there exist integers $m(a_1+k_1) \leq i < j < m(a_1+k_1) + 2h$ such that $f\restriction_{\mathcal{S}_1 + (0,i)} = f\restriction_{\mathcal{S}_1 + (0,j)}$. Since $\tilde{f}\restriction_{\partial_1^{m'}(R,k_1,k_1)}$ is vertically periodic of period at most $h$ and each vertical line intersecting $\mathcal{S}_1$ intersects it in at least $h$ points (since $\mathcal{S}_1$ is a vertically balanced set), it follows that $f\restriction_{\tilde{\mathcal{S}}_1 + (0,i+t)} = f\restriction_{\tilde{\mathcal{S}}_1 + (0,j+t)}$ for each $t$ such that $(\tilde{\mathcal{S}}_1 + (0,i+t)) \cup (\tilde{\mathcal{S}}_1 + (0,j+t)) \subset \partial_1^{m'}(R,k_1,k_1)$. But since each endpoint of $E(\mathbf{e}_2,\mathcal{S}_1)$ is $\eta$-generated, an easy induction argument shows that $\tilde{f}\restriction_{\mathcal{S}_1 + (0,i+t)} = \tilde{f}\restriction_{\mathcal{S}_1 + (0,j+t)}$. By (2), the union of all such translates of $\mathcal{S}_1$ contains $\partial_1^{m'+1}(R,k_1,0)$, so $\tilde{f}$ is periodic of period at most $2h \leq P$ on this set. $\square$

We have now assembled the tools to prove Lemma 4.1:

*Proof of Lemma 4.1.* Let $R = ([0,n_1-1] \times [0,k-1]) \cap \mathbb{Z}^2$ and let $T_{m,k} = \mathrm{ext}_1^m(R)$. Note that $B_{m,k} \subset T_{m,k}$. Fix an $\eta$-coloring $f$ of $T_{m,k}$. Let $\mathcal{S}_1, \mathcal{S}_2$ be the sets defined in Lemma 5.5; meaning that $\mathcal{S}_1$ is $\mathbf{e}_2$-balanced and $\mathcal{S}_2$ is $-\mathbf{e}_2$-balanced.

Let $\tilde{\mathcal{S}}_1 = \mathcal{S}_1 \setminus E(\mathbf{e}_2, \mathcal{S}_1)$ and $\tilde{\mathcal{S}}_2 = \mathcal{S}_2 \setminus E(-\mathbf{e}_2, \mathcal{S}_2)$. Suppose the restriction of $f$ to $[0, n_1 - 1] \times [0, k-1]$ does not extend uniquely to an $\eta$-coloring of $T_{m,k}$, so that $(i)$ does not hold for this restriction (since $B_{m,k} \subset T_{m,k}$). Then there exists $0 \le i \le m - n_1'$ such that the restriction of $f$ to $W_i \overset{\text{def}}{=} \partial_1^i(R)$ does not extend uniquely to an $\eta$-coloring of $W_i \cup W_{i+1} = \text{ext}_1^1(W_i)$. Since the height of $W_i$ is $k - 1 - i(k_1 - 1 - b_1 - a_1) \ge k - mk_1 - 1 \ge 4mk_1 - 1$, Lemma 5.8 implies that $f\restriction\partial_1(W_i, 0, k_1)$ is vertically periodic with period at most $|E(\mathbf{e}_2, \mathcal{S}_1)| - 1 \le k_1$.

Let $R' = \partial_1(W_i, 0, k_1)$. Since $|E(-\mathbf{e}_2, \mathcal{S}_2)| \le k_1$, by Lemma 5.10 we have that $f$ is vertically periodic, with period at most $(2k_1)!$, on $\text{ext}_2^t(R', k_1)$ and $\text{ext}_1^t(R', k_1)$ for each $t \le \dfrac{k - mk_1 - 2k_1}{4k_1}$. Now, $B_{m,k}$ is a subset of the union of these two sets for $t = m$, and since $4k_1 m = 7k_1 m - 3k_1 m \le k - 3k_1 m \le k - mk_1 - 2k_1$, we have $m \le \dfrac{k - mk_1 - 2k_1}{4k_1}$, so $f$ is vertically periodic on $B_{m,k}$, with period independent of $m$ and $k$. □

The proof of Lemma 4.2 requires the following technical lemma:

**Lemma 5.11.** *Let $\eta$, $n_i$, $k_i$, and $p$ be as in Lemma 4.2, and let $w \in \mathbb{N}$. Let $n_i^* = \lfloor n_i/3 \rfloor$ for $i \in \mathbb{N}$, $h_i = 2(p + k_i)$, $R_{i,w}$ be the integer points in $[0, w-1] \times [0, h_i - 1]$ and $S_i$ be the integer points in $[0, n_i^* - 1] \times [0, h_i - 1]$.*

*There exists a constant $C$ independent of $w$ and $i$ such that for any $i \in \mathbb{N}$ with $k_i > 4p$, there exist $\eta$-colorings $g_1, \dots, g_C$ of $S_i$ such that if $n_i \le w$, $(x_0, y_0) \in \mathbb{Z}^2$, and $\eta\restriction R_{i,w} + (x_0, y_0)$ is vertically periodic with period at most $p$, then the following hold:*

- *(a) Either there exists minimal $y_1 \ge y_0 + h_i$ such that for some $x_1 \in \{x_0, x_0 + 1, \dots, x_0 + w - 1\}$, $\eta(x_1, y_1) \ne \eta(x_1, y_1 - p)$, or*
- *(b) there exists maximal $y_1 < y_0$ such that $\eta(x_1, y_1) \ne \eta(x_1, y_1 + p)$ for some $x_1 \in \{x_0, x_0 + 1, \dots, x_0 + w - 1\}$,*

*and exactly one of the following holds:*

- *(i) $x_1$ can be chosen to lie in $\{x_0 + n_1, x_0 + n_1 + 1, \dots, x_0 + w - n_1 - 1\}$, in which case $\eta$ is horizontally periodic on $[x_0 + (2p+1)n_1, x_0 + w - 1 - (2p + 1)n_1] \times [0, h_i - 1]$ with period at most $(2n_1)!$,*
- *(ii) $x_1$ cannot be chosen to lie in $\{x_0 + n_1, x_0 + n_1 + 1, \dots, x_0 + w - n_1 - 1\}$ but can be chosen to lie in $\{x_0, x_0 + 1, \dots, x_0 + n_1 - 1\}$, in which case $\eta\restriction S_i + (x_0, y_0) = g_j$ for some $1 \le j \le C$, or*
- *(iii) $x_1$ can only be chosen to lie in $\{x_0 + w - n_1, x_0 + w - n + 1 + 1, \dots, x_0 + w - 1\}$, in which case $\eta\restriction S_i + (x_0 + w - n_i^*, y_0) = g_j$ for some $1 \le j \le C$.*

*Proof.* If $\eta$ is vertically periodic on some strip of width $n_1$, then by Lemma 5.10 it is periodic on all such strips, with bounded period, and so $\eta$ is vertically periodic. Hence, $\eta$ is not periodic on any vertical strip of width $n_1$, meaning that either (a) or (b) holds. We assume throughout the rest of the proof that (a) holds; the argument in the other case is similar.

Let $p' \le p$ be the vertical period of $\eta\restriction R_{i,w} + (x_0, y_0)$, and for convenience set $R = R_{i,w}$, $S = S_i$, and $h = h_i$. Let $\mathcal{S}_3$ be the $-\mathbf{e}_1$ balanced set of Lemma 5.5. First suppose $x_1$ may be chosen to lie in $\{x_0 + n_1, x_0 + n_1 + 1, \dots, x_0 + w - n_1 - 1\}$. Then the restriction of $\eta$ to $[x_0, x_0 + w - 1] \times [y_1 - k_1, y_1 - 1]$ extends nonuniquely to an $\eta$-coloring of $\text{ext}_3^1([x_0, x_0 + w - 1] \times [y_1 - k_1, y_1 - 1])$, and hence also extends nonuniquely to an $(\mathcal{S},\eta)$-coloring of that set. By Lemma 5.8, it follows that $\eta$ is horizontally periodic on $R' \stackrel{\text{def}}{=} \partial_3([x_0,x_0+w-1]\times[y_1-k_1,y_1-1],0,n_1)$ with period at most $|E(-\mathbf{e}_1,\mathcal{S}_3)|-1<n_1$. Therefore by Lemma 5.10, it is horizontally periodic with period at most $(2n_1)!$ on $\mathrm{ext}_4^p(R',n_1)$. It follows that $\eta$ is horizontally periodic with period at most $(2n_1)!$ on the rectangle

$$[x_0+(2p+1)n_1,x_0+w-1-(2p+1)n_1]\times[y_1-p,y_1-1].$$

Thus by the vertical periodicity assumption, $\eta$ is horizontally periodic on

$$[x_0+(2p+1)n_1,x_0+w-1-(2p+1)n_1]\times[0,h-1]$$

with period at most $(2n_1)!$.

Otherwise, $x_1$ cannot be chosen to lie in $\{x_0+n_1,x_0+n_1+1,\dots,x_0+w-n_1-1\}$ but can be chosen to lie in either $\{x_0,x_0+1,\dots,x_0+n_1-1\}$ or $\{x_0+w-n_1,c_0+w-n_1+1,\dots,x_0+w-1\}$. Let us assume it is the former; the argument in the other case is similar. Let $x_0',y_0'$ be other integers such that $\eta\restriction R+(x_0',y_0')$ is vertically periodic with period $p'$. Assume that (a) holds for $(x_0',y_0')$ as well and that $y_1'$ is as in (a). Suppose also that $x_1'$ cannot be chosen in $\{x_0'+n_1,x_0'+n_1+1,\dots,x_0'+w-n_1-1\}$ but can be chosen in $\{x_0',x_0'+1,\dots,x_0'+n_1-1\}$. Assume further that $x_1'-x_0'=x_1-x_0$. We claim that $\eta(x_0+x,y_1+y)=\eta(x_0'+x,y_1'+y)$ for $(x,y)\in[0,n_i^*-1]\times[-p+1,0]$. Indeed, let $B=[0,n_i-1]\times[0,k_i-1]$ and, for an integer vector $\mathbf{t}\in[0,n_i-n_i^*-1]\times[1,k_i-p-1]$, let $B_{\mathbf{t}}=B+(x_0,y_1-k_i)+\mathbf{t}$ and $B_{\mathbf{t}}'=B+(x_0',y_1'-k_i)+\mathbf{t}$. For a coloring $\alpha\colon B\to\mathcal{A}$, define $y(\alpha)$ to be the minimal integer $p'\le y\le k_i-1$ such that $\alpha(x,y)\ne\alpha(x,y-p')$ for some $0\le x\le n_i-1$, and let $x(\alpha)$ be the maximal such $x$. If we set $\alpha_{\mathbf{t}}=\eta\restriction B_{\mathbf{t}}$ then $(x(\alpha_{\mathbf{t}}),y(\alpha_{\mathbf{t}}))=(x(\alpha_{\mathbf{t}'}),y(\alpha_{\mathbf{t}'}))$ if and only if $\mathbf{t}=\mathbf{t}'$ and so the colorings $\alpha_{\mathbf{t}}$ are all distinct. Similarly, setting $\alpha_{\mathbf{t}}'=\eta\restriction B_{\mathbf{t}}'$ these colorings of $B$ are also distinct from one another. Since there are $(n_i-n_i^*)(k_i-p)$ choices of $\mathbf{t}$, we have $\alpha_{\mathbf{t}}=\alpha_{\mathbf{t}'}'$ for some $\mathbf{t}\ne\mathbf{t}'$. If not, instead we have

$$2(n_i-n_i^*)(k_i-p)\ge 4/3n_i(k_i-p)>4/3n_i(k_i-k_i/4)=n_ik_i$$

distinct $\eta$-colorings of $B$, a contradiction. However, since we assume that $x_1-x_0=x_1'-x_0'$, we can have $\alpha_{\mathbf{t}}=\alpha_{\mathbf{t}'}'$ only if $\mathbf{t}=\mathbf{t}'$. Since $[x_0,x_0+n_i^*-1]\times[y_1-p+1,y_1]\subset B_{\mathbf{t}}$ for all $\mathbf{t}$, it follows that $\eta(x_0+x,y_1+y)=\eta(x_0'+x,y_1'+y)$ for $(x,y)\in[0,n_i^*-1]\times[-p+1,0]$, as claimed. By the vertical periodicity assumptions, there exists $0\le j\le p'$ such that $\eta(x_0+x,y_0+y)=\eta(x_0'+x,y_0'+y+j)$ for all $(x,y)\in[0,n_i^*-1]\times[0,h-p-1]$. Thus, for a pair $(x_0,y_0)$ such that (a) and (ii) hold, $\eta\restriction S+(x_0,y_0)$ is determined by

- the vertical period $p'\le p$ of $\eta\restriction R+(x_0,y_0)$,
- the integer $x_1-x_0\in[0,n_1-1]$, and
- the integer $(y_1-y_0) \mod p'\in[0,p-1]$.

Thus, there are at most $p^2n_1$ possibilities for $\eta\restriction S+(x_0,y_0)$. Arguing similarly, we can bound the number of possibilities if (b) and (ii) hold, and if (iii) holds, all independent of $w$ and $i$. Taking $C$ to be the sum of these bounds completes the proof. □

We now use this to complete the proof of Lemma 4.2:

*Proof of Lemma 4.2.* Again for convenience, write $R=R_{i,w}$, $S=S_i$, and $h=h_i$. Let $f$ be an $\eta$-coloring of $R$ that is vertically periodic with period $p'\le p$, and let

$x_0, y_0$ be such that $f(x,y) = \eta(x + x_0, y + y_0)$ for all $(x,y) \in R$. Define a finite sequence of rectangles in the following way. Let $R_0 = R = R_{i,w}$. For each $0 \le j$ until the process terminates, apply Lemma 5.11 to $R_j + (x_0, y_0)$. If case (i) holds, terminate the process. If case (ii) holds, let $R'_{j+1}$ be the translate of $S_i$ that shares a left edge with $R_j$ and let $R_{j+1} = R_j \setminus R'_{j+1}$, which is a rectangle to which the claim also applies, so long as $w - (j+1)n_i^* \ge n_i$. (If this inequality fails, terminate the process instead.) If case (iii) holds, let $R'_{j+1}$ be the translate of $S_i$ that shares a right edge with $R_j$ and let $R_{j+1} = R_j \setminus R'_{j+1}$, which is also a rectangle to which the claim applies for $w - (j+1)n_i^* \ge n_i$. The coloring $f$ is completely determined by the following data:

- The length $m$ of the sequence of rectangles, which satisfies $m \le \left\lfloor \dfrac{w - n_i}{n_i^*} \right\rfloor + 1$.
- Whether $R'_{j+1}$ is on the right or left side of $R_j$ for each $0 \le j < m$.
- The indices $1 \le a_j \le C$ for which $\eta\restriction_{R'_j + (x_0, y_0)} = g_{a_j}$ for $1 \le j \le m$.
- The restriction of $\eta$ to $R_m + (x_0, y_0)$.

Since $\left\lfloor \dfrac{w - n_i}{n_i^*} \right\rfloor + 1 \le 4w/n_i$, the number of colorings $f$ with these properties is at most

$$\frac{4w}{n_i} 2^{4w/n_i} C^{4w/n_i} \max\{C_1, C_2\},$$

where $C_1$ is the number of $\eta$-colorings of $R$ that are horizontally periodic on $[x_0 + (2p+1)n_1, x_0 + w - 1 - (2p+1)n_1] \times [0, h-1]$ with period at most $(2n_1)!$ and vertically periodic on $R$ with period at most $p$ and $C_2$ is the number of $\eta$-colorings of $[0, n_i - 1] \times [0, h-1]$. Clearly we have

$$C_1 \le p(2n_1)! |\mathcal{A}|^{p(2n_1)! + 2(2p+1)n_1 h}$$

and

$$C_2 \le |\mathcal{A}|^{n_i h}.$$

In particular, $C_1$ and $C_2$ are independent of $w$, and so the number of $\eta$-colorings of $R$ that are vertically periodic with period at most $p$ is at most $K_i w (2C)^{4w/n_i}$, where $K_i$ is independent of $w$ and $C$ is independent of both $w$ and $i$. Choose $i$ large enough such that $(2C)^{4/n_i} < \sqrt{\lambda}$. Then for large enough $w$, the number of colorings $f$ of $R$ that are vertically periodic with period at most $p$ is less than

$$K_i w (2C)^{4w/n_i} < K_i w \lambda^{w/2} < \lambda^w.$$

Now note that for $h \ge \max\{h_i, p+1\}$, the number of such colorings of $[0, w-1] \times [0, h-1]$ is nonincreasing in $h$, so the lemma follows. □

## 6. Further directions

We conjecture a stronger result than Theorem 1.2, namely that it holds under the same assumption as that in Nivat's Conjecture:

**Conjecture 6.1.** *For $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \le nk$, then the directional entropy of every nonexpansive direction of $X_\eta$ is zero.*

If the answer is no, this would provide a counterexample to the Nivat Conjecture, and if the answer is yes, this is further evidence in favor of the conjecture. A weaker conjecture would be that under the same hypothesis, $X_\eta$ has some direction with zero directional entropy. Both statements follow from Theorem 1.2 under the stronger assumptions on the complexity.

Alternately it is likely easier to show that a generalization of Nivat's Conjecture, but with a stronger complexity assumption, holds (recall Notation 3.1):

**Conjecture 6.2.** *If there exist $K_i \subset \mathbb{R}^2$ compact and convex with $\lim_{i\to\infty} \frac{\log P_\eta(K_i)}{\tau_{\mathbf{u}}(K_i)} = 0$, then $\eta$ is periodic.*

Closely related, we ask:

**Question 6.3.** *Say that $\eta$ has an isolated, rational direction of zero directional entropy and that $P_\eta(n,k) \leq nk$. Must $\eta$ be periodic?*

The following example shows that if the complexity assumption is removed, then the answer is no:

**Example 6.4.** *Let $\alpha\colon \mathbb{Z} \to \{0,1\}$ such that $P_\alpha(n) = 2^n$ for all $n \in \mathbb{N}$. Define $\eta\colon \mathbb{Z}^2 \to \{0,1,2,3\}$ by $\eta(n,k) = \alpha(n)$ for each $k \neq 0$, and $\eta(n,0) = \alpha(n)+2$. Then for the $\mathbb{Z}^2$ action on $X_\eta$ by translations, $h(\mathbf{e}_2) = h(-\mathbf{e}_2) = 0$, but $h(\mathbf{u}) > 0$ for all other unit vectors $\mathbf{u}$.*

*Proof.* We first prove $h(\mathbf{e}_2) = 0$ (the proof that $h(-\mathbf{e}_2) = 0$ is analogous). Fix $t > 0$. There are $2^{2t+1}$ $\alpha$-colorings of $[-t,t]$. For each of these $\alpha$-colorings $f$, there are at most $s+2t+2$ $\eta$-colorings of $[-t,t]\times[-t,s+t]$ for which $\eta(i,j) = f(i)$ for some $-t \leq j \leq s+t$. Hence,

$$P_\eta(L^{(t)}_{s\mathbf{e}_2}) \leq P_\eta([-t,t]\times[-t,s+t]) \leq 2^{2t+1}(s+2t+2).$$

Thus,

$$\limsup_{s\to\infty} \frac{\log P_\eta(L^{(t)}_{s\mathbf{e}_2})}{s} \leq \limsup_{s\to\infty} \frac{(2t+1)\log 2 + \log(s+2t+2)}{s} = 0.$$

By Lemma 2.6, it follows that $h(\mathbf{e}_2) = 0$.

For $\mathbf{u} \neq \pm\mathbf{e}_2$, let $m = \frac{1}{\|\operatorname{proj}_{\mathbf{e}_1}\mathbf{u}\|}$ where $\operatorname{proj}_{\mathbf{v}}$ is the projection onto the direction $\mathbf{v}$. By assumption $m < \infty$. Then $\|\operatorname{proj}_{\mathbf{e}_1} m\mathbf{u}\| = 1$, and so $L^{(1)}_{ms\mathbf{u}} \cap \{i\}\times\mathbb{Z} \neq \emptyset$ for each $0 \leq i \leq s$. For any set $K \subset \mathbb{R}^2$, if $|\{i \in \mathbb{Z}\colon K \cap \{i\}\times\mathbb{Z} \neq \emptyset\}| = k_1$ and $|\{j \in \mathbb{Z}\colon K \cap \mathbb{Z}\times\{j\} \neq \emptyset\}| = k_2$, then $P_\eta(K) = (k_2+1)2^{k_1}$. Hence,

$$\limsup_{s\to\infty} \frac{\log P_\eta(L^{(1)}_{ms\mathbf{u}})}{ms} \geq \limsup_{s\to\infty} \frac{\log(2^s)}{ms} = \frac{\log 2}{m}. \qquad \square$$

Finally, we can ask how much of this holds in higher dimensions. While there are examples [8] showing that the analog Nivat's Conjecture is false for dimension $d \geq 3$, it is possible that the results on directional entropy generalize. We remark that Cassaigne [2] constructed aperiodic examples in higher dimensions which satisfy the higher dimensional analog of the complexity assumptions used in our results. These all have zero directional entropy in all directions, and so do not rule out a higher dimensional version of our theorem:

**Question 6.5.** *Does the analog of Theorem 1.2 hold for $\eta\colon \mathbb{Z}^d \to \mathcal{A}$, where $d \geq 3$?*

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