

VIEW CAMERA GEOMETRY

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1. INTRODUCTION

This article started as an exposition for mathematicians of the relation between projective geometry and view camera photography. The two subjects share a common history, and problems in view camera geometry lead to some interesting mathematics. In the process of writing the article, I realized that some technical problems in using a view camera could be approached profitably by the projective approach. A considerable part of the article, particularly the section on depth of field, deals with such matters.

The article is largely original, but I did receive helpful advice from various sources. In particular, I want to thank Jeff Conrad, who looked at my many earlier drafts and kept bringing me back to the really interesting questions. I've also received help from comments made in the Large Format Forum at www.lfphoto.info. Of course, I am not the first to look into many of the topics considered in this article, and I have tried to include other sources of useful information in the text, mainly in footnotes. I apologize if I've left anyone or anything out.

2. SOME HISTORY

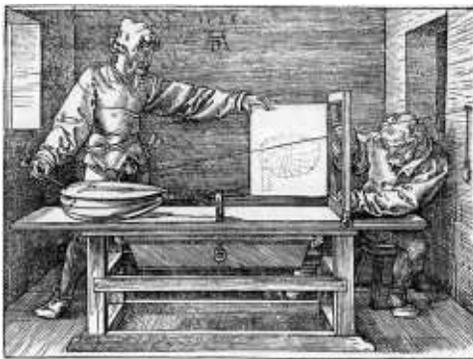


FIGURE 1. Durer Woodcut: Man Drawing a Lute 1523

The history of *projective geometry* is intimately related to that of representational art. In the 15th and 16th century artists studied how to draw in perspective. (Alberti, della Francesca, Durer, etc.). Imagine a frame through which the artist looks at a scene, and follows a ray from a center of perspective to a point in the

scene. Such a ray intersects the rectangular region bounded by the frame in an image point, and a picture results from transferring those images to paper.

A tool designed to help in this endeavor was the *camera obscura*, which is Latin for ‘dark room’.¹ Originally, this was indeed a darkened room with a small hole in one side. Rays from the scene through the hole impinge on the opposite wall forming an image of the scene which can be traced on the wall. The image is upside down. If you traced it on paper and rotated the paper through 180 degrees, you would find that left and right are reversed as in a mirror. It has been suggested that 17th century Dutch masters such as Vermeer may have used the camera obscura as an aid,² but this remains controversial. Adding a lens produces a much brighter image, and through the use of various devices, it became possible to view the image outside the camera, right side up without the left-right reflection. The camera obscura played an important role in the history of representational art, and photography arose through efforts to ‘fix’ its image.

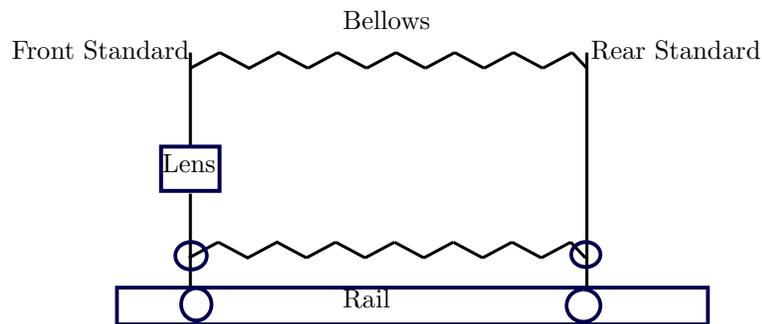


FIGURE 2. View Camera Schematic

A modern camera employs basically the same principle, with the photographer outside the camera, and the image recorded by a light sensitive material, either film or a digital sensor array. View cameras are a special class of camera, usually using formats such as 4 x 5 inches or larger. Figure 2 shows a scheme for such a camera, which consists of two *standards*, connected by a *bellows*, and mounted on a *rail*.

A lens is mounted in a *lensboard* attached to the front standard. The rear standard has a ground glass screen onto which the image is projected, and the photographer views that image from behind the camera rather than from inside it. It is rotated 180 degrees, but if you rotate it back you will find that left and right are preserved. The ground glass can be moved out of the way in order to mount a film holder or other recording device. In addition, either standard is capable of movements such as rise, fall, and shift to one side as well as rotations about two perpendicular axes. Through the use of such movements, one can adjust the relative positions and orientation of the image frame with respect to the lens and both of

¹en.wikipedia.org/wiki/Camera_obscura.

²See www.essentialvermeer.com/camera_obscura/co_one.html.

them with respect to the scene. The bellows, which keeps out stray light, and the rail, which serves to hold the standards in place, play no role in the geometry, so we shall largely ignore them.

3. PROJECTIVE GEOMETRY

Parallel lines in a scene receding into the distance appear to the viewer to converge to a point, but of course there is no such point in the scene. Like the proverbial pot of gold at the end of a rainbow, it recedes from you as you try to approach it. But the images of such lines in a two-dimensional perspective representation actually do converge to a common point, which is called a *vanishing point*. So it is easy to imagine adding to the scene an extra *point at infinity* to correspond to the vanishing point. The formalization by mathematicians of such ideas in the 17th, 18th, and 19th centuries led to the idea of *projective space* and *projective geometry*.

In real projective space $\mathbb{P}^3(\mathbb{R})$, three non-collinear points determine a plane, two distinct planes intersect in a line, and three non-collinear planes intersect in a point. There are some other axioms which are necessary to specify the dimension and characterize the associated field, which in our case is the real number field.³ Those maps of the space which carry points into points, lines into lines, planes into planes and also preserve incidence are called projective transformations. (We also require that such a transformation yield the identity on the associated field—in general it may be a field automorphism—but that is always the case for \mathbb{R} .)

We also may specify one plane $\Omega = \Omega_\infty$ as the plane at infinity. With that plane deleted, the resulting space satisfies the axioms of affine geometry, which is concerned with parallelism. Two planes, a line and a plane, or two lines, assumed distinct, are parallel if they meet at infinity. If we specify a point O in affine space, the directed line segments \overrightarrow{OP} from that point may be added and multiplied by scalars using suitable geometric constructions using parallel lines. The result is a vector space, in our case isomorphic to \mathbb{R}^3 . Given two points O and O' , there is an obvious way to translate the vector space at O to that at O' via the directed line segment $\overrightarrow{OO'}$. (That makes affine space into a flat 3-dimensional real manifold with an affine connection.) We can use this to define a map \mathcal{A} called the antipodal map relative to O and Ω . Namely just require that \mathcal{A} fix O and that the corresponding directed line segments (vectors) satisfy $\overrightarrow{OA(P)} = -\overrightarrow{OP}$. (This structure makes affine space into a symmetric space.) It is not hard to check that if we extend \mathcal{A} to the \mathbb{P}^3 by fixing all points in the plane at infinity, the resulting map is a projective transformation. Note also that P, O , and $\mathcal{A}(P)$ are always collinear.

There are various models for $\mathbb{P}^3(\mathbb{R})$. The first is obtained by taking Euclidean 3-space and adjoining an additional plane at infinity by adding one point for every family of parallel lines. Another approach is to take $\mathbb{R}^4 - \{(0, 0, 0, 0)\}$ and identify vectors by the equivalence relation which identifies vectors differing by a non-zero scalar multiplier. The resulting quotient space is $\mathbb{P}^3(\mathbb{R})$. A plane through the origin in \mathbb{R}^4 yields a projective line, and a 3-space through the origin yields a projective plane. \mathbb{R}^4 has a distinguished canonical basis, but we don't give it any special prominence. Projective transformations of \mathbb{P}^3 are exactly those that arise from nonsingular linear transformations of \mathbb{R}^4 .

³See any good book on projective geometry such as *Foundations of Projective Geometry* by Robin Hartshorne.

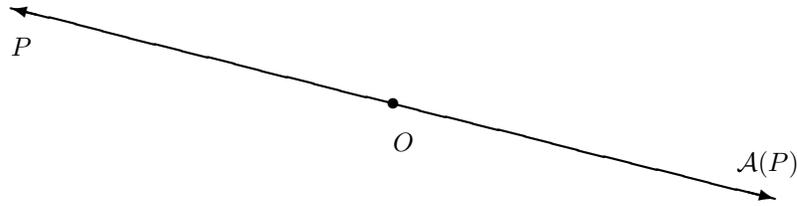


FIGURE 3. Antipodal Map

Any basis in \mathbb{R}^4 yields a coordinate system and we may attach so called *homogeneous coordinates* (x_0, x_1, x_2, x_3) to a point P in \mathbb{P}^3 , except that such coordinates are only unique up to multiplication by a non-zero scalar. If we arrange the coordinates so that $x_0 = 0$ is the equation of the specified plane Ω at infinity, then for a point in the associated affine space, we may take the coordinates to be $(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})$ which are unique. The last three coordinates are the affine coordinates of the point.

We all carry in our heads a model of Euclidean space, which adds a metric to affine space. This may be specified by attaching consistent inner products to the real vector spaces at each point. It is surprising how far we will be able to get without making use of the Euclidean structure,

Because of gravity, we also carry in our heads a distinction between vertical and horizontal. This may be formalized as follows. We pick a point called the *zenith* in the plane at infinity. Then all lines passing through that point, are called *vertical*. Similarly, we choose a line in the plane at infinity which we call the *horizon*. All planes containing that line are called horizontal. In the Euclidean context, we also assume the horizontal planes are perpendicular to the vertical lines.

4. PINHOLE CAMERA GEOMETRY

Let us now think of how this framework may be used to study image formation. We shall start just with the case of a “pinhole camera” which is essentially a rectangular box with a pinhole in one side and the film on the opposite side. Note that any view camera may be made into a pinhole camera by substituting a plate with a small hole in it for the lens, and we thereby get the extra flexibility of movements.

The light rays emanating from a point in the scene which pass through the hole will form a solid cone. If we intersect that cone of light with the frame in which we capture the image, we will get a small blob of light. If it is small enough, the human visual system can’t distinguish it from a point. As a result, we may treat the pinhole as a point for purposes of geometric ray tracing. That point will be called the *principal point* of the camera. It is the *center of perspective*, also called the point of view.

In deciding how large to make the pinhole, one must balance brightness against sharpness. (If the hole is very small, the image will be sharp, but it will also be very dim, which may require very long exposures of the photosensitive material.) If the subject point is relatively distant compared to the dimensions of the camera enclosure, the size of the blob of light won’t vary much over the box, so the image will be equally sharp over the course of the light ray, and the sharpness of the

image won't depend much on just where you place the film frame, but the size of the image will change. Usually, we assume the film frame is in a vertical plane, but it need not be. If you place it closer the image will be smaller, and if you place it further, it will be larger. The ratio of the image distance v to the subject distance u is called the *magnification* or *scale of reproduction*. It tells us how much measurements at distance u in the subject space are reduced at the corresponding distance v in the image space. It is usually less than 1. If the film frame is tilted, the relative positions of vanishing points will change, so shapes will also be affected.

As noted earlier, the image inside the camera when viewed from the front has orientation reversed. But the process of making an image to be viewed, either by making a print or by projection on a screen or showing it on a monitor, reverses that so the orientation in the final image is correct. The traditional method was to make a negative image in the film, develop and fix it chemically, and then print it by contact printing or with an enlarger. Darkroom workers had to be careful about which way they positioned the negative to make sure the resulting print didn't reverse right and left.

We now formalize these concepts as follows. As before, we assume a plane Ω at infinity has been specified (with specified zenith and horizon). In addition, we specify a different plane Λ called the lens plane—although we don't yet have a lens—with a point O called the principal point. The lens plane and the plane at infinity divide projective space into two compartments. Finally, we specify a simple closed planar curve, not coplanar with the principal point, and we call the plane region it bounds the *frame*. We shall assume the frame is a rectangle, but it need not be. We further stipulate that the frame lies entirely in one of these two compartments which we call the *image space*. The other compartment is called the *subject space*. The plane Π containing the frame is called the *image plane* or *film plane*.

See Figure 4 for a diagrammatic representation. *It is important to remember that the geometry is three-dimensional, and that a diagram typically only shows a two-dimensional cross section. So don't begin to think of the planes as lines.* On the left we show how it looks conventionally, and on the right how it would look if we could move ourselves to where we could see the plane at infinity. Note that although it may look as if the two planes divide space into four regions, there are actually only two regions involved.

Consider an affine point P in the subject space. The line determined by O and P intersects the image plane in a unique point P' called the corresponding *image point*. If the image plane is parallel to the lens plane, the image P' is in the image space. If the image plane is not parallel to the lens plane, then the image point P' is in the image space or at infinity. It makes sense also to restrict attention to those points in the subject space with images in the frame, but since in principle the frame may be anywhere in the image space, this is no restriction. For simplicity we have ignored the camera enclosure, but if we incorporated other bounding planes in addition to the lens plane, it would further restrict possible positions of the frame and which subject points could have images in the frame. Being able to control the position of the frame is probably the most useful feature of a view camera.

The map $P \mapsto P'$ is called a perspective transformation, or just a *perspectivity* through the principal point O . It takes lines into lines and preserves incidence, but of course it is far from being a bijection, so it is not a projective transformation.

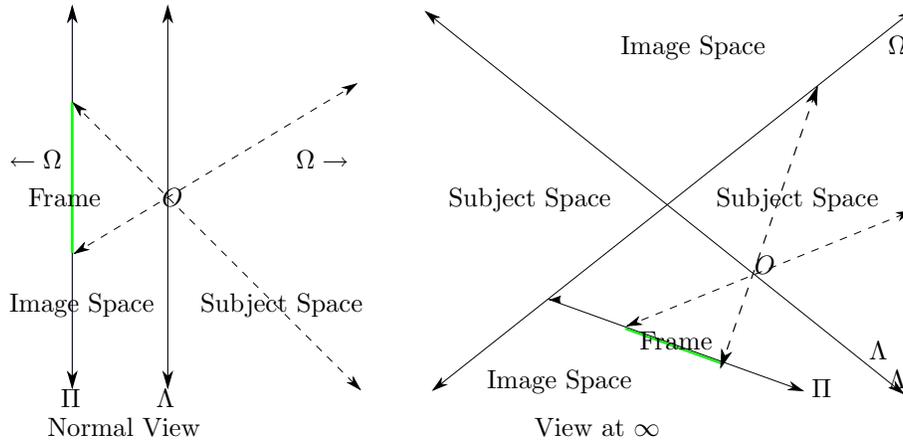


FIGURE 4. Two views of Subject and Image spaces

It is completely determined by the principal point and the position of the image plane and it is what creates the image.

As noted earlier, if two lines intersect at infinity, their images intersect in a vanishing point in the image plane. Generally this is an affine point, but if the lines are parallel to the image plane, the corresponding vanishing point in the image is also at infinity, which means the affine sections of the image lines are parallel.

This has important implications in photography. Consider a camera pointing upward at a building with vertical sides as sketched in Figure 5. Photographers will often do that to include the top of the building in the frame.

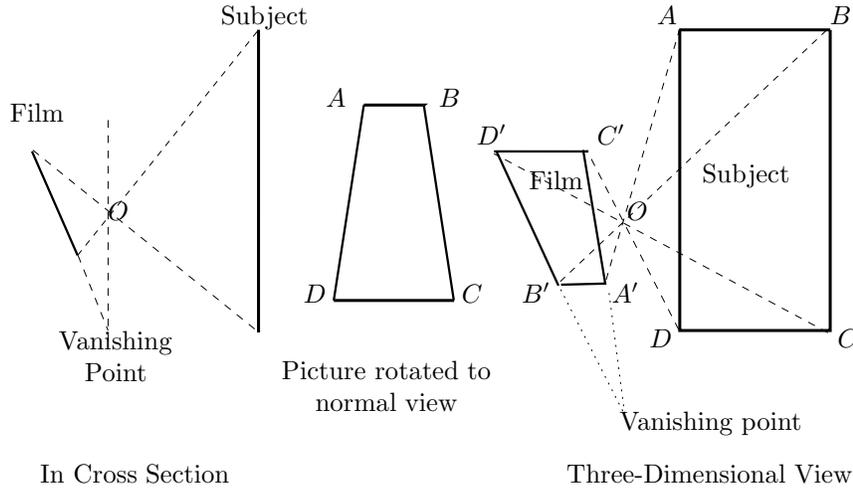


FIGURE 5. Convergence of verticals when film plane is tilted upward

The camera back, i.e., the image plane, is tilted with respect to the vertical, so the images of the sides of the building will converge to a vanishing point. It may not be in the frame, but the sides of the building will still appear to converge, downwards on the ground glass, but, upwards, after we rotate the picture to view it right side up. (In a conventional SLR or other such camera, this is done for you.) There has been a lot of discussion about what people actually see when they look at such a scene. The human visual system does a lot of processing, and we don't just "see" the projected image on the retina. Some have argued that whatever the human visual system once expected due to our evolutionary history of hunting and gathering, recently, decades of looking at snapshots with sides of buildings converging, have conditioned us to find such representations normal. Since structures with vertical sides presumably didn't play a direct role during our evolution, it is not clear just what we expect "naturally", but I would argue that if you don't have to turn your head too much to see the top of the building, the sides will appear to be parallel. On the other hand, if you crane your head a lot, the sides will appear to converge. In any case, many people do prefer to have verticals appear parallel, and it is considered essential in most architectural photography. View camera photographers usually start with the film plane vertical, and only change that when it is felt to be necessary for technical or aesthetic reasons. Of course, users of conventional cameras don't have that choice, so the problem is ignored, resulting in a multitude of photographs with converging verticals. In the past, when necessary, any correction was done in the darkroom by tilting the print easel in the enlarger, and today it is done by digital manipulation of the image. But such manipulation, if not done correctly, can produce other distortions of the image.

In a view camera, we make sure verticals are vertical simply by keeping the frame and image plane vertical, which, as noted above, puts the vanishing point at infinity. If the vertical angle subtended by the frame at the principal point (called the vertical angle of view) is large enough, we can ensure that both the top and bottom of the building are included in the frame by moving the frame down far enough, i.e., by dropping the rear standard. If the position of the principal point, in relation to the scene, is not critical, which is usually the case, then we may instead raise that principal point by raising the front standard. If we were to leave the frame centered on the horizontal, the top of the building might not be visible and there might be an expanse of uninteresting foreground in front of the building in the picture.

The ability of a view camera to perform maneuvers of this kind is one thing that makes it so flexible.

5. ADDING A LENS

If we add a lens to the mix we gain the advantage of being able to concentrate light. A lens has an opening or aperture, and the rays from a subject point passing through the aperture are bent or refracted so they form a cone with vertex at the image point. The image will be much brighter than anything obtainable in a pinhole camera. In addition, for each subject point, there is a unique image point, on the ray through the principal point, which is in focus. In a pinhole camera, as noted earlier, any point on that ray is as much in focus as any other.

Unfortunately the situation is more complicated than this description suggests. First of all, because of inevitable lens aberrations, there is no precise image point,

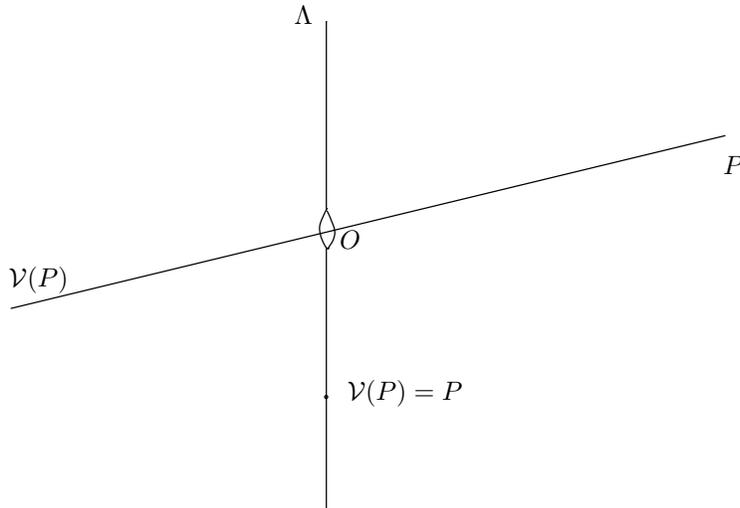


FIGURE 6. Lens Map

but rather, if one looks closely enough, a three-dimensional blob. In addition, a plane in the subject space is generally taken into a curved surface in the image space, a phenomenon called *curvature of field*. If this were not enough, the fact that light is also a wave, leads to a phenomenon called *diffraction*. Light passing through an aperture, with or without a lens present, does not produce a point image but rather a diffuse central disc surrounded by a series of diffuse rings. (The positions of the rings are determined by the zeroes of a Bessel function.) It is not possible for a simple lens to produce a sharp image. Fortunately, over the years designers have developed complex lenses which overcome the aberration and field curvature problems so well that in practice we may act if the image of a subject point is just a point and planes get carried into planes.⁴ Diffraction is still a problem, even for a perfect lens, but it becomes a serious issue only for relatively small apertures. We shall pretty much ignore diffraction because, while it is important, it is based on physical optics (wave optics) rather than geometric optics, the main focus of this article.

So, we shall propose a simple abstract model of a lens, which tells us pretty well how geometric optics works for a photographic lens.

We start as before with a lens plane Λ and a principal point O in that plane. We call a projective transformation \mathcal{V} of \mathbb{P}^3 a *lens map* if it has certain additional properties. Note first that since it is a projective transformation, it necessarily takes each plane Σ into another plane $\Pi = \mathcal{V}(\Sigma)$. These planes are called respectively the *subject plane* and the *image plane*. We also require that

- (1) \mathcal{V} fixes each point in the lens plane Λ .
- (2) The points P, O and $P' = \mathcal{V}(P)$ are collinear.

⁴This article is concerned with the consequences of applying the rules of geometric optics. Because a real lens only obeys those laws approximately, the conclusions we draw must be taken with a grain of salt.

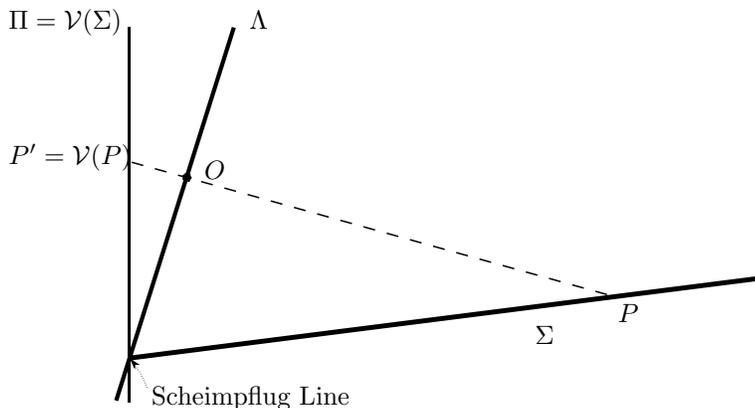


FIGURE 7. Scheimpflug Rule

Note that one consequence of (2) is that \mathcal{V} carries every plane through O into itself.

(3) $\mathcal{V}\mathcal{A}\mathcal{V} = \mathcal{A}$ where \mathcal{A} is the antipodal map in O .

(3) is a symmetry condition, the significance of which we shall explain later. But notice for the present that since \mathcal{A} is an involution, we have that $\mathcal{V}\mathcal{A}\mathcal{V} = \mathcal{A}$ is equivalent to $(\mathcal{V}\mathcal{A})^2 = Id$ or $(\mathcal{A}\mathcal{V})^2 = Id$, i.e., both $\mathcal{A}\mathcal{V}$ and $\mathcal{V}\mathcal{A}$ are involutions. It is also clear by taking inverses that $\mathcal{V}^{-1}\mathcal{A}\mathcal{V}^{-1} = \mathcal{A}$. These facts will later turn out to have important consequences.

Note that a trivial consequence of (1) is the following rule

(1') The subject plane Π , the lens plane Λ , and the image plane $\Pi' = \mathcal{V}(\Pi)$ are collinear.

(Note also that (1') and (2) together imply (1).)

(1') is called the *Scheimpflug Rule*, and I shall call the common line of intersection the *Scheimpflug line*.⁵

Scheimpflug's Rule is usually derived as a consequence of the laws of geometric optics by means of Desargues Theorem in $\mathbb{P}^3(\mathbb{R})$.⁶ One aim of this article is to show you that it is simpler in many ways to start with (1) or (1') and derive the laws of geometric optics. See Appendix A for a description of Desargues Theorem and a derivation of Scheimpflug's Rule using geometric optics.

Before proceeding further, let me say a bit about how Scheimpflug's Rule is used in practice. When the camera is set up in standard position with the lens plane and film plane parallel to one another—the only possible configuration for a conventional camera—they intersect at infinity. so that is where the Scheimpflug

⁵The rule appears in Scheimpflug's 1904 patent application, but apparently he didn't claim credit for it. See

www.trenholm.org/hmmerk/SHSPAT.pdf and www.trenholm.org/hmmerk/TSBP.pdf.

⁶See Wheeler (www.bobwheeler.com/photo/ViewCam.pdf) or

Luong (www.largeformatphotography.info/scheimpflug.jpeg).

line is. That means the corresponding subject plane also intersects those planes at infinity and so it is also parallel to them. In some circumstances, it is useful to have the subject plane oriented differently. For example we might have a field of wildflowers extending horizontally away from the camera, and it might not be possible to get it all in focus by the usual mechanism (stopping down). On the other hand, we might not care much whether anything significantly above or below the field was in focus. With a view camera, we can accomplish this by reorienting the front and rear standards with respect to one another so they are no longer parallel. As a result, they will intersect somewhere within reach, i.e., the Scheimpflug line will be brought in from infinity. This may be done by tilting the rear standard, but that may produce undesirable effects on shapes in the scene, as noted previously. So it is often considered better to tilt the front standard. In any case, what turns out to be important is the line through the lens perpendicular to the rear standard. Tilting the rear standard is the same as rotating the rail (hence that line) and tilting the front standard, followed perhaps by shifts of the standards. So, in this article, with a few exceptions, we shall discuss only movements of the front standard.

Note. Strictly speaking the term *tilt* should be restricted to rotations of the standard about a horizontal axis. We shall use the term more generally in this article to refer to a rotation of a standard about any axis. In Section 9, we shall discuss in greater detail how this is actually done in view cameras.

Let me also emphasize here that I am talking about the *exact* subject plane Σ and the *exact* image plane Π which correspond to one another via \mathcal{V} . If a point P is not in Σ , its image P' won't be in Π , but it will still produce a blurred extended image in Π . If that extended image is small enough, an observer may not be able to distinguish it from a point. We don't have the liberty of simultaneously using more than one image plane Π , so we have to make do with one exact subject plane. The set of all subject points, such that the extended images in Π are small enough, form a solid region in the subject space called the *depth of field*. We shall study depth of field later, but for the moment we just address exact image formation.

Let's now return to the abstract analysis. Consider the image $\Pi_F = \mathcal{V}(\Omega)$ of the plane at infinity. By the Scheimpflug Rule, Ω , Λ , and Π_F intersect in a line, necessarily at infinity, i.e., they are parallel. Π_F is called the *rear focal plane* or often just the *focal plane*. A point P at infinity is associated with a family of parallel lines, all those which intersect at P . One these lines OP passes through the principal point O , and $P' = \mathcal{V}(P)$, where OP intersects the rear focal plane, is the vanishing point associated with that family of parallel lines. All other light rays parallel to OP which pass through the lens are refracted to converge on P' . (If you've studied optics before, you will find this configuration in Figure 8 familiar.)

Consider next the subject plane Σ_F corresponding to the image plane Ω , i.e., $\mathcal{V}(\Sigma_F) = \Omega$ or $\Sigma_F = \mathcal{V}^{-1}(\Omega)$. By definition, all rays emanating from a point P in Σ_F come to focus at infinity. Σ_F is called the *front focal plane*.

Proposition 1. *The front and rear focal planes are antipodal.*

Proof. Let P be a point in the front focal plane. We want to show that $\mathcal{A}(P)$ is in the rear focal plane. But condition (3) tells us that $\mathcal{V}(\mathcal{A}(\mathcal{V}(P))) = \mathcal{A}(P)$. Since $\mathcal{V}(P)$ is at infinity, it is fixed by \mathcal{A} . That means that $\mathcal{V}(\mathcal{V}(P)) = \mathcal{A}(P)$. But, since $\mathcal{V}(P)$ is at infinity, $\mathcal{V}(\mathcal{V}(P))$ is in the rear focal plane. Similarly, using the condition $\mathcal{V}^{-1}\mathcal{A}\mathcal{V}^{-1} = \mathcal{A}$, we conclude that the antipodal point of a point in the rear focal plane lies in the front focal plane. \square

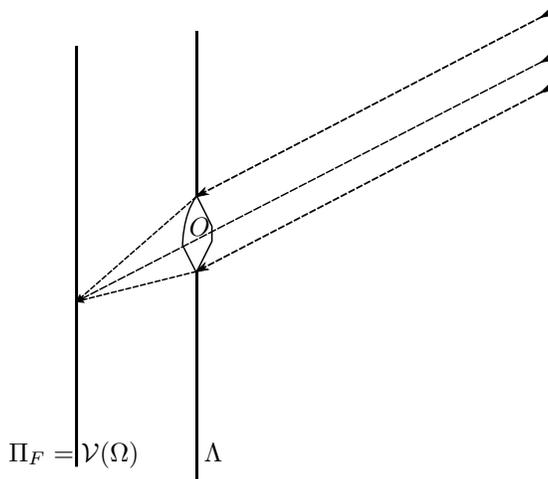


FIGURE 8. Rear Focal Plane

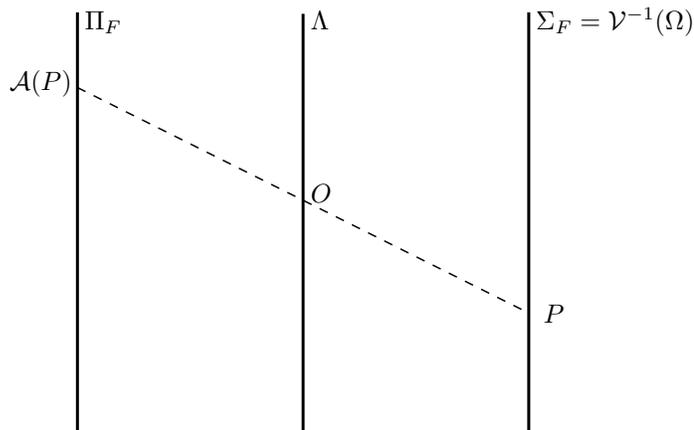


FIGURE 9. Front and Rear Focal Planes

Remark 1. *The statement in the Proposition implies condition (3) and so is equivalent to it.*

Here is an outline of the proof. First show that if the front and rear focal planes are antipodal then $\mathcal{V}\mathcal{A}\mathcal{V}$ and \mathcal{A} are the same on the front focal plane. That is not hard to show using the definitions and the fact that P, O , and $\mathcal{V}(P)$ are collinear. Similarly, it is not hard to see that $\mathcal{V}\mathcal{A}\mathcal{V}$ and \mathcal{A} fix the plane at infinity. Since they also fix the principal point, it follows that we can choose five points, no three of which are collinear on which both transformations agree. But a theorem in projective geometry asserts that two projective transformations are the same if and only if they agree on such a set of points.

As a consequence, we get the following important result.

Proposition 2. *The lens map \mathcal{V} is completely determined by the lens plane Λ , the principal point O , and the focal plane Π_F .*

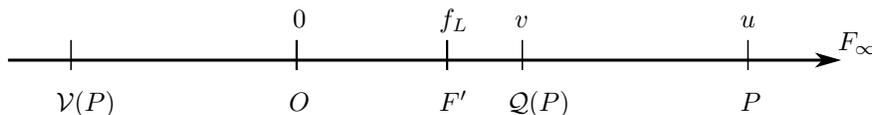
Proof. The (rear) focal plane and the front focal plane are antipodal, so if we know the first we know the second and vice versa. Any line through the origin is sent into itself by a lens map \mathcal{V} , since if P_1 and P_2 are collinear with O , then so are $\mathcal{V}(P_1), \mathcal{V}(P_2), P_1$, and P_2 . Fix such a line L let it intersect the rear focal plane at F , the front focal plane at F' and the plane at infinity at F_∞ . It is easy to see that $\mathcal{Q} = \mathcal{AV}$ interchanges F' and F_∞ , and it fixes O .

We now use the concept of the cross ratio, $\{P_1, P_2; P_3, P_4\}$, which is a number defined for any four collinear points. Given a projective coordinate system, we may assign an affine coordinate p to each point on such a line, and supposing p_i is the coordinate of P_i , the cross ratio is given by the formula

$$\{P_1, P_2; P_3, P_4\} = \frac{p_1 - p_3}{p_1 - p_4} \bigg/ \frac{p_2 - p_3}{p_2 - p_4}.$$

(One deals with $p = 0$ and $p = \infty$ in this this formula in the obvious ways.) The most important property of the cross ratio is that it is an invariant preserved by projective transformations.

Let P be any point on the line $OF_\infty = OF'$. Assume the coordinate system on that line has been chosen so that O has coordinate 0, F' has coordinate f_L , F_∞ has coordinate ∞ , P has coordinate u and $\mathcal{Q}(P)$ has coordinate v . Then $\{P, O; F', F_\infty\} = \{\mathcal{Q}(P), O; F_\infty, F'\}$.



In terms of coordinates that means

$$\frac{u - f_L}{u - \infty} \bigg/ \frac{-f_L}{-\infty} = \frac{v - \infty}{v - f_L} \bigg/ \frac{-\infty}{-f_L}.$$

The terms with infinities mutually cancel and we get

$$\frac{u - f_L}{-f_L} = \frac{-f_L}{v - f_L}$$

or

$$(LQ) \quad (u - f_L)(v - f_L) = (f_L)^2.$$

That means that $\mathcal{Q}(P)$ is completely determined by P and F' (since $\mathcal{Q}(F') = F_\infty$). Thus \mathcal{Q} is completely determined under the hypotheses, which means that $\mathcal{V} = \mathcal{AQ}$ is. \square

With a bit more algebra, formula (LQ) reduces to

$$(ALQ) \quad \frac{1}{u} + \frac{1}{v} = \frac{1}{f_L}$$

which should be recognized by anyone who has taken a basic physics course. Note that v here represents the coordinate of the antipodal point $\mathcal{A}(\mathcal{V}(P))$, so $\mathcal{V}(P)$ is actually on the other side of the origin. That means that v must be measured so its *positive* direction is *to the left*. The coordinate u of P is measured as usual to the right. (See Appendix B for a more geometric derivation of the lens equation using similar triangles.)

Note that generally, the values f_L will be different for different lines L through O , and with the formalism used so far, no one line is distinguished.. So there is no single lens equation, which applies uniformly. If you fix one line L to make the calculation, the lens equation for that line will only tell you what happens for points on that line. For a point P not on L , you must do the following. First project P onto L by seeing where the plane through P parallel to the lens plane intersects L . Use the equation for L with the value f_L to determine an image point on L . Finally see where the plane through that point parallel to the lens plane intersects the line OP , and that is $\mathcal{V}(P)$.

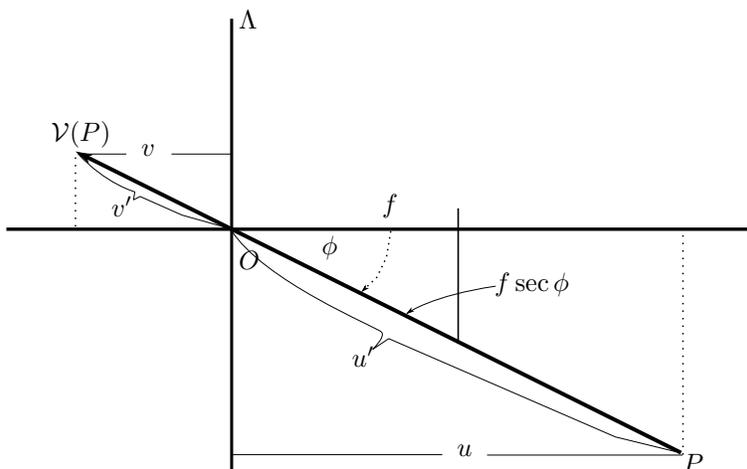


FIGURE 10. The Lens Equation

Of course, a real lens, in Euclidean space, has a *lens axis*, which is characteristic of the lens, so it makes sense to choose the lens axis as the line L , and to define the lens plane as the plane through the principal point perpendicular to the lens axis. The focal planes, which are parallel to the lens plane, are also perpendicular to the lens axis, and their intersections F and F' , which are equidistant from the principal point O , are called the *focal points of the lens*. Their distance f from the principal point is called the *focal length* of the lens.

From now on, we shall assume that the Euclidean structure is available, with distance and angles defined as usual, and the focal points and focal length are defined as above.

For points on the lens axis, equation (ALQ) holds, with u , v , and $f_L = f$, the focal distance, measured as Euclidean distances. For points not on the lens axis—see Figure 10—we must apply the more general rule discussed above to obtain

$$\text{(Lens Equation)} \quad \frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

where u and v refer respectively to the *perpendicular distances* of the subject point P and the image point $\mathcal{V}(P)$ to the lens plane, not their Euclidean distances u' and v' to the principal point O . People often misunderstand this. To obtain the appropriate equation relating u' and v' , note that the line $L = P\mathcal{V}(P)$ intersects the focal planes at Euclidean distance $f_L = f \sec \phi$, where ϕ is the angle between the lens axis and $P\mathcal{V}(P)$. Hence,

$$\text{(GLE)} \quad \frac{1}{u'} + \frac{1}{v'} = \frac{1}{f \sec \phi} = \frac{\cos \phi}{f}$$

6. THE HINGE LINE

In a camera without movements, the image plane is parallel to the lens plane, as is the subject plane, and the Scheimpflug line is at infinity. But, in a view camera, when the lens plane is tilted with respect to the image plane, knowing how to use the Scheimpflug line when focusing is crucial.

Beginning view camera photographers often think that the Scheimpflug Rule completely determines the positions of the subject and image planes, and this confuses them when trying to apply it. If you specify the image plane, all you know from Scheimpflug is that the subject plane passes through the intersection of the image plane with the lens plane. But there are infinitely many such planes and without further information, you don't know which is the actual subject plane. Similarly given the subject plane, there are infinitely many planes with pass through the Scheimpflug line and only one is the actual image plane.

Fortunately, Herr Scheimpflug provided us with another rule, which helps to determine exactly where either plane is, given the other.

Consider a setup where the rear standard is kept vertical and we focus by moving it parallel to itself along the rail, so that all potential image planes Π are parallel to one another. As we do, the line where Π intersects the plane at infinity stays fixed. Call that line I_∞ . Consider the plane through the principal point O which is parallel to the rear standard. We call that plane the *reference plane* and generally denote it by Δ .⁷ It also passes through that same line I_∞ at infinity. See Figure 11.

The front focal plane intersects the reference plane Δ in a line H which we shall call the *hinge line*. It is the inverse image of the line I_∞ under the lens map. It depends *only* on the lens map and the reference plane. It doesn't change as we focus by translating the image plane Π parallel to itself. Let Σ be the subject plane associated to Π , and consider the line H' where it intersects the front focal plane Σ_F . That line is carried by \mathcal{V} into the intersection of Π with the plane at infinity, which by our assumptions is I_∞ . Hence $H' = H$, the hinge line. So *all relevant subject planes pass through the hinge line*. In other words, we can envision

⁷For a real lens, which has physical extent, the location of the reference plane can be tricky. For most lenses used in view camera photography, it will be where the lens axis intersects the plane through the front of the lensboard, or close enough that the difference won't matter. For some telephoto lenses, this can get more complicated. See Section 10 for a bit more discussion.

the subject plane rotating about the hinge line as we focus, which explains the terminology.

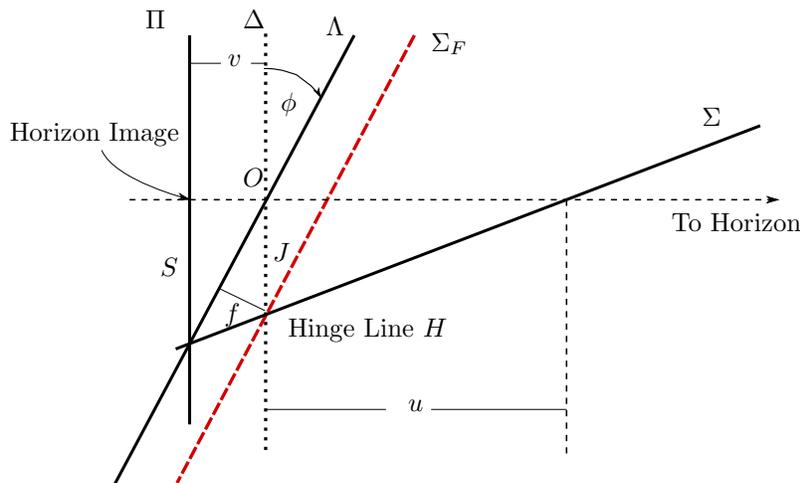


FIGURE 11. The Hinge Line

The hinge line appears in Scheimpflug's 1904 patent application, although he didn't claim credit for it, and used different terminology. Most photographers, including the current author, first learned about it, its history, and its uses from Merklinger.⁸ It is also called the pivot or rotation axis.

Important Note. Figure 11 is typical of the diagrams used in this subject. It implicitly suggests that the lens plane is tilted downward, and the image plane is vertical. This is by far the most common situation, but it doesn't by any means cover all possible cases. The lens plane could be tilted upward, with the Scheimpflug line and the hinge line above the lens. More to the point, the diagrams also apply to swings to one side or the other, where neither side is distinguished. In that case, the standard diagram would show things as they would appear if the observer were above the camera, looking down, and the lens plane was swung to the right. It gets even more complicated if the lens plane is tilted with respect to a skew axis, which is neither horizontal nor vertical, or if we tilt the back, hence, the reference plane, in which case, the whole diagram is tilted with respect to the vertical. However, whatever the tilt orientation, we may assume the standard diagram presents an accurate picture, simply by changing the orientation of the observer in space. So doing may require some verbal gymnastics. For example, for tilts about a horizontal axis, the horizon provides a reference to which the subject plane can be compared. There is no comparable reference when you swing about a vertical axis, but you could look at the vertical plane perpendicular to the image plane and its intersection with the plane at infinity. (Perhaps it should be called the 'verizon'— \downarrow .) In general, assuming the standards are not parallel, consider the line through the principal point parallel to intersection of the lens plane with the image plane (determined by the rear standard). The rotation of the front standard

⁸www.trenholm.org/hmmerk.

about that axis is what we are calling the tilt. Instead of the actual horizon, which exists independently of the orientation of the camera, you would use the plane perpendicular to the image plane and parallel to the axis of rotation, as well as its intersection with the plane at infinity. I leave it to the reader to deal with such matters. Just keep in mind, in any situation, the positions of image plane, the hinge line, the Scheimpflug line, and the rotation axis will guide you about how to interpret the words.

Introducing the hinge line simplifies considerably the problem of obtaining a desired subject plane. We first concentrate on where that plane intersects the reference plane, i.e., the desired hinge line. If we get that right, we need only focus the rear standard until the resulting subject plane swings into place.

With the front standard untitled and parallel to the rear standard, the hinge line is at infinity, but as we tilt the lens plane (forward), the hinge line moves (up) in the reference plane to a finite position.

The perpendicular distance J in the reference plane Δ from the principal point O to the hinge line H is called the *hinge distance*. It is clear geometrically that it can't be smaller than the focal length, which would be the case if the lens plane were turned 90 degrees so it was perpendicular to the image plane. In practice you never get anywhere near that, with typical rotation angles being a few degrees and never exceeding 15 or 20 degrees. From Figure 11, we have

$$(HD) \quad J = \frac{f}{\sin \phi}$$

where ϕ is the tilt angle, i.e., the angle that the lens plane makes with the reference plane and the image plane. It follows that the the tilt angle depends only on the position of the hinge line.

Note also that the distance S from the image of the horizon line to the Scheimpflug line is given by

$$(SchD) \quad S = v \cot \phi$$

But from equation (GLE), we have $\frac{\cos \phi}{f} = \frac{1}{v} + \frac{1}{u}$. (In Figure 11, u and v are now the *oblique* distances to O .) So

$$v \cos \phi = \left(1 + \frac{v}{u}\right)f = (1 + M)f \quad \text{or}$$

$$(VE) \quad v = f(1 + M) \sec \phi$$

where M is the magnification just for objects at distance u at which the subject plane intersects the *reference normal*. i.e., the line through O perpendicular to the reference (image) plane.⁹ Elsewhere in the subject plane, the magnification will be different.

Note that u and M may be negative, which would be the case when the subject plane tilts away from the reference normal. In that case, there is no subject point corresponding to u .

It now follows that

$$(SchDa) \quad S = v \cot \phi = v \frac{\cos \phi}{\sin \phi} = \frac{(1 + M)f}{\sin \phi} = (1 + M)J$$

⁹That line is usually drawn as if it were horizontal, the most common case. But as we noted before, it need not be.

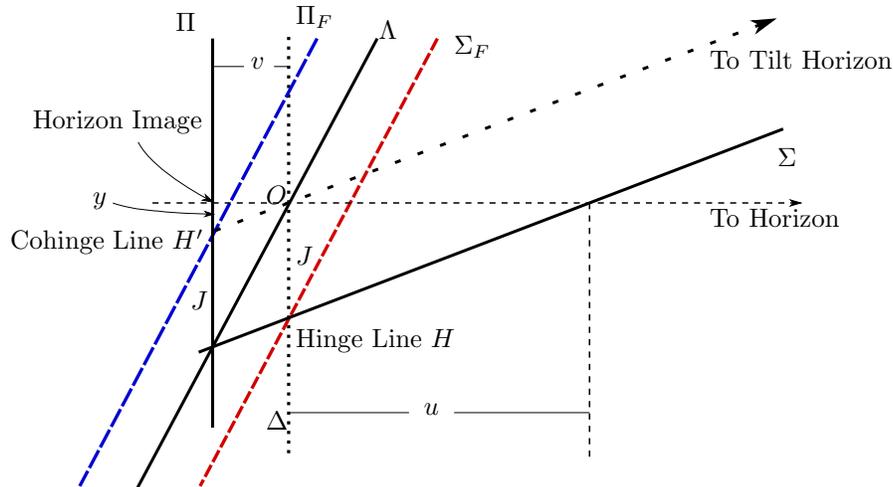


FIGURE 12. Cohinge Line

Note one important consequence of equation (VE). Since usually $|M| \ll 1$ and ϕ is pretty small, typically $v \approx f$. The exception would be in close-up situations when M can be a fairly large, possibly even larger than 1.

For a given orientation of the lens—which of course fixes the positions of the front and rear focal planes, any family of parallel subject planes determines a coreference plane and its intersection with the rear focal plane should be called the *cohinge line*. (See Figure 12 again.) It plays a role dual to that of the hinge line, but there aren't many circumstances in which it remains fixed as you manipulate the camera, so it is not used the same way as the hinge line.

But, the cohinge line can also be thought of as the *image* of the intersection of the subject plane with the plane at infinity, which we call the *tilt horizon*.¹⁰ All parallel line pairs, in the subject plane or parallel to the subject plane, have vanishing points on the cohinge line, and every such point is obtainable in that way. This can sometimes be useful as the photographer tries to visualize the placement of the subject plane by looking at the image on the ground glass. Also, nothing in the image plane below the cohinge line can be exactly in focus, i.e., no point in the subject above the tilt horizon can be in focus. See Figure 12.

From the diagram and equations (SchDa) and (HD), we get the following equation for the distance of the cohinge line, i.e., the image of the tilt horizon, below the image of the horizon.

$$(TH) \quad y = S - J = (1 + M)J - J = MJ$$

Note that if $M < 0$, then $y < 0$, which means that the tilt horizon image is on the opposite side of the reference normal from where we usually picture it.

We can now begin to see why large tilt angles ϕ are not common. Namely, except in close-up situations, $|M| \ll 1$, so, unless $J = \frac{f}{\sin \phi}$ is large enough, i.e., ϕ small

¹⁰It should be distinguished from the horizon, which is the intersection with the plane at infinity with the plane through O , perpendicular to the reference plane, and parallel to the hinge and cohinge lines. Of course, this language only makes sense in case of a pure tilt.

plane Λ . At that point the tilt angle will satisfy the relation $\sin \phi = \frac{f}{J}$. Let A and B two points in Σ , and suppose they come to focus, with the lens plane untilted, at A' and B' at distances v_A and v_B from O . Let $s = |v_B - v_A|$ be the focus spread between them, and let t be the vertical distance between A' and B' on the ground glass, which during this process may be translated parallel to itself when focusing. Then

$$\tan \alpha = \frac{s}{t} = \frac{v}{J} = \frac{v f}{f J} = (1 + M) \sin \phi$$

$$(WR) \quad \sin \phi = \frac{1}{1 + M} \frac{s}{t}$$

Usually, M is very small, and $1 + M \approx 1$. So

$$(AWR) \quad \sin \phi \approx \frac{s}{t}$$

Since the focus spread s and the distance t between the image points on the ground glass can be measured directly at the camera, this gives us a convenient method for calculating the required tilt angle ϕ .¹³ Wheeler simplified this by using the very rough approximation $\sin \phi \approx \frac{\phi \text{ (in degrees)}}{60}$ to yield

$$\phi \approx 60 \times \frac{s}{t} \text{ degrees}$$

But he also pointed out that one could avoid dealing with the angle ϕ altogether. Namely, if R is the distance from the tilt axis to the top of the standard, then $\sin \phi = \frac{d}{R}$ where d is the distance you should move the top of the standard forward from the vertical. Then the rule becomes

$$d \approx R \frac{s}{t}$$

Note that after tilting the lens through angle ϕ and refocusing as necessary, the distance v' of the image plane from O will be slightly different from v because of the tilt. In fact, for the distance u , to where the subject plane crosses the line of sight, fixed we have

$$v = \frac{uf}{u - f}$$

$$v' = \frac{uf}{u \cos \phi - f} \quad \text{and, after lots of algebra}$$

$$(FocSh) \quad \frac{v' - v}{v} = \frac{1 - \cos \phi}{\frac{1}{1+M} - (1 - \cos \phi)}$$

Tilt angles are usually pretty small, and as noted above M is usually much less than 1. Using $\cos \phi \approx 1 - \phi^2/2$ and $1/(1 + M) \approx 1 - M$, after some additional fiddling, we find that the relative shift is bounded approximately by $\frac{\phi^2(1 + M)}{2}$. So it turns

¹³In practice, few experienced view camera photographers will take the time to use such a rule. Usually one can make a pretty good first guess, and then home in on the final tilt by reducing the focus spread between A' and B' . The simplest way to do this is to focus first on the far point A , and then refocus on the near point B . If that requires increasing the distance between the standards, increase the tilt, and if it requires decreasing the distance, decrease the tilt. Usually two or three iterations suffice.

out that the relative shift, while noticeable, particularly for larger tilts, is pretty small. For example, for $\phi = 0.25$ radians (about 14.3 degrees) and $M = 0.01$, the relative shift turns out to be about 0.03 or 3 percent. For $f = 150$ mm, $v = 151.5$ mm, the shift would be about 5 mm. For a more modest tilt of 0.1 radians (about 5.7 degrees), the relative shift would be about 0.005 or 0.5 percent, and for the same f and v , the shift would be about 0.8 mm, which would be just barely detectable.

7. REALIZABLE POINTS

There are certain forbidden regions which are not available for subject or image points. Look at Figure 12 once more, and concentrate on the two focal planes.

Subject points to the right of the front focal plane Σ_F in the diagram map to image points to the left of the rear focal plane Π_F . As you approach the front focal plane, the image point moves to infinity. Subject points between the front focal plane and the lens plane, even if they are in the subject plane, map to points in the subject space. Such points cannot be captured by film or a device. Sometimes such image points are called virtual images. You “see” a magnified virtual image if you put a magnifying glass closer to a subject than its focal length, but it can’t be captured without additional optics to turn it into a real image. In effect that is what your eye does when you use a magnifying glass. In any case, this plays no role in photography. The points to the left of the lens plane get mapped by the lens map to the part of the image space between the lens plane and the rear focal plane. For such points the subject and image points are on the same side of the lens plane, so the light ray would have to double back on itself, and there is no image which can be captured. We see from this that the region between the focal planes is forbidden and no point in it corresponds to a realizable subject or image point.

That means that no point between the hinge line and the lens plane produces a real image, and no point between the lens plane and the cohinge line is a real image. In particular, no point in the frame below the cohinge can be exactly in focus, so no subject point above the tilt horizon can produce an image exactly in focus. But it is possible that some points not in the exact subject plane have slightly blurry images which are still sharp enough. (See the extensive discussion of depth of field in Section 8.3.)

As we shall see later, the finite dimensions of the camera also put further restrictions on which points can be realized.

8. DEPTH OF FIELD

As mentioned earlier, we have to put the film (image plane) Π at a specific location, so there is one and only one subject plane Σ corresponding to it. A subject point P not in Σ will produce an image point P' not in Π , but it may still be close enough that it won’t matter. This effect is called *defocus*.

The lens is not a point, although, for the purposes of geometric optics, we can treat it as such. In reality, there is a lens opening or *aperture* through which light rays pass. As the diagram shows the light rays which start at a subject point P will emerge from the lens as a solid cone with vertex at the image point P' , and base the aperture. The aperture is usually a regular polygon with five or more sides, and it is simplest to assume it is a circular disc, although that is not essential. Under that assumption, the cone will intersect the desired exact image plane Π in

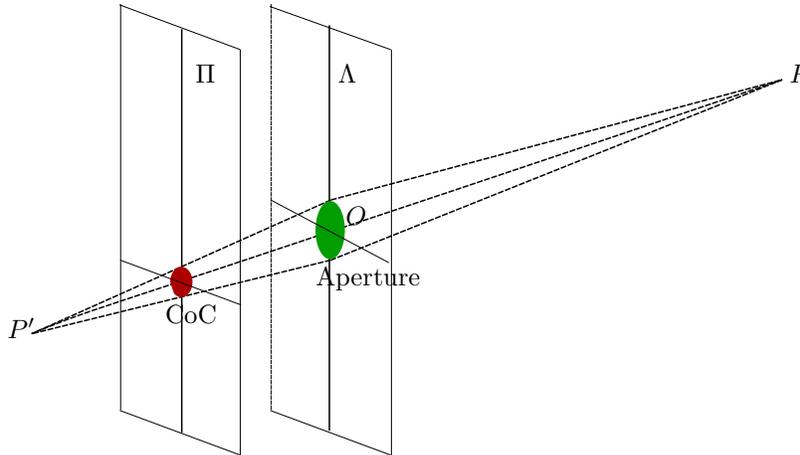


FIGURE 14. Circle of Confusion

a region, which, if the film plane is parallel to the aperture, will be a circular disc, but in general it will be an ellipse. So we call that region the *circle of confusion*—abbreviated *CoC*, whether it is actually a circle or not. If the circle of confusion is not too large, it can't be distinguished from the point where the axis of the cone intersects Π . For all such points it is as if we were using a pinhole camera with the pinhole at the center of the aperture. The *pinhole image* of P' will be the intersection of the line connecting the center of the aperture, i.e., the principal point O , to P' .

Usually we set a criterion for how large the circle of confusion can be and still be indistinguishable from a point. That will depend on factors such as the intended final image, how much it must be enlarged from the image recorded in the camera, and how, and by whom, it is viewed. For a given position of the subject plane (and corresponding image plane), there will be a region about it, called the *depth of field region* or, often, just the *depth of field*. and it is abbreviated 'DOF'. It should be emphasized that there generally won't be a sharp cutoff at the boundaries of depth of field. Other factors, such as contrast and edge effects, affect perception of sharpness, so detail outside the region may still appear sharp. But, at least in principle, everything within the region should appear to be in focus, and details well outside it should appear to be out of focus¹⁴.

Most of the rest of the article will be concerned with determining the shape of this region under different circumstances. It turns out that if the subject plane, lens plane, and image plane are all parallel, i.e., no tilt, the DOF will be contained between two planes parallel to both. If the lens plane is tilted, then the region is *almost* a wedge shaped region between two bounding planes. By far, the greatest part of this article will be concerned with quantifying the term 'almost', which is something that is usually ignored.

¹⁴Analysis of depth of field usually ignores diffraction, which has an overall blurring effect on the image. See the article by Jeff Conrad at www.largeformatphotography.info/articles/DoFinDepth.pdf, for a discussion of diffraction.

Before beginning that discussion, it is worthwhile saying something about how a photographer might approach the problems presented by a given scene. Typically, one wants certain parts of the scene to be in focus and will first try to achieve that, without tilt, by stopping down far enough. If that doesn't work because the scene extends from very near in the foreground to the distant background, one will then try to decide if the situation may be improved by tilting the lens somewhat. That may work provided that the vertical extent one needs in focus close to the lens is not too great. In the process, one will at some point settle on a narrow range for the desired subject plane, and that, as we have seen, *will determine the tilt angle within certain narrow limits. So normally, we will consider the tilt angle essentially fixed and proceed from there.*

Suppose we specify as before that the image plane Π may only be adjusted by translation parallel to itself. Then, all corresponding subject planes meet in the hinge line (which may be at infinity). Suppose we pick two such subject planes, or equivalently, two corresponding image planes. The image plane farther from the lens is called the *outer image plane* and that closer to the lens is called the *inner image plane*. The corresponding subject planes Σ' and Σ'' are called respectively the *upper and lower subject planes* or sometimes the *near and far subject planes*. The latter terminology is more appropriate in the case of zero tilt, for, then all the planes are parallel and the former plane is indeed nearer to and the latter plane is further from the lens plane. But it can be misleading if the tilt is not zero, since the upper limit of what can be in focus in the distance is determined by the upper or 'near' plane, and the lower limit of what can be in focus in the foreground is controlled by the lower or 'far' plane. By assumption, the image planes are all parallel, so there is less ambiguity. The outer image plane corresponds to the upper (near) subject plane, and the inner image plane corresponds to lower (far) subject plane. But, there is still another confusion in terminology. The distances v', v, v'' of the image planes to the reference plane determine positions on the rail. Depending on the context, the position v' should properly be called either the near point or the upper point, and v'' should be called the far point or lower point. See Figure 15.

In the diagram, we assume we have already tilted the lens plane. Note that while the hinge line stays fixed, the Scheimpflug lines differ for the two planes.

There are now two interesting questions to ask.

(1) Given Σ' and Σ'' (equivalently Π' and Π'') as above, where shall we focus, i.e., place the exact subject plane Σ so that points in Σ' and Σ'' are "equally out of focus" in Π in a sense to be discussed. (Some people have argued for favoring one plane over the other, but we shall not get into that controversy in this article.)

(2) Suppose we have specified a criterion for sharpness, and for whatever reason, we have decided on an exact image plane Π . How far can we place Π' and Π'' on either side of Π so all points between them will satisfy that criterion?

A problem associated with (2) is

(2a) Suppose we start with Π' and Π'' as in (1) and find the best place to place Π . How can we control the circles of confusion in Π from points between Π' and Π'' and ensure they satisfy our criterion for sharpness?

Note that if we start with Π and find Π' and Π'' solving problem (2), then, intuitively, at least, Π *should be* the plane solving problem (1) for those two planes. The argument would be that if Π were not already in the position which best

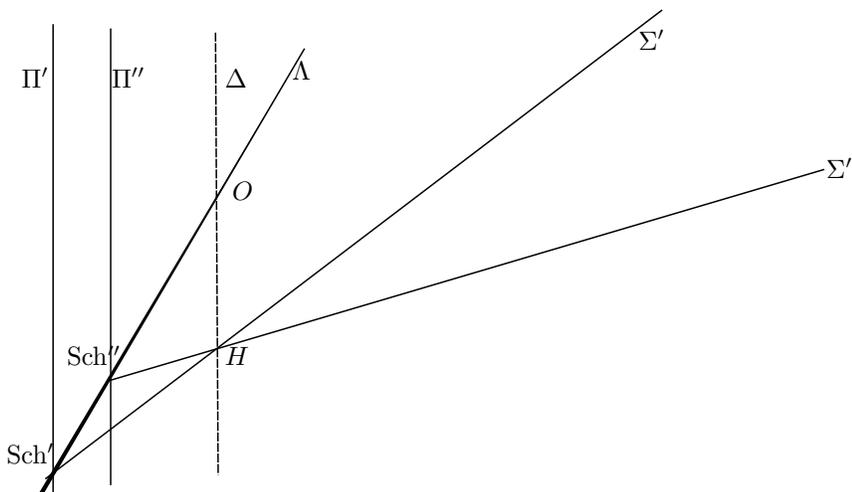


FIGURE 15. Upper (Near) and Lower (Far) Planes

balanced the focusing, then we could improve the balance by moving it. But such a movement should upset the sharpness condition for one of the two planes, so we already were at the right location. Unfortunately, we shall see that it requires some care to say just what we mean by “equally out of” or “equally in” focus, and for the case of tilted lens plane, both problems and their solutions become rather murky.

In principle, the aperture need not be in the lens plane, but it usually is or is close to it, so we shall start off under the assumption that it is a disc in the lens plane centered on the principal point. (We shall discuss cases where it is not in Section 10.1.) The aperture is formed by a number of blades, usually at least six, and, as noted earlier, it is actually a regular polygon rather than a circle. Its size can be varied by moving the blades, a process called *stopping down* when the aperture is made smaller or *opening up* when it is made larger. The ability to do this plays a crucial role in trying to obtain desired depth of field.

Before proceeding, let us review some facts about the cross ratio. We already discussed the cross ratio for a set of four collinear points. Given four distinct planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4$, which, intersect in a common line, we may define the cross ratio $\{\Pi_1, \Pi_2; \Pi_3, \Pi_4\}$ to be $\{P_1, P_2; P_3, P_4\}$ where P_1, P_2, P_3, P_4 are the intersections with the four planes of any sufficiently general line. (It is shown in projective geometry that this cross ratio is independent of the line.) The following facts about cross ratios are useful.

(i) Reversing either the first two or last two arguments in a cross ratio inverts the cross ratio.

(ii) $\{t_1, t_3; t_2, t_4\} = 1 - \{t_1, t_2; t_3, t_4\}$.

(i) and (ii) together allow us to determine the cross ratio of four arguments in any order from that in one specific order. the possible values are $t, \frac{1}{t}, 1-t, \frac{1}{1-t}, 1-\frac{1}{t}$, and $\frac{t}{t-1}$.

(iii) Given 5 collinear points (planes) t_0, t_1, t_2, t_3, t_4 , we have

$$\frac{\{t_1, t_0; t_3, t_4\}}{\{t_2, t_0; t_3, t_4\}} = \{t_3, t_4; t_1, t_2\}.$$

8.1. Case I. The lens is not tilted with respect to the reference plane. As we shall see, things work out very nicely in this case, and both problems have clear solutions.

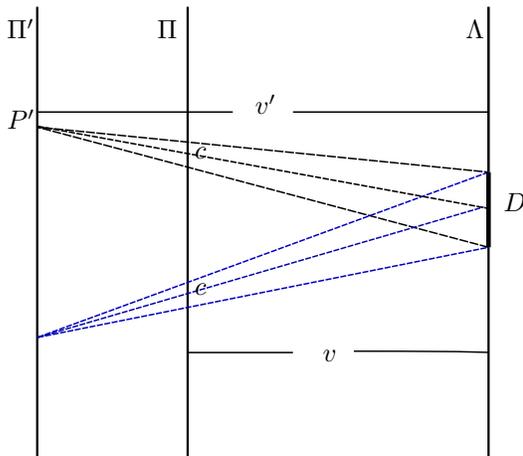


FIGURE 16. The Basic Diagram

8.1.1. *Problem (1).* Assume the aperture has diameter D . Choose a plane Π between the outer and inner planes Π' and Π'' . Fix attention on Π' . Choose a point P' in Π' , and form the solid cone with vertex P' and base the aperture. Let Q be the pinhole image in Π ; i.e., the intersection of Π with OP' . Let c be the diameter of the disc in which the cone intersects Π . Note that the ratio $\frac{c}{D}$ remains fixed as long as the plane Π' remains fixed. That is clear by considering the similar triangles in Figure 16, which shows a slice by a p plane through the principal point.

The ratios of corresponding sides in similar triangles are equal. So, we have

$$(A) \quad \{\Pi', \Omega; \Pi, \Lambda\} = \{v', \infty; v, 0\} = \frac{v' - v}{v'} = \frac{c}{D}$$

Now let P'' be a point on OP' in the outer image plane Π'' on the same side of Π as the lens plane. Clearly, Π will be in the optimum position between Π' and Π'' when the circle of confusion produced by P'' is identical with that produced by P' .

So we also have

$$(B) \quad \{\Pi'', \Omega; \Pi, \Lambda\} = \{v'', \infty; v, 0\} = \frac{v'' - v}{v''} = -\frac{c}{D}.$$

Hence, by dividing (A) by (B), we obtain

$$(H) \quad \{\Pi, \Lambda; \Pi', \Pi''\} = \{v, 0; v', v''\} = \frac{v - v'}{v - v''} \bigg/ \frac{-v'}{-v''} = -1$$

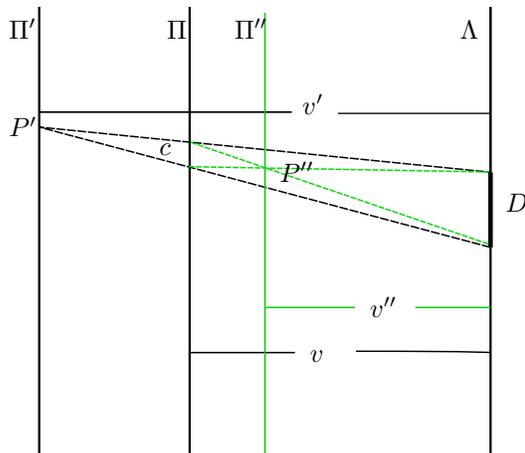


FIGURE 17. Balance

It is easy to check that this is true if and only if

$$(H') \quad \frac{1}{v} = \frac{1}{2} \left(\frac{1}{v'} + \frac{1}{v''} \right)$$

(H') by definition says that v is the *harmonic mean* of v' and v'' , and similarly we say that a plane Π is the harmonic mean of Π' and Π'' relative to a plane Π_0 if $\{\Pi, \Pi_0; \Pi', \Pi''\} = -1$. Note that this relation doesn't depend on the order of Π', Π'' since reversing order of either the first two or the last two arguments in a cross ratio just inverts the cross ratio.

It is sometimes more useful to rewrite equation (H') as

$$(H'') \quad v = \frac{2v'v''}{v' + v''}$$

which it reduces to by some simple algebra.

In practice, it might be tricky determining the position of the harmonic mean, but fortunately there is a simple approximation which makes focusing much easier. It is based on the relation (H'), which may also be written

$$(Ha) \quad \frac{v'' - v}{v - v'} = \frac{v''}{v'}$$

For normal photography where all interesting parts of the image are relatively distant, the values of v' and v'' are not much greater than the focal length, and hence are relatively close to one another, so it follows from (Ha) that $v'' - v \approx v - v'$, i.e., that *we can place the image plane halfway between the near (outer) and far (inner) points on the rail* without being very far off, and that simplifies focusing enormously. In fact, detailed calculations show that the error made in so doing is negligible, even in close-up situations except for very short focal length (wide angle) lenses, a situation almost never met in practice.

8.1.2. *Problem(2)*. Specify the criterion by setting a maximum possible diameter c for a circle of confusion. We want to determine just how far from Π we may put Π'

and Π'' so any point between them will yield a circle of confusion in Π of diameter not greater than c .

To this end, we use the same equations (A) and (B). Start at Π and move away from it in both directions until the two cross ratios are equal to $\mu = \frac{c}{D}$ and $-\mu$ respectively. (Note also that in this case Π will end up being the harmonic mean of Π' and Π'' by the same argument as before.) Solving those equations for v' and v'' yields

$$(V) \quad v' = \frac{v}{1 - \mu} \quad v'' = \frac{v}{1 + \mu}.$$

We also see how to solve problem (2b). Namely, the cross ratios in (A) and (B) are fixed by the positions of the planes, so the ratio $\mu = \frac{c}{D}$ is determined by that information. Now, given c , simply choose D , by stopping down, so that $\frac{c}{D} = \mu$ or $D = \mu c$.

In photography, one seldom uses the actual diameter of the aperture but instead the *f-number* $N = \frac{f}{D}$ where f is the focal length.¹⁵ (This is because the f-number is more directly related to the light intensity at P' .) Hence, it is more useful to rewrite the crucial ratio $\mu = \frac{Nc}{f}$. So, given a maximum possible diameter c for the circle of confusion, we need to choose $N = \frac{f\mu}{c}$

On the other hand, when $v = \frac{2v'v''}{v' + v''}$ is chosen as the harmonic mean of v' and v'' , we have

$$\mu = \frac{v' - v}{v'} = 1 - \frac{v}{v'} = 1 - \frac{2v''}{v' + v''} = \frac{v' - v''}{v' + v''},$$

so

$$(N) \quad N = \frac{v' - v''}{c} \frac{f}{v' + v''}.$$

But from (E), we get

$$v' + v'' = v \left(\frac{1}{1 + \mu} + \frac{1}{1 - \mu} \right) = \frac{2v}{1 - \mu^2}$$

So

$$N = \frac{v' - v''}{2c} \frac{f}{v} (1 - \mu^2).$$

On the other hand, from the lens equation, as in Section 6, we get

$$\frac{v}{f} = \frac{v}{u} + 1 = M + 1$$

¹⁵The f-number is usually denoted symbolically as the denominator of a symbolic ‘fraction’ as in ‘f/32’ for f-number 32. f is not suppose to have a specific value, and you are not supposed to perform a division, but if it were the actual focal length then f/N would be the actual diameter of the aperture, so this terminology can be confusing. We shall abuse the terminology at times in this article by referring to ‘aperture N ’ when we mean that the f-number is N .

where $M = \frac{v}{u}$ is the *magnification* or *scale of reproduction* at the exact plane of focus. Hence,

$$(1) \quad N = \frac{v' - v''}{2c(1 + M)}(1 - \mu^2) < \frac{v' - v''}{2c(1 + M)}.$$

Overestimating N is usually innocuous, so the quantity on the right is a reasonable estimate for the appropriate f-stop. In any case, μ^2 is typically small,¹⁶ so we may say

$$(N') \quad N \approx \frac{v' - v''}{2c(1 + M)}$$

Except in close-up situations, the scale of reproduction M is quite small and can be ignored, in which case the estimate becomes $N \approx \frac{v' - v''}{2c}$. In any case, if we ignored M , that would just lead us to stop down more than we need to. Usually that is innocuous since we are interested in adequate depth of field and don't care if we get a bit more than we expected¹⁷. But there are occasions when one wants to limit depth of field to precisely what is needed, in which case one would keep the factor $1 + M$.

8.1.3. *The Subject Space.* What does all this say about what is happening in the subject space? If we remember that the lens map and its inverse are projective transformations and so preserve cross ratios, it follows that $\{\Sigma, \Lambda; \Sigma', \Sigma''\} = -1$, i.e., *the best place to focus is at the harmonic mean of the near and far¹⁸ subject planes with respect to the lens plane.* In particular, if the distances of the near and far subject planes from the lens plane are respectively u' and u'' , then you should focus at the harmonic mean u of those distances where

$$(SH) \quad \frac{1}{u} = \frac{1}{2} \left(\frac{1}{u'} + \frac{1}{u''} \right).$$

For scenes in which u' and u'' reasonably close, we may use the analogue of equation (Ha)

$$(SHa) \quad \frac{u'' - u}{u - u'} = \frac{u''}{u'}.$$

to conclude that $u - u' \approx u'' - u$, i.e., that the front depth of field and the rear depth of field are approximately equal. That holds true not only for close-up photography but also for normal portraiture. A reasonable estimate for that distance—see 8.2—is

$$\frac{cN(1 + M)}{M^2}$$

so it depends only on the circle of confusion c , the f-number N and the scale of reproduction M . In particular, it is essentially independent of the focal length as

¹⁶An extreme estimate for μ with $N = 64$, $c = 0.1$ and $f = 65$ is $\frac{NC}{f} < 64 \times 0.1/65 \approx 0.1$, so $\mu^2 < .01$ or 1 percent. That works for 4 x 5 format. For 8 x 10, with $N = 128$, $c = 0.2$ and $f = 130$ mm, it would be less than 4 percent. Either would amount to an insignificant difference in setting the f-stop.

¹⁷Stopping down does require increasing the time setting to maintain exposure, and in some cases that may create problems because of subject movement.

¹⁸We switch from 'upper' and 'lower' to 'near' and 'far' since they are more appropriate in this case.

long as the scale of reproduction is kept fixed. One often sees this rule stated as if it held in general, whatever the subject distance, but it clearly fails for subjects sufficiently remote from the lens. In particular, as we see below, the far depth of field is often infinite.

8.1.4. *The Hyperfocal Distance.* If the far plane is at infinity, i.e., $u'' = \infty$, then equation (SH) tell us that $u = 2u'$, in other words the correct place to focus is at twice the distance of the near plane.

On the other hand, suppose we stop down to aperture (f-number) N . At which distance should we focus so that far plane bounding the depth of field is at infinity? Since the plane at infinity is mapped by the lens map to the focal plane, that amounts to setting $v'' = f$, so equation (B) reduces to $\frac{v-f}{f} = \mu = \frac{c}{D} = \frac{f}{Nc}$, which yields $v - f = Nc$ or $v = Nc + f$. The corresponding subject plane distance may be derived from the relation $(u - f)(v - f) = f^2$ which yields

$$(Hyp) \quad u_H = f + \frac{f^2}{Nc} \approx \frac{f^2}{Nc}$$

(where the approximation holds as long as Nc is much smaller than f which is *always the case, except for extremely short focal length lenses.*) The distance u_H is called the *hyperfocal distance*. If you focus at the hyperfocal distance, then infinity will just barely be in focus. To do so, you place the rear standard back from the focal plane by the quantity Nc , which is independent of the focal length.

But, as we saw above, $u'' = \infty$ implies that $u = u_H = 2u'$, the near distance. In other words, *if you focus at the hyperfocal distance, everything from half that distance to infinity will be in focus.*

Let's look at a typical example. $c = 0.1$ mm is a typical value taken for the diameter of the maximum allowable circle of confusion in 4 x 5 photography, and $f = 150$ mm is a typical "normal" focal length. A typical f-number would be $N = 22$. So, in this case $Nc = 2.2$ mm. Similarly, $f = 150 \gg 2.2$, so the hyperfocal distance is given accurately enough by $\frac{f^2}{Nc} = \frac{150^2}{2.2} \approx 10227$ mm ≈ 10 meters. Half the hyperfocal distance will be about 5 meters. An extremely short focal length for such a camera would be $f = 50$ mm, and in this case $\frac{50^2}{2.2} \approx 1136$ mm. In this case, it might be more accurate to add the focal length to get $1186 = 1.186$ meters. Everything from infinity down to about 0.593 meters should be in focus.

Notice that the hyperfocal distance only makes sense given a choice of relative aperture N . If you stick with the same aperture but focus so that $u > u_H$, you will place the inner image plane Π'' in the forbidden region, so 'image points' between Π'' and the focal plane will not correspond to real subject points. (See Section 7.) The result will be that you will lose some far depth of field. See Section 8.2 below for formulas which allow you to determine the near depth of field in that case.

It is hard to measure small distances that precisely along the rail, but most view cameras employ a gearing mechanism which magnifies motions along the rail for fine focusing. Using a scale on the focusing knob in such cases makes it much easier. The alternative is to calculate the hyperfocal distance, or get it from a table, and then focus on some object in the scene at that distance. Rangefinder devices are available for measuring such distances, but it seems overkill to use them, since the camera itself effectively does it for you.

8.2. Depth of Field Formulas. Assume we focus at distance u , at aperture (f-number) N , with circle of confusion c , and let $u_H = H + f$, where $H = \frac{f^2}{Nc}$, be the hyperfocal distance. Recall formulas (A) and (B)

$$\begin{aligned}\{\Pi', \Omega; \Pi, \Lambda\} &= \frac{Nc}{f} \\ \{\Pi'', \Omega; \Pi, \Lambda\} &= -\frac{Nc}{f}\end{aligned}$$

As before, since cross ratios are preserved, and \mathcal{V} carries the front focal plane Σ_F into the plane at infinity Ω , we obtain

$$\begin{aligned}\{\Sigma', \Sigma_F; \Sigma, \Lambda\} &= \frac{Nc}{f} \\ \{\Sigma'', \Sigma_F; \Sigma, \Lambda\} &= -\frac{Nc}{f}\end{aligned}$$

or

$$\begin{aligned}\{u', f; u, 0\} &= \frac{Nc}{f} \\ \{u'', f; u, 0\} &= -\frac{Nc}{f}\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{u' - u}{u'} &= \frac{Nc}{f} \frac{f - u}{f} \\ \frac{u'' - u}{u''} &= -\frac{Nc}{f} \frac{f - u}{f}\end{aligned}$$

After some algebra, these lead to the equations

$$\begin{aligned}u' &= \frac{u}{1 + \frac{Ncu}{f^2} - \frac{NC}{f}} \\ u'' &= \frac{u}{1 - \frac{Ncu}{f^2} + \frac{NC}{f}}\end{aligned}$$

In each case, multiply the numerator and denominator by the quantity $H = \frac{f^2}{NC}$. This yields

$$\text{(NDOF)} \quad u' = \frac{uH}{H + (u - f)}$$

$$\text{(FDOF)} \quad u'' = \frac{uH}{H - (u - f)}$$

Since $u > f$, the denominator in equation (NDOF) is positive, and the formula always makes sense. But the denominator in equation (FDOF) may vanish or even be negative, so that formula requires a bit of explanation.

If $u = H + f$, the formula yields $u'' = \infty$, which just says the Σ'' is at infinity as in the previous section (8.1.4).

Suppose on the other hand that $u > u_H$, so the denominator is negative. In that case, the formula yields a negative value for u'' . As we saw in the previous section, that means that $v'' < f$, i.e., Π'' is in the forbidden zone. Restricting

attention to real subject points, we ignore the second equation and conclude that the depth of field extends from u' , given by equation (NDOF), to ∞ . In the extreme case in which one focuses at $u = \infty$, equation (NDOF) tells us $u' = H$, which for all practical purposes is the hyperfocal distances. In other words, *if you focus at infinity, then everything down to the hyperfocal distance will be in focus.*

If one's aim is to maximize depth of field, there is little point in focusing at a distance greater than the hyperfocal distance, so it is common to add the restriction $u \leq u_H$.¹⁹

If $u \gg f$, i.e., we are not in a close-up situation, then the above equations yield the approximations

$$u' \approx \frac{uH}{u+H}$$

$$u'' \approx \frac{uH}{u-H}$$

It is also useful to determine the front and rear depth of field

$$\text{(FDOF)} \quad u - u' = \frac{u(u-f)}{H+(u-f)} \approx \frac{u^2}{H+u}$$

$$\text{(RDOF)} \quad u'' - u = \frac{u(u-f)}{H-(u-f)} \approx \frac{u^2}{H-u}$$

where the approximations hold when we are not in a close-up situation. In the case of close-ups, it is better to divide the numerator and denominator by f^2 to obtain

$$u - u' = \frac{\frac{u}{f}(\frac{u}{f} - 1)}{\frac{1}{Nc} + \frac{1}{f}(\frac{u}{f} - 1)}$$

$$u'' - u = \frac{\frac{u}{f}(\frac{u}{f} - 1)}{\frac{1}{Nc} + \frac{1}{f}(\frac{u}{f} - 1)}$$

If we use the relation $\frac{u}{f} = \frac{1}{M} + 1$, we obtain after lots of algebra

$$u - u' = \frac{Nc(1+M)}{M^2(1 - \frac{Nc}{fM})}$$

$$u'' - u = \frac{Nc(1+M)}{M^2(1 + \frac{Nc}{fM})}$$

If $\frac{Nc}{fM} \ll 1$, then, we may ignore the second term in parentheses in the denominators to obtain

$$\text{(Ha)} \quad u - u' \approx u'' - u \approx \frac{Nc(1+M)}{M^2}$$

which we mentioned earlier.

¹⁹There may be circumstances in which one wants to favor the background relative to the foreground. In that case, one might sacrifice some or all of the rear depth of field.

8.3. Case II. Tilted lens plane. In this case, the aperture, which we have assumed is in the lens plane, is tilted with respect to the reference plane Δ , and this complicates everything.²⁰

To see why, fix a potential image plane Π parallel to the reference plane, and choose an image point P' not in Π . Consider the solid cone with vertex P' , and base the aperture, which is a circular disc in the lens plane of radius $R = D/2$. Let Q be the intersection of OP' with Π . The cone will intersect the plane $\tilde{\Pi}$ through Q , parallel to the lens plane, in a circular disc, but the circle of confusion in the desired image plane Π will be an elliptical region. In Figure 18, we show some examples of how the ellipse compares to the circle for different values of the parameter. (Note that the circle and the ellipse are in different planes, tilted with respect to one another, so the diagrams don't reflect what you would see if you looked at them in space.)

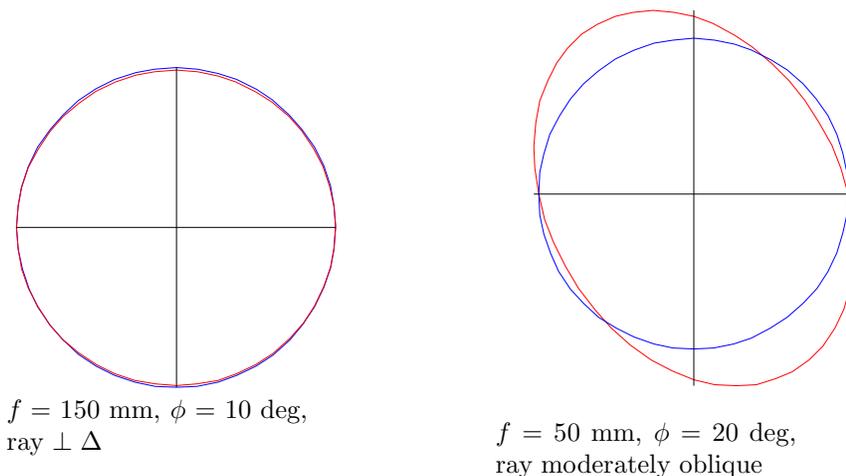


FIGURE 18. Circles of Confusion (circle in blue, ellipse in red)

The ellipse won't be centered on Q , although, as in the examples in Figure 18, the shift is usually quite small. On the other hand, its size, shape, and orientation will depend strongly on the focal length, the tilt, the distance from P' to O and the orientation of the ray OP' relative to the reference plane. Moreover, if we specify outer and inner image planes Π' and Π'' , as before, and points, P' and P'' in each of them, there will be no simple relation between the corresponding circles of confusion, even in the simplest case that the points in the two planes are on the same ray through O . So it is not clear precisely how to analyze Problem (1), and it is even harder to analyze Problem (2). We shall see, however, that, if we take into

²⁰My interest in this subject was piqued by reading Section 10.5. of Wheeler's Notes [/www.bobwheeler.com/photo/ViewCam.pdf](http://www.bobwheeler.com/photo/ViewCam.pdf) where he considers the problem, two-dimensionally, in the central plane ($x_1 = 0$ in our notation below). Unfortunately, the problem is fundamentally three-dimensional, and ignoring the third-dimension, transverse to the central plane, as we shall see, can lead to misperceptions. I have always been amazed at Wheeler's success in making such calculations using just algebra, trigonometry, and simple geometry. Indeed, I had no luck extending his results until I thought of using projective geometry.

account the limitations imposed by the scene and the mechanical structure of the camera and remember that the tilt angle ϕ is almost always relatively small, then things don't get too bad. But the analysis is rather involved, so we now address it.

In all that follows, we assume $\phi \neq 0$, i.e., that the lens plane really is tilted with respect to the reference plane.

With P' and Π as above, consider a plane cross section through P' as in Figure 19.

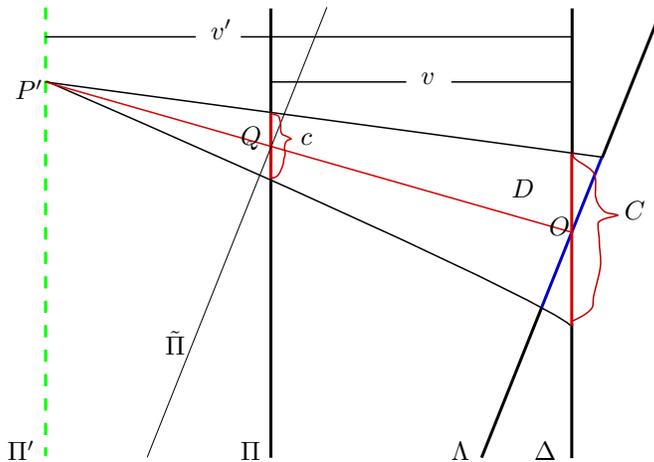


FIGURE 19. Circle of confusion in Image Plane and Aperture

Here c is the length of the segment of the circle of confusion cut off by the cross sectional plane, and C is the corresponding length cut off in the reference plane Δ . v and v' are as before the perpendicular distances of P' and Π to the reference plane Δ . Π' is the plane through P' parallel to the reference plane.

As in the untilted case, we have the following analogue of equation (A).

$$(A') \quad \{\Pi', \Omega; \Pi, \Delta\} = \{v', \infty; v, 0\} = \frac{v' - v}{v'} = \frac{c}{C}$$

where C has replaced D and the reference plane Δ has replaced the lens plane Λ . Note the following crucial fact: the cross sectional plane in the Figure 19 need not contain O (nor the pinhole image Q). So, C can be any chord in the reference plane ellipse, and c the corresponding chord of the circle of confusion in Π , and that includes the major axes of the two ellipses!

The difficulty is that the chord length C depends strongly on both the tilt angle and the point P' , so although the ratio $\frac{c}{C}$ is fixed by the positions of Π and P' , unlike the untilted case, the value of c is not.

So, to make further progress we have to spend some time analyzing the relation between the ellipse in the reference plane Δ , and the aperture, which is a circular disc of diameter $D = 2R$ in the lens plane Λ .

As indicated in the Figure 20, we choose an orthogonal coordinate systems x_1, x_2, x_3 in the reference plane, and one X_1, X_2, X_3 in the lens plane, which share the common origin O and the same $x_1 = X_1$ -axis, which is the line of intersection of the reference plane with the lens plane. The x_3 and X_3 axes are perpendicular

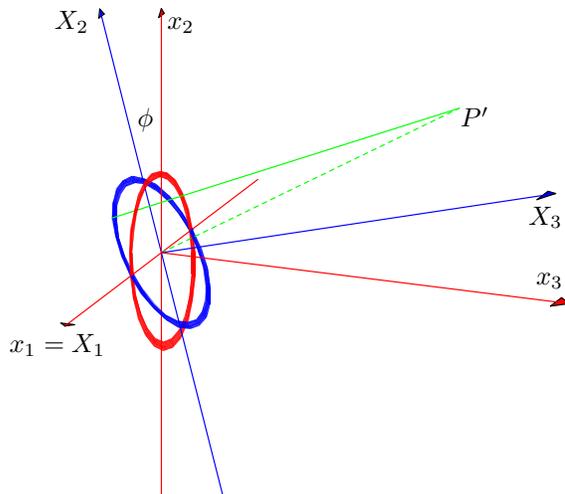


FIGURE 20. The Basic Coordinate Systems

to the reference plane and lens plane respectively, so the angle ϕ between them is just the tilt angle. Note that we have chosen left hand orientations, which is more appropriate for the image space. In the subject space, we would choose right hand orientations by reversing the direction of the x_3 and X_3 axes. We sometimes also use the notations v and u to denote the coordinates along the respective x_3 axes in the subject and image spaces respectively, to be consistent with the terminology used in the lens equation. Both v and u are non-negative in their respective spaces.

Let the point P' have coordinates (p_1, p_2, p_3) in the reference plane coordinate system. Put $p = \sqrt{(p_1)^2 + (p_2)^2 + (p_3)^2}$ and $(p_1, p_2, p_3) = p(j_1, j_2, j_3)$, so $\mathbf{j} = (j_1, j_2, j_3)$ defines the orientation of the ray OP' , and its components are its direction cosines. We must have $p_3 > 0$ (hence $j_3 > 0$), since, otherwise the circle of confusion in the image plane produced by P' will be *outside the camera*. Also, P' must be above the lens plane Λ , so for the ray OP' , we must have $j_2 > -j_3 \cot \phi$. But using $\sin \phi = \frac{j_3}{\sqrt{j_2^2 + j_3^2}}$, this can be rewritten $j_2 > -\sqrt{1 - j_1^2} \cos \phi$. It may also be rewritten $j_2 \sin \phi + j_3 \cos \phi > 0$, and that will be useful below. In fact, there are much more stringent restrictions on j_2 and j_3 which we shall get to later.

Since $x_3 = 0$ in the reference plane and $X_3 = 0$ in the lens plane, points in those planes are described by coordinate pairs (x_1, x_2) and (X_1, X_2) . In Appendix C, we investigate the perspectivity \mathcal{P} through P' which maps the lens plane to the reference plane and consider how it is expressed in terms of these coordinates.

The following facts follow from that analysis.

(a) The point at infinity on the common $x_1(X_1)$ -axis is fixed since every point on that axis is fixed. So also are the points O , and $(\pm R, 0)$.

(b) The point at infinity on the X_2 -axis in the lens plane gets mapped to the point $P_\infty = (p_1, p_2 + p_3 \cot \phi)$ in the reference plane.

(c) Parameterizing the aperture circle by $\mathbf{X} = (X_1 = R \cos \theta, X_2 = R \sin \theta)$, $0 \leq \theta \leq 2\pi$, leads to the following parametrization of the reference ellipse: $\mathbf{x} = (x_1, x_2)$

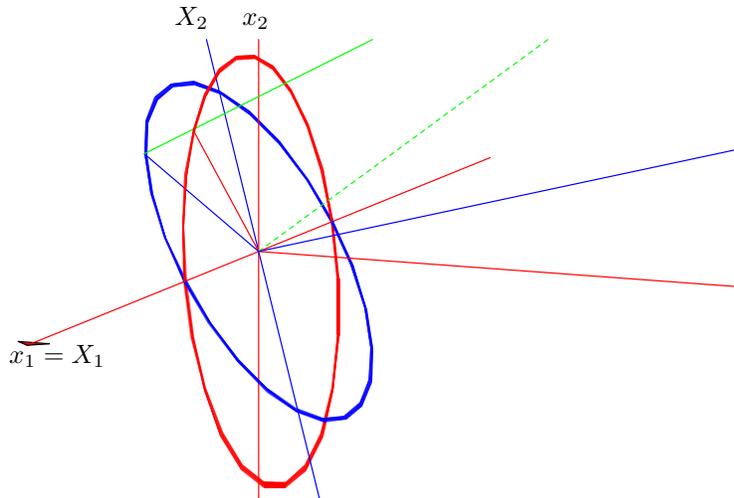


FIGURE 21. Mapping the Circle to the Ellipse

where²¹

$$(P1) \quad x_1 = \frac{p_3 R \cos \theta + p_1 \sin \phi R \sin \theta}{p_3 + R \sin \phi \sin \theta}$$

$$(P2) \quad x_2 = \frac{(p_2 \sin \phi + p_3 \cos \phi) R \sin \theta}{p_3 + R \sin \phi \sin \theta}$$

(See Figure 21.)

It is useful to rewrite these equations as follows. Divide both numerator and denominator by p_3 , so the denominator becomes $1 + \frac{R \sin \phi \sin \theta}{p_3} = 1 + \epsilon \sin \theta$, where

$$\epsilon = \frac{R \sin \phi}{p_3} = \frac{f \sin \phi}{2Np_3}$$

where $N = \frac{f}{2R}$ is the f-number discussed earlier. Next, factor out the common magnitude p , and express everything in terms of (j_1, j_2, j_3) to get

$$(P1') \quad x_1 = \frac{j_3 R \cos \theta + j_1 \sin \phi R \sin \theta}{j_3} \left(\frac{1}{1 + \epsilon \sin \theta} \right)$$

$$(P2') \quad x_2 = \frac{(j_2 \sin \phi + j_3 \cos \phi) R \sin \theta}{j_3} \left(\frac{1}{1 + \epsilon \sin \theta} \right)$$

The quantity ϵ will play a crucial role in our analysis.

The polar coordinate θ' of (x_1, x_2) , i.e., the angle the vector from O to that point makes with the positive x_1 -axis is determined by $\tan \theta' = \frac{x_2}{x_1}$, so, because the

²¹We regularly omit dependence on the variables $\theta, \mathbf{j} = (j_1, j_2, \sqrt{1 - j_1^2 - j_2^2}), \phi, R$, etc. Such abuse of terminology, which includes identifying dependent variables with the functions they depend on, is innocuous, and simplifies notation, as long as the reader is aware of it. Also, in any given case, some of the variables such as \mathbf{j}, ϕ , and R , are parameters, meaning they are temporarily kept constant for that part of the discussion.

factors involving ϵ cancel, θ' is independent of ϵ , and we have

$$(T1) \quad \tan \theta' = \frac{(j_2 \sin \phi + j_3 \cos \phi) \sin \theta}{j_3 \cos \theta + j_1 \sin \phi \sin \theta} = \frac{(j_2 \sin \phi + j_3 \cos \phi) \tan \theta}{j_3 + j_1 \sin \phi \tan \theta}$$

$$(T2) \quad \theta' = \arctan \left(\frac{(j_2 \sin \phi + j_3 \cos \phi) \tan \theta}{j_3 + j_1 \sin \phi \tan \theta} \right) \quad -\pi/2 < \theta' < \pi/2$$

In (T2), θ' , as noted, is returned in the usual range for the arctangent function, so one must correct appropriately for other values. Note also that the factors R cancelled.

Note, however, that replacing θ by $-\theta$ doesn't replace θ' by $-\theta'$, so generally $\mathbf{x}(-\theta)$ doesn't have direction opposite to $\mathbf{x}(\theta)$, but there is one exception, as we shall see below, $\theta = \pm \frac{\pi}{2}$.

(d) In particular, for $\theta = \pm \pi/2$, the points $\bar{Q}_+ = (0, +R)$ and $\bar{Q}_- = (0, -R)$ in the lens plane get mapped respectively to the points $Q_+ = (q_1^+, q_2^+)$ and $Q_- = (q_1^-, q_2^-)$ where

$$(E1) \quad q_1^+ = \frac{j_1 R \sin \phi}{j_3} \frac{1}{1 + \epsilon}$$

$$(E2) \quad q_2^+ = \frac{(j_2 \sin \phi + j_3 \cos \phi) R}{j_3} \frac{1}{1 + \epsilon}$$

and

$$(E3) \quad q_1^- = -\frac{j_1 R \sin \phi}{j_3} \frac{1}{1 - \epsilon}$$

$$(E4) \quad q_2^- = -\frac{(j_2 \sin \phi + j_3 \cos \phi) R}{j_3} \frac{1}{1 - \epsilon}$$

Note that the line Q_+Q_- passes through O and has slope

$$m = \frac{j_2 \sin \phi + j_3 \cos \phi}{j_1 \sin \phi}$$

which one easily checks is the same as that of OP_∞ , so the points P_∞, Q_+, O, Q_- are collinear. Also, the slope m of this line, is independent of ϵ , i.e., depends only on the direction of the ray OP' and not on the position of P' on it.

Note also that since the numerator in (E2), $j_2 \sin \phi + j_3 \cos > 0$, Q_+ is in fact above the x_1 -axis, as indicated in the diagram. So according to (E1), it is in the first quadrant if $j_1 > 0$ and in the second quadrant if $j_1 < 0$, and, of course, Q_- is in the opposite quadrant below the x_1 -axis. (If $j_1 = 0$, Q_+ is on the positive x_2 axis, and Q_- is on the negative x_2 -axis.) It is geometrically clear that the major axis must be in the same quadrant as Q_+ , so this leads to the following conclusion: for $j_1 > 0$, the major axis extends from the first quadrant to the third quadrant, and for $j_1 < 0$, it extends from the second quadrant to the fourth quadrant. In words, *the major axis of the reference ellipse extends from closer to the lens plane opposite the ray to further from the lens plane on the same side as the ray.* This holds true whatever the sign of j_2 , which is surprising, from a cursory examination of the three-dimensional geometry, but it does become clear if you look hard enough.

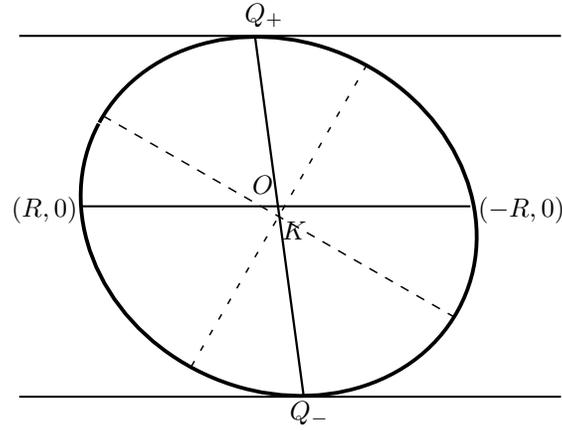
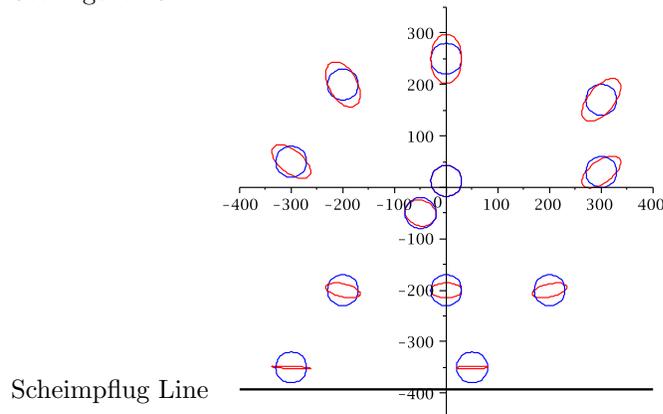


FIGURE 22. Points with Horizontal Tangents

Finally, note that as $j_2 \sin \phi + j_3 \cos \phi \rightarrow 0$, the points Q_+ and Q_- approach each other, so that in the limit, the ellipse approaches a line segment along the x_1 axis.²² See Figure 23.

FIGURE 23. Sample Circles and Ellipses of Confusion in the Image Plane under a Microscope, for $v = 100$ mm and $\phi \approx 14.3$ degrees

(e) The tangents at \bar{Q}_+ and \bar{Q}_- are parallel to the $X_1(x_1)$ -axis. Since tangents to a conic are preserved by a projective transformation, so also are the tangents to the reference ellipse at Q_+ and Q_- . (See Figure 22.) Since two points on an ellipse have parallel tangents if and only if they are antipodal, it follows that the midpoint

²²If one is in doubt about the suitability of using the major axis alone to estimate the effect of the CoC on sharpness, one place to look would be rays with P' close to the lens plane. But for P' to be close to the the lens plane, the upper bounding surface in the subject space would have to be close to the front focal plane, which is highly implausible. The other place to look would be where the major axis of the reference ellipse is large compared to the diameter of the aperture, but, as we shall see, such points are trouble anyway. In any event, they may be excluded because of mechanical limits on possible shifts.

$K = (k_1, k_2)$ of Q_+Q_- is the center of the ellipse. Hence,

$$\begin{aligned} k_1 &= \frac{q_1^+ + q_1^-}{2} \\ &= \frac{j_1 R \sin \phi}{2j_3} \left(\frac{1}{1 + \epsilon} - \frac{1}{1 - \epsilon} \right) \\ &= -\frac{j_1 R \sin \phi}{j_3} \frac{\epsilon}{1 - \epsilon^2} \end{aligned}$$

Similarly

$$\begin{aligned} k_2 = mk_1 &= -\frac{j_2 \sin \phi + j_3 \cos \phi}{j_1 \sin \phi} \frac{j_1 R \sin \phi}{j_3} \frac{\epsilon}{1 - \epsilon^2} \\ &= -\frac{(j_2 \sin \phi + j_3 \cos \phi) R}{j_3} \frac{\epsilon}{1 - \epsilon^2} \end{aligned}$$

Note that although the distance between the principal point O and the center K of the reference ellipse looks pretty small in Figure 22, it has been exaggerated for effect, so typically it is even smaller than indicated. From the above formulas, since $\frac{\epsilon}{1 - \epsilon^2} \approx \epsilon$ for small epsilon, it is usually so small that it would not be visible in an accurate graph, and that was clear in typical examples using Maple to plot the graphs.

To proceed, we need to see how the reference ellipse varies as we change the projection point $P' = (p_1, p_2, p_3) = p(j_1, j_2, j_3)$. As noted earlier, the parameter $\epsilon = \frac{f \sin \phi}{2Np_3}$ will play an important role. We shall study it and its effect in detail in Appendices D.1 and D.3, where we consider the various factors which may come into play. The worst possible estimate, under plausible assumptions is $\epsilon \leq 0.02$ meaning that if we ignore it, we make an error of at most 2 percent. In practice, it is usually much less than that, but the discussion in the appendices gives us an idea of when it might be large enough to matter.

If we fix the ray and let P' go to infinity, then $\epsilon \rightarrow 0$. Hence, the reference ellipse approaches the limiting reference ellipse centered at O , with parametrization

$$(L1) \quad x_1 = R \frac{j_3 \cos \theta + j_1 \sin \phi \sin \theta}{j_3}$$

$$(L2) \quad x_2 = R \frac{(j_2 \sin \phi + j_3 \cos \phi) \sin \theta}{j_3}$$

In general, it is still not true for the limiting reference ellipse that antipodal points on the circle get mapped to antipodal points on the ellipse. But it is true for $\theta = \pm \frac{\pi}{2}$. In that case, the line through O with slope

$$m = \frac{j_2 \sin \phi + j_3 \cos \phi}{j_1 \sin \phi}$$

intersects the limiting ellipse at antipodal points with horizontal tangents and such that

$$q_2^\pm = \pm R \frac{(j_2 \sin \phi + j_3 \cos \phi)}{j_3}.$$

Also, the tangents at $(\pm R, 0)$ are parallel to this line and so have the same slope.

For the limiting ellipse, the equations (T1) and (T2) describing the relation between the polar angle θ in the lens plane and the polar angle θ' in the reference

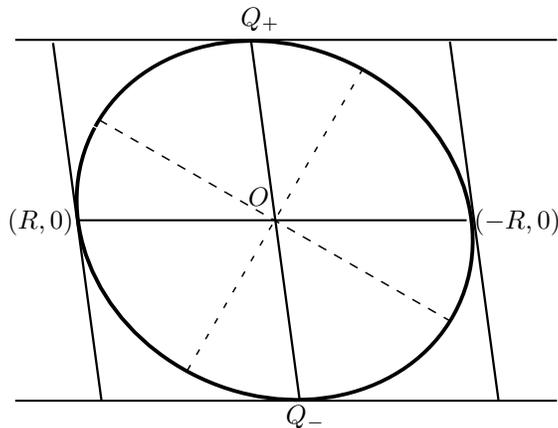


FIGURE 24. The Limiting Reference Ellipse

plane remain true because they were independent of ϵ . Clearly, it suffices to examine that dependence for the upper half of the ellipse, i.e., for $0 \leq \theta \leq \pi$. It is geometrically clear that, for $j_1 > 0$, except at $0 = \theta = \theta'$ and $\pi = \theta = \theta'$, we have $0 < \theta < \theta' < \pi$, and for $j_1 < 0$ the inequality is reversed. With some difficulty it may also be verified analytically.

For any finite P' , since ϵ is quite small, the corresponding reference ellipse differs only slightly from this limit. As we shall see in Appendices D.1 and D.2, the direction cosines (j_1, j_2, j_3) are severely limited by the possible placement of the image plane because of bounds on movements of the rear standard and also by the nature of plausible scenes. But they could vary enough to make a significant difference in the size and shape of the limiting reference ellipse, and that complicates the analysis.

8.3.1. *Problem (1)*. As before, suppose outer and inner image planes Π' and Π'' , each parallel to the reference plane Δ , are given. Fix a ray with direction cosines (j_1, j_2, j_3) and let that ray intersect the two planes in P' and P'' respectively. Because of the above analysis, the reference ellipses for P' and P'' are both quite close to the limiting reference ellipse for that ray, so *we shall treat them as identical in all that follows*. (See Appendices D.1 and D.3 for a discussion of how much that affects the calculations.)

Consider now a plane Π between the outer and inner image planes, and look at the circles of confusion in Π from P' and P'' respectively, which will be similar to the limiting reference ellipse, or so close that we can ignore any difference. We may conclude as in the untilted case, that the circles of confusion will agree just when Π is the harmonic mean of Π' and Π'' with respect to the reference plane. See Figure 17, entitled “Balance”, but note that D in the diagram gets replaced here by C as in Figure 19.

As before, we have

$$(C) \quad \{\Pi', \Omega; \Pi, \Delta\} = \{v', \infty; v, 0\} = \frac{v' - v}{v'} = \frac{c}{C} = \frac{v - v''}{v''} = -\{\Pi'', \Omega; \Pi, \Delta\}$$

where $\frac{c}{C}$ is fixed for the ray and the choice of Π .

Consider now what happens in the subject space. Saying the image points are on the same ray is the same thing as saying that the corresponding subject points are also on the same ray through O , i.e., they are along the same line of sight through O , which is normally the center of perspective (or generally very close to it).

Thus, since the lens map \mathcal{V} sends the reference plane into itself, and since cross ratios are preserved, it follows that that if, the subject plane Σ is set at the harmonic mean of the upper and lower planes with respect to the reference plane, then points in those planes, *which are along the same ray through the principal point*, will be equally in focus or out of focus. (Since one such subject point will obscure the other, we would in practice look at points close to the a common ray, but not necessarily exactly on it.) However, unlike case I (lens plane not tilted), where it didn't make any difference, this won't assure us that the same is true for other points in the two planes, which are not in line with one another. The problem of finding the best position for Σ in that case doesn't appear to have a nice general solution, so we will to content ourselves, for the while, with solving the more restricted problem.

The problem is that if, for each pair of points P' and P'' in the upper and lower bounding planes, you choose a point P between them which produces the best balance of circles of confusion for those points, then the locus of such points P won't even be a plane, but just a fuzzy region centered on the harmonic mean plane. We shall try to say a little more about these matters when we look at Problem (2).

Choosing Σ to be the harmonic mean of the upper plane Σ' and the lower plane Σ'' turns out to be quite simple.

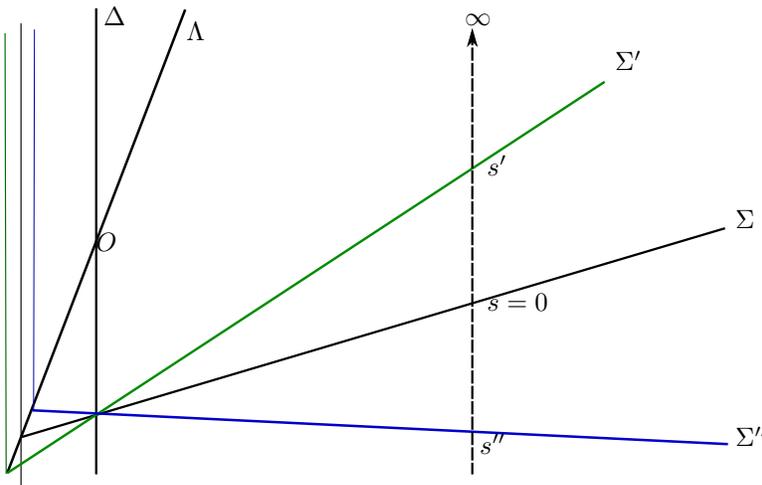


FIGURE 25. Vertical Split

Consider any plane in the subject space parallel to the reference plane, and look at a cross section perpendicular to it, and an axis in the intersection. (See Figure 25.) Choose a coordinate s on that axis so the upper and lower subject planes are positioned at s' and s'' respectively, and the harmonic mean is at $s = 0$. The

coordinate of the reference plane will be ∞ . So we have

$$-1 = \{\Sigma, \Delta; \Sigma', \Sigma''\} = \{0, \infty; s', s''\} = \frac{s'}{s''}$$

so $s'' = -s'$. In other words,

Proposition 3. *The harmonic mean of the upper and lower planes relative to the reference plane bisects the plane region between those planes in any cross sectional plane parallel to the reference plane.*

8.3.2. *Problem (2).* With the same setup as before, suppose we pick a subject plane Σ and its corresponding image plane Π . Let v be the perpendicular distance from Π to the reference plane, and consider a point P' not on Π at perpendicular distance v' from the reference plane. (v' is what we called p_3 before.) Refer to Figure 26 which shows the limiting circle of confusion and the limiting reference ellipse, and compare it to Figure 19. Here C is the major axis of the limiting reference ellipse and c is the corresponding major axis of the limiting circle of confusion. The centers O' and Q' will be shifted slightly from O and Q , but, as noted before, the shifts will be proportional to ϵ and hence quite small. *In all that follows, we shall ignore the small error made by using the limiting reference ellipse and the limiting circle of confusion.*

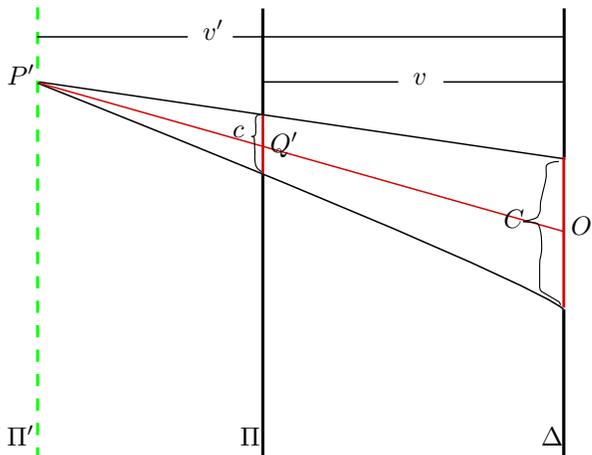


FIGURE 26. Limit Circle of confusion and Limit Reference Ellipse

We first consider the case where P' is on the far side of Π from the reference plane, as Figure 26. We have

$$\frac{v' - v}{v'} = \frac{c}{C} = \frac{c}{D} \frac{D}{C} = \frac{Nc}{f} \frac{D}{C} = \frac{Nc}{f} \kappa.$$

where $\kappa = \frac{D}{C}$. Note that the quantity on the left is the cross ratio $\{\Pi', \Omega; \Pi, \Delta\}$ where Π' is the plane parallel to Π containing P' . Since c is the major axis of the circle of confusion, if we make sure that its diameter (i.e., major axis) is always

small enough, the circle of confusion won't be distinguishable from a point.²³ Let c be the maximum such diameter, and solve for v' to get

$$(SN) \quad v' = \frac{v}{1 - \kappa \frac{Nc}{f}}.$$

This equation defines a surface in the image space, which we shall call the *outer surface of definition*. All points between the plane Π and that surface will produce sufficiently sharp images in Π . Points beyond it will appear blurred. Unfortunately, this simple equation hides a lot of complexity. If κ were constant, the surface of definition would be a plane, but, as we saw previously, κ is far from constant, being highly dependent on the direction \mathbf{j} of the ray OP' .

So, let us look more carefully at $\kappa = \frac{D}{C}$.

Since C is the major axis of the reference ellipse and D is the length of a chord, it follows that $\kappa \leq 1$.²⁴ First suppose that $j_1 = 0$ (so $q_1^\pm = 0$). In this case the reference ellipse is symmetrical about the $x_1 = X_1$ -axis, and $2q_2^+$ is its minor axis provided it is less than $D = 2R$ and otherwise it is its major axis. (When $2q_2^+ = D$, the ellipse is a circle.) So, $\kappa = 1$ as long as $2q_2^+ \leq D$. i.e.,

$$\frac{j_2 \sin \phi + j_3 \cos \phi}{j_3} \leq 1.$$

Otherwise,

$$\frac{1}{\kappa} = \frac{j_2 \sin \phi + j_3 \cos \phi}{j_3} > 1$$

so $\kappa < 1$ and $\kappa \rightarrow 0$ as $j_2 \rightarrow 1, j_3 \rightarrow 0$.²⁵

To see what happens as we move laterally away from the $j_1 = 0$ plane, we look at where

$$\frac{dx_1}{d\theta} = R \frac{-j_3 \sin \theta + j_1 \sin \phi \cos \theta}{j_3} = 0.$$

This occurs when

$$\begin{aligned} \tan \theta &= \frac{j_3}{j_1 \sin \phi} \\ \sin \theta &= \frac{j_3}{\sqrt{j_3^2 + j_1^2 \sin^2 \phi}} \\ \cos \theta &= \frac{j_1 \sin \phi}{\sqrt{j_3^2 + j_1^2 \sin^2 \phi}} \end{aligned}$$

²³One might question whether the major axis is the right thing to look at. Possibly, some other measure associated with the ellipse might be more appropriate. Presumably, if the minor axis is much smaller than the major axis, there will be an astigmatic-like effect with a loss of sharpness along one direction and an actual improvement transverse to it. Since one never knows just what kind of detail will be present in a scene, it seems to me that one must worry about the worst case effects, not some sort of average effect. To settle the matter would require experimental investigations beyond the scope of this article. Unless someone show definitively that some other measure works better, I will use the major axis.

²⁴If $\kappa = 1$, then $v' = \frac{v}{1 - Nc/f}$ which is the estimate which would apply in the untilted case.

²⁵A simple geometric argument shows the dividing point is where the ray (defined by the unit vector $\mathbf{j} = (0, j_2, j_3)$) makes an angle $\frac{\phi}{2}$ with the x_3 -axis.

That defines two points on the ellipse, and taking the one with $0 \leq \theta \leq \pi$, we get

$$x_1 = R \frac{\sqrt{j_3^2 + j_1^2 \sin^2 \phi}}{j_3} = \sqrt{1 + \left(\frac{j_1 \sin \phi}{j_3}\right)^2}.$$

The semi major axis is greater than this quantity, so its reciprocal $\kappa < 1$, and $\kappa \rightarrow 0$ as $j_3 \rightarrow 0$, i.e., $\sqrt{j_1^2 + j_2^2} \rightarrow 1$, as long as $j_1 \neq 0$.

Similarly, the surface with equation

$$(SF) \quad v'' = \frac{v}{1 + \kappa \frac{Nc}{f}}.$$

describes the surface formed by all points on the side closer to the lens for which the circles of confusion in Π will have diameter c . We shall call it the *inner surface of definition*. It approaches the plane Π from the other side as the ray becomes more oblique.

We show some examples of the what the outer surface of definition looks like in Figure 27.

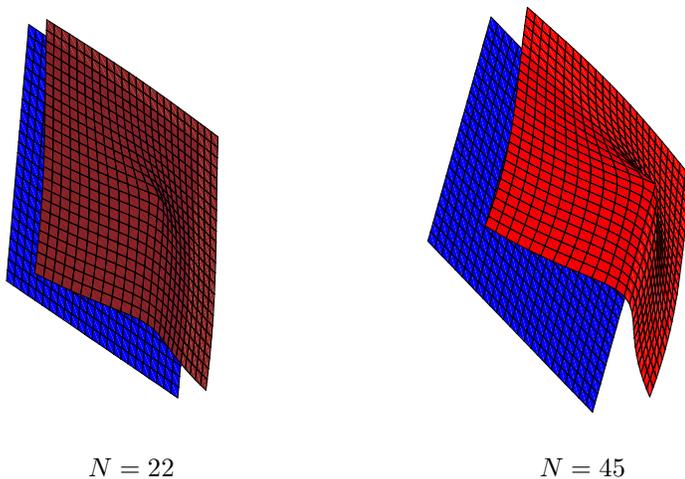


FIGURE 27. Outer Surface of Definition, $f = 150$ mm, $c = 0.1$ mm, $\phi = 14.3$ degrees, approaching exact image plane

As the diagram indicates, these surfaces depend on N as well as c (and of course f .) As N increases, the surface moves farther away from Π . It approaches the plane Π (in blue in Figure 27) as the ray becomes more oblique.

To measure the effect of κ 's deviation from 1, note that in the fraction $\frac{N\kappa c}{f}$, we may always compensate for κ by increasing N appropriately, i.e., by stopping down. It is usual to measure changes in the aperture in terms of stops or fractions of a stop rather than with the f-number. So, in this case, we have

$$\text{Number of stops} = \frac{2 \log(\kappa)}{\log(2)}.$$

To proceed further, we must make some estimates for κ in typical situations encountered in view camera photography. We save the detailed calculations for Appendix D, but we shall outline some of the results below.

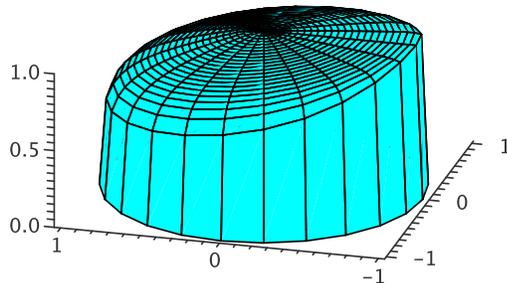


FIGURE 28. κ as a function of the unit vector (j_1, j_2) for $\phi = 14.3$ degrees

κ may be noticeably less than one if the tilt angle is large and/or the vector $\mathbf{j} = (j_1, j_2, j_3)$ departs significantly from the normal to the reference plane. Figure 28 shows the graph of κ as a function of (j_1, j_2) for the case $\phi = 0.25$ radians ≈ 14.3 degrees.²⁶ That is larger than one usually needs in practice, but still, we see that κ remains fairly close to 1 over much of the range.

In Figure 29, we show the contour curves for $\kappa = 0.9$ and $\kappa = 0.8$, enclosed within the unit circle in the j_1, j_2 -plane.

It is clear from the figure that, for those examples, the smallest values of $j = \sqrt{j_1^2 + j_2^2}$ occur on the j_2 axis. Solving numerically, we find that the crucial values are $j_2 \approx 0.5$ for $\kappa = 0.9$ and $j_2 \approx 0.75$ for $\kappa = 0.8$. The first corresponds to an angle of about 30 degrees with respect to the normal to the reference plane, and the second to an angle of about 48.6 degrees. To compensate for $\kappa = 0.8$, we would have to stop down almost two thirds of a stop, whereas to compensate for $\kappa = 0.9$, we would have to stop down less than one third of a stop.²⁷

While it is helpful to know κ as a function of (j_1, j_2) , it is even more enlightening to understand how it varies as a function of (x_1, x_2, v) representing a point in the image plane at distance v from the reference plane. In Section D, I shall prove Proposition 5 which asserts the following

²⁶The graph is not entirely accurate since it includes points for many excluded rays, e.g. such that $j_2 < -\sqrt{1 - j_1^2} \cos \phi$. Also the rate of change of κ at the boundary is too great for accurate plotting, particularly near $(0, -1)$. Indeed, as j_2 approaches -1 along the j_2 -axis, κ goes from 1 at $j_2 = -\cos(\phi/2)$, which is very close to -1 , to zero at $j_2 = -1$.

²⁷Despite appearances, each contour curve crosses the j_2 -axis, not at $(0, -1)$, but slightly above it. Also, as above, points such that $j_2 < -\sqrt{1 - j_1^2} \cos \phi$, while contributing to the graph, are associated with excluded rays.

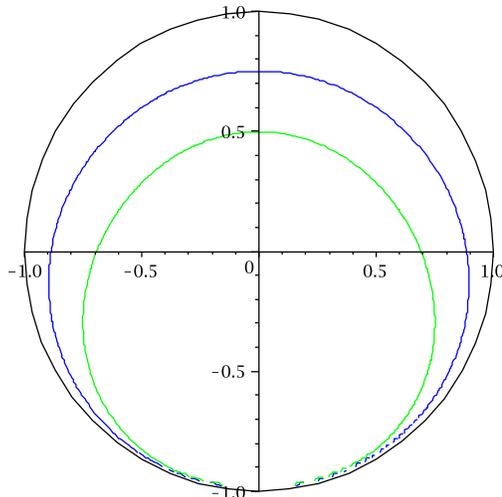


FIGURE 29. Contour Curves for $\phi = 14.3$ degrees, $\kappa = 0.8, 0.9$.

(a) If $0 < \kappa_0 < 1$, then the projection from the sphere $|\mathbf{j}| = 1$ to the image plane, of the contour curve defined by $\kappa = \kappa_0$, is an ellipse, centered at the Scheimpflug point $(0, -S)$, ($S = v \cot \phi = (1 + M)J$), where the x_2 -axis intersects the Scheimpflug line. Its semi-major axis and semi-minor axes are given by

$$(Mn) \quad d_2 = S \frac{\sec \phi}{\kappa_0} \quad d_1 = d_2 \sqrt{1 - \kappa_0^2}$$

In particular, the ellipse crosses the positive x_2 -axis at

$$S \left(\frac{\sec \phi - \kappa_0}{\kappa_0} \right).$$

(b) If $\kappa_0 = 1$, the projection is the line segment on the x_2 -axis from $-S(\sec \phi + 1)$ to $S(\sec \phi - 1)$.

(c) For κ_0 fixed, the set of all such contour ellipses forms an elliptical cone centered on the line through O in the lens plane, perpendicular to the x_1 -axis (the tilt axis).

In Figure 30, we show these contour ellipses for $\phi = 0.25$ radians, $v = 100$ mm, and $\kappa_0 = 0.6, 0.7, 0.8, 0.9, 1.0$.²⁸ As indicated only the portions above the Scheimpflug line are shown, since nothing outside the camera is relevant. Actually, nothing below the image of the upper tilt horizon²⁹ is relevant, since such points will necessarily have too large CoCs in the image plane. Usually, that line will be well above the Scheimpflug line, but placing an upper limit on it requires knowing details about the scene and plausible small apertures. We shall investigate this in some detail later.

For fixed ϕ , the contour ellipses scale linearly with v , so to obtain the corresponding curves for different positions of the image plane, just leave the curves where

²⁸The view is as it would appear from the vantage point of the lens, since I couldn't figure out how to get Maple to reverse the direction of an axis in a plot.

²⁹The upper tilt horizon is where the upper bounding plane Σ' intersects the plane at infinity. Similarly the lower tilt horizon is where the plane Σ'' intersects the plane at infinity.

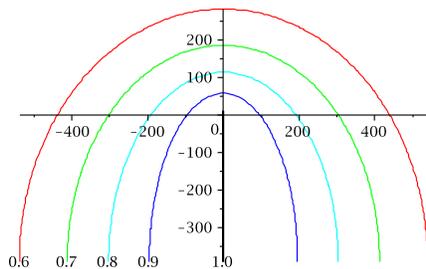


FIGURE 30. Contour Ellipses in Image plane, $\phi = 14.3$ degrees, $v = 100$ mm, $\kappa_0 = 0.6, 0.7, 0.8, 0.9, 1.0$ (Only portions above Scheimpflug line are shown.)

they are and multiply the units on the axes by the ratio $\frac{v}{100}$. The dependence on the focal length f or the tilt angle ϕ is more complicated, since they also affect the likely value of v . We have generally

$$S = J(1 + M) = \frac{f}{\sin \phi}(1 + M)$$

$$d_2 = \frac{S \sec \phi}{\kappa_0} = \frac{f}{\kappa_0 \sin \phi \cos \phi}(1 + M) = \frac{2f}{\kappa_0 \sin 2\phi}(1 + M)$$

$$d_1 = d_2 \sqrt{1 - \kappa_0^2}$$

But, for small ϕ , $\sin \phi \approx \phi$ and $\sin 2\phi \approx 2\phi$, so these equations become

$$S \approx \frac{f}{\phi}(1 + M)$$

$$d_2 \approx \frac{1}{\kappa_0} \frac{f}{\phi}(1 + M)$$

$$d_1 \approx \frac{\sqrt{1 - \kappa_0^2}}{\kappa_0} \frac{f}{\phi}(1 + M)$$

Of course, in the process of changing f and ϕ , we may also need to change M to accommodate a particular scene, but except for situations where the subject plane rises very steeply, which almost never occur, M is very small, so we may treat it as essentially constant. With that in mind, we can say that the diagram scales roughly linearly with f and inversely with ϕ . If we ignore M , $v = 100$ and $\phi = 0.25$ implies that $f \approx 103$. So to use the same diagram with different values of f or ϕ , leave the contour ellipses where they are and multiply the units on the axes by $\frac{f}{103}$ and by $\frac{0.25}{\phi}$.

As $\kappa_0 \rightarrow 1$, the ellipse approaches the indicated segment on the x_2 axis. As κ_0 decreases, the ellipses get larger and the ratio $\frac{d_1}{d_2} \rightarrow 1$. In the limit, as $\kappa_0 \rightarrow 0$, the ellipse approaches a circle of infinite radius.

We now begin to see how to attack the problems we have set.

We assume that the subject plane has been specified to within fairly narrow limits, which, as noted previously, will severely restrict the position of the hinge line and the tilt angle ϕ necessary to achieve it.

For any given position of the rectangular frame, we need only find the smallest value κ_0 of κ which can occur for rays that pass through that frame. To do that we look at the family of contour ellipses in the image plane, as in Figure 30, and determine the smallest one containing the frame, thereby setting κ_0 . We call the cone defined by $\kappa = \kappa_0$ the *cone of accessibility*. It is clear from the geometry that the frame will hit that cone at one of the two corners furthest from the Scheimpflug point, which simplifies finding the relevant value of κ_0 .

Physical and optical restrictions place strict bounds on the possible positions of the frame. For example, there is a limit to how close the standards can be, which places a lower bound on focal length, which in turn places a lower limit on how close Π can be to the reference plane, i.e., on the value of v . There is also a limit to how far apart the standards can be, which places an upper limit on the focal length. The region between the reference plane and the rear focal plane is optically forbidden. Moreover, for any possible choice of the image plane Π , limitations on camera movements restrict possible positions of the frame within that plane, and hence on rays passing through the frame. Finally, we can ignore highly implausible scenes or scenes which would require stopping down beyond what the lens allows or such that diffraction would dominate considerations from geometric optics.

All of these conditions place a lower bound on κ_0 , i.e., an upper bound on the size of the cone of accessibility. Given N, c , and κ_0 , define

$$v'_0 = \frac{v}{1 - \frac{Nc\kappa_0}{f}}.$$

It is clear that

$$v' = \frac{v}{1 - \frac{Nc\kappa}{f}} = v'_0 \iff \kappa = \kappa_0$$

so, the plane Π'_0 at distance v'_0 from the reference plane intersects the outer surface of definition along its intersection with the cone of accessibility. Similarly, if we define

$$v''_0 = \frac{v}{1 + \frac{Nc\kappa_0}{f}}$$

and let Π''_0 be the plane at distance v''_0 from the reference plane, the same thing holds for the intersection of the inner surface of definition with that plane and the cone of accessibility. See Figure 31 which illustrates the situation for the outer surface.

Any point in Π'_0 on the cone of accessibility will yield a circle of confusion c in Π such that $\frac{Nc\kappa_0}{f} = 1 - \frac{v}{v'_0}$, but, if N is fixed, then for a point in Π'_0 inside the

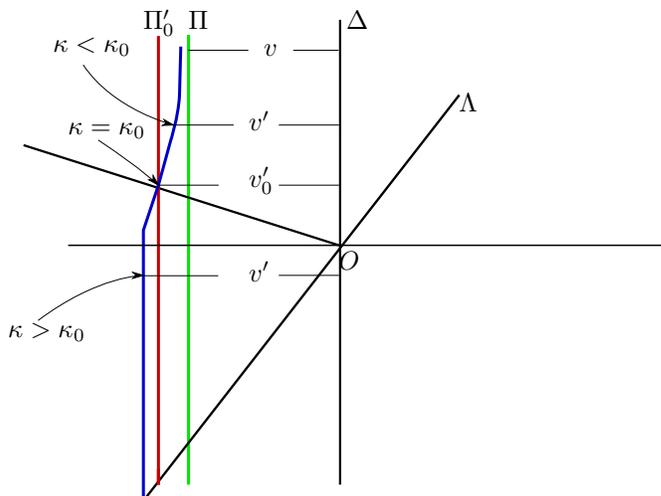


FIGURE 31. Cone of Accessibility

cone of accessibility, we will get a circle of confusion \tilde{c} in Π such that

$$\frac{N\tilde{c}\kappa}{f} = 1 - \frac{v}{v'_0} = \frac{Nc\kappa_0}{f}.$$

Since $\kappa > \kappa_0$ inside the cone, it follows that $\tilde{c} < c$. In other words, if a sharpness criterion is satisfied for points in Π'_0 on the ellipse, it is certainly met for such points within the ellipse. (The interplay between N , c , and κ can be confusing—the author admits having had his share of such confusions—and we shall return to it later.)

Similar remarks apply to the inner surface of definition and Π''_0

We may now use the above analysis to address Problems (2) and (2a).

Suppose we have a good estimate for κ_0 determined by the worst possible case for the camera, or by the details of the specific scene, or just from what our experience tells us works.

First we address Problem (2). Suppose an image plane Π has been chosen by some means, and we give a criterion for sharpness set by a value c for the circle of confusion, and also an aperture with f-number N . Then, as above

$$(VnVf) \quad \begin{aligned} 1 - \frac{v}{v'_0} &= \frac{Nc\kappa_0}{f} = \frac{v}{v''_0} - 1 \\ v'_0 &= \frac{v}{1 - \frac{Nc\kappa_0}{f}} & v''_0 &= \frac{v}{1 + \frac{Nc\kappa_0}{f}} \end{aligned}$$

tell us how to determine v'_0 (Π'_0) and v''_0 (Π''_0) and from these the corresponding upper and lower subject planes.

Next consider Problem (2a), in which we can suppose we have specified upper and lower subject planes and the corresponding image planes Π'_0 and Π''_0 , i.e., that we are given v'_0 and v''_0 , which we can determine, as in the untilted case, by the positions on the rail corresponding to the upper and lower subject planes. Choose Π to be their harmonic mean with respect to the reference plane. Then we want

to know how large we need to choose N so that the sharpness criterion set by c is met everywhere within the cone of accessibility determined by κ_0 .

To get some idea of the orders of magnitude involved, note that stopping down by one third of a stop corresponds to increasing the f-number by about 12 percent. One third of a stop would not usually be considered significant when making depth of field corrections. Often a photographer will stop down an entire additional stop or more to compensate for the lack of precision inherent in the work.

We shall investigate lower limits on κ in detail in Appendix D.2, but we shall outline the conclusions below.

From Figure 30, it is clear that κ is going to be larger for rays closer to the tilt horizon, and smaller for rays pointing further from the tilt horizon, i.e., those imaging points in the foreground. Similarly, κ is larger for rays closer to the line $x_1 = 0$, and smaller for rays pointing further from that line. In either case, the smaller values of κ will require shifting the frame either further in the direction of positive x_2 , i.e., away from the lens plane, or, more to one side or the other. But, limitations of shifts inherent in the structure of the camera will determine just how small κ can be in possible scenes. Moreover, even if the camera allowed very large shifts, the resulting scenes which require that everything in the subject plane from just beyond the lens to infinity be in the frame would be unusual, to say the least.

If the hinge line is skew, the x_1 and x_2 axes will be rotated with respect to the sides of the frame, which makes it a bit difficult to interpret all this, but for a pure tilt or a pure swing, it is clear what they mean. For example, for a pure tilt downward, κ may get too small for points in the extreme foreground, either below the camera or far to one side. For everything else fixed, limitation on vertical rise of the rear standard relative to the lens and limitations on shifts to one side or the other will limit how small it can get. Similarly, for a pure swing to the right, we may encounter problems for points in the extreme foreground to the right, or, too high or too low, but limitations on shifts of the frame to the left relative to the lens, or, vertical shifts in either direction will limit how small κ can be.

Unfortunately, the position of the frame and how this relates to the focal length, the tilt angle, and the position of the subject plane can be something of a wild card. Let us illustrate by an example. Let $f = 90$ mm, a common short focal length for 4 x 5 photography. Suppose we tilt downward from the horizontal by 0.25 radians (about 14.3 degrees), and suppose the subject plane intersects the reference normal line at distance $u = 5$ m = 5,000 mm from the lens. The hinge distance would be $J = \frac{f}{\sin \phi} \approx 364$ mm, and the slope of the subject plane would be $\frac{J}{u} \approx 0.0173$.

From the lens equation $v = \frac{fu}{u \cos \phi - f} \approx 94.6$ mm, and $M = \frac{v}{u} \approx 0.19$. Since v is not much less than 100, Figure 30 gives a pretty accurate rendition of the contour ellipses in the image plane. The scale will be reduced by about 5 percent, meaning that the units at the tick marks on the axes should be about 5 percent smaller, while the curves remain where they are. My view camera in these circumstance will allow an effective vertical rise of at most 40 mm. (Many cameras, particularly with a bag bellows, will allow considerably larger shifts.) The 4 x 5 frame has dimensions just about 96 x 120 mm. Suppose the frame is in landscape orientation and lifted 40 mm, but left symmetric about the line $x_1 = 0$. Then its top will be $40 + 48 = 88$ mm above the line $x_2 = 0$, and its upper corners will be 60 mm on

either side of that. Looking at Figure 30, we see that κ will be between 0.8 and 0.9, and much closer to the former than the larger in those corners. Taking into account the reduction in scale will reduce κ slightly, so we can guess that it will be about 0.82 in the corners.³⁰ That would require stopping down about three fifths of a stop, which, in a critical situation, might be more than one is comfortable with. Of course for most of the frame κ would be much closer to 1, so it would only be an issue if those corners had significant detail.

Note, however, that the scene corresponding to this position of the frame would be somewhat unusual. For one thing, the hinge distance at 364 mm is rather short. For another, the position of the frame is rather unusual. First, it is almost entirely above the center of the field, so nothing much above the horizon would be in view. In addition, the questionable corners would correspond to points in the scene very close to the lens. Given the data above, since the subject plane slopes so gently upward, the corners would be about 364 mm (≈ 14.3 inches) below the lens and $\frac{88}{90}364 \approx 356$ mm or (14 inches) into the scene. Such a scene is unusual, but not entirely implausible.³¹ Still, this indicates why the plausibility of the scene might come into play before the limits of shift were encountered. For example, if one shifted the frame even higher, thus reducing κ further, the foreground would come even closer to the lens. On the other hand, for longer lens, where v typically would be much larger, limitations on shift may play a more important role.

It turns out in the end—see Appendix D.2—that, *provided the Scheimpflug distance is large compared to E_1 and E_2 , κ is not much less than 1, and the necessary correction is less than one third of a stop.* Since $S = J(1 + M)$, it usually suffices to look at the hinge distance, but in some odd situations, we might have to consider the Scheimpflug distance directly.

The hinge distance is given by $J = \frac{f}{\sin \phi}$, so it will be extremely large for very small tilts. For example, for $\phi = 1$ degree and f ranging from 65 mm to 450 mm, the hinge distance would range from 3.7 meters to over 25 meters. Any plausible values of E_1 and E_2 would be a very small fraction of that. So we need only consider moderate to large tilts when trying to decide if κ might be small enough to matter.

Usually E_1 and E_2 are smaller than the dimensions of the frame, but when extreme shifts are involved, they might be somewhat larger. Usually the camera won't support such large shifts, and even if it did, they would only rarely be needed. For example suppose $f = 150$ mm, often considered the 'normal' focal length, and we have a pure tilt with $\phi = 0.15$ radians (≈ 8.6 degrees), still a substantial tilt. Then $J \approx 1,004$ mm, or about 1 meter, and assuming, as usually is the case, that M is small, $v \approx f \sec \phi \approx 152$ mm. Suppose also that the frame is centered on the horizontal line of sight and it is in landscape orientation. Then $E_1 = 60$ mm and

³⁰An exact computation using our formulas gives $\kappa \approx 0.8144$, requiring stopping down ≈ 0.5925 stops. But such accuracy is overkill, and the graphical method more than suffices for our needs.

³¹For example, the camera might be placed just behind a retaining wall, the top of which could be that distance below the lens, with the surface beyond the wall starting at the same level and sloping slightly upward, and we might want everything from a foot or so beyond the lens to a much greater distance in focus. I've taken pictures at the Chicago Botanic Garden not too different from this.

$E_2 = 48$ mm,³² which are both small compared to the hinge distance. Calculating κ from the formula³³ yields $\kappa \approx 0.95$ which would require less than one sixth of a stop correction. Putting the frame in portrait orientation and raising it by 20 mm, so $E_1 = 48$ mm and $E_2 = 80$ mm, would only reduce κ to about 0.93, and require about one fifth of a stop correction. Even reducing the focal length to 90 mm, which would reduce the hinge distance to about 600 mm, would in the latter case, only reduce κ to 0.88, and require a little over one third of an f-stop correction.

Our approach does indicate when κ might be small enough to matter. That would be for large tilts and either when the hinge distance was too short or the frame shifted so much that the relevant quantities were not sufficiently small in comparison to it. This is more likely to occur for very short focal length lenses. This is all subject to the assumption that M is relatively small, which is almost always the case, except possibly in close-ups.

For cases where the above conditions are not met, e.g., if E_1 or E_2 are a sizable fraction of S , there is no alternative but to calculate κ for the explicit situation under consideration. In the example above, we had $E_1 = 60$ and $E_2 \approx 90$ mm, which are still significant fractions of $S \approx 400$ mm, and we found κ might require a significant f-stop adjustment to compensate.

As we saw, in typical situations, κ only requires stopping down a negligible fraction of an f-stop, so the whole issue can be ignored. In any case, there are many other sources of uncertainty in the whole methodology, which may affect the situation more, thus swamping the effect of κ . First, there is the fact that actual lenses only behave approximately like the ideal lenses discussed in this article. Second, even for an ideal lens, difficulties in positioning the standards, setting the f-number and judging when something is in focus on the ground glass would limit precision. So, in any event, a prudent photographer may be wise to stop down as much as a stop or more than the theory would suggest.

In addition, diffraction, which we have ignored entirely in this article, increasingly comes into play as the aperture decreases (f-number increases). This effect is opposite of that of geometric defocus, which decreases as the aperture gets smaller. There is some aperture in between where the two effects are balanced, but characterizing it is not easy.³⁴

8.4. Determining the Proper f-stop in the Tilted Case. To determine the proper f-number in terms of the focus spread, we use essentially the same reasoning as in Section 8.1.2, but to avoid missing some subtle point, we repeat the argument. First, it is easy to see that v is the harmonic mean of v'_0 and v''_0 if and only if

$$1 - \frac{v}{v'_0} = \frac{v}{v''_0} - 1 = \frac{Nc\kappa_0}{f}.$$

³²This means that half the frame will be below that line, which means the frame might extend below the image of the upper tilt horizon. But, typically, that part of the scene would be open sky, and so we wouldn't care what was in focus there.

³³We could also use Figure 30. Just remember that the numbers at the tick marks must be multiplied by about $\frac{150}{103} \frac{0.25}{0.15} \approx 2.4$.

³⁴There are rules of thumb which try to estimate how to choose the aperture which best balances diffraction against geometric defocus, but they are usually based on fairly crude analyses. See Jeff Conrad's previously referenced article for a nuanced discussion of this issue.

Using the equality of $\frac{Nc\kappa_0}{f}$ with either of the two other expressions yields, after some algebra,

$$\begin{aligned} \frac{v'_0 - v''_0}{v'_0 + v''_0} &= \frac{Nc\kappa_0}{f} \quad \text{so} \\ \text{(NE)} \quad N_0 = N\kappa_0 &= \frac{v'_0 - v''_0}{c} \frac{f}{v'_0 + v''_0} \end{aligned}$$

But using $v'_0 = \frac{v}{1 - (Nc\kappa_0)/f}$ and $v''_0 = \frac{v}{1 + (Nc\kappa_0)/f}$, it is easy to see that

$$v'_0 + v''_0 = \frac{2v}{1 - (Nc\kappa_0/f)^2} \approx 2v$$

since $\left(\frac{Nc\kappa_0}{f}\right)^2$ is very small, except in highly unusual circumstances, as noted in Section 8.1.2. Hence,

$$N_0 = N\kappa_0 \approx \frac{v'_0 - v''_0}{2c} \frac{f}{v}.$$

So far, this exactly what we got in the untilted case, with $\kappa_0 = 1$. But there are some major differences. The first is the factor κ_0 , and the second is that, from the lens equation, we obtain, in the tilted case, $\frac{v}{f} = (1 + M) \sec \phi$, where M is the magnification for the distance at which the subject plane crosses the reference normal line. That leads to the estimate

$$\text{(TK)} \quad N_0 = N\kappa_0 \approx \frac{v'_0 - v''_0}{2c(1 + M)} \cos \phi$$

This can be interpreted as follows. We start by finding the f-number N_0 which would we get if we assumed $\kappa = 1$ were true. This is done by a modification of the rule used in the untilted case: divide the focus spread $v'_0 - v''_0$ by twice the maximum acceptable CoC c . Correct this as before by dividing by $1 + M$, where M is the above magnification, but also multiply by $\cos \phi$. Finally correct for the departure of κ from unity to obtain $N = \frac{N_0}{\kappa_0}$. Equivalently, just stop down by an additional $\frac{2 \log \kappa_0}{\log 2}$ stops.

Note that we can simplify this rule quite a bit. First note that for typical scenes, M is very small, so it won't make much difference if we ignore the factor $1 + M$. (In any case doing so just overestimates N , which is usually innocuous.) Similarly $\cos \phi$ is usually very close to 1 because for small ϕ , $\cos \phi \approx 1 - \phi^2/2 \approx 1$.

8.4.1. *What happens in the subject space.* Let Σ be the desired subject plane and let Π be the corresponding image plane. Fix the f-number N and a maximal acceptable CoC c . Then, instead of the upper and lower planes Σ' and Σ'' we would get for $\kappa = 1$, we get upper and lower surfaces of definition, corresponding to the outer and inner surfaces of definition discussion in the previous section. The upper surface of definition lies between Σ and the Σ' , and the lower surface of definition lies between Σ and Σ'' . But, as we shall see, the picture, is a bit more complicated than that in the image space. See Figure 32. The coordinates are as before, except that in the subject space, we have reversed the direction of the x_3 axis. The diagram shows a cross-section for the various planes and surfaces for a

Fix a plane Γ parallel to the reference plane. Since, Δ intersects that plane at infinity, we have from equation (HMS)

$$\{x'_2, x_2^F; x_2, \infty\} = \frac{x'_2 - x_2}{x_2^F - x_2} = \frac{Nc}{f} \quad \text{or}$$

$$x'_2 - x_2 = (x_2^F - x_2) \frac{Nc}{f} = \left(\cot \phi - \frac{J}{u} \right) x_3 \frac{Nc}{f}$$

But,

$$\begin{aligned} \frac{J}{u} &= J \left(\frac{\cos \phi}{f} - \frac{1}{v} \right) \\ &= \frac{f \cos \phi}{\sin \phi} \frac{1}{f} - \frac{J}{v} \\ &= \cot \phi - \frac{J}{v} \end{aligned}$$

Hence,

$$(Spl) \quad x'_2 - x_2 = \frac{J Nc}{v} \frac{1}{f} x_3 = \frac{Nc}{v \sin \phi} x_3$$

and the total depth of field parallel to the reference plane would be twice that. But, $v = f \sec \phi(1 + M)$, where M is the magnification at distance u , and so

$$\frac{J Nc}{v} \frac{1}{f} = J \frac{Nc}{f^2 \sec \phi(1 + M)} = \frac{J \cos \phi}{H M + 1}$$

where $H = \frac{f^2}{Nc}$ is (the approximation to) the hyperfocal distance in the untilted case for f , N , and c . So (Spl) can also be written

$$(SplM) \quad x'_2 - x_2 = \frac{J \cos \phi}{H M + 1} x_3$$

$$(TSplM) \quad \text{Total DOF} = 2 \frac{J \cos \phi}{H M + 1} x_3$$

Often, M is small, i.e., the exact subject plane doesn't depart significantly from the horizontal, and $\cos \phi \approx 1$ since ϕ is small. If that is the case, then the factor $\frac{\cos \phi}{M + 1}$ is close to one, and we conclude that the difference in the slopes of the

upper bounding plane and the subject plane is close to $\frac{J}{H}$, (and similarly for the subject plane and the lower bounding plane). For example, suppose $f = 135$ mm, and we used a tilt of 5 degrees ≈ 0.09 radians. Then $J \approx 1549$ mm. Suppose also that the grade was as high as 15 percent (a slope angle of about 8.5 degrees). Then $u = \frac{1549}{0.15} \approx 10,326$ mm, $M \approx 0.0133$, and $\cos \phi \approx 0.996$. Thus the difference in the slopes would be reduced from $\frac{J}{H}$ by approximately 0.983, or by about 2.27 percent.

One way to summarize all this in words is to say that *at the hyperfocal distance, the transverse split above and below the exact subject plane is approximately equal to the hinge distance*³⁵.

³⁵We should also, reduce this by the factor κ as below.

Thus, for extremely small tilt angles³⁶, the vertical split will be enormous, and it is only for somewhat larger tilts that this rule becomes useful.

For larger tilts, the rule of thumb is not hard to apply in practice to help estimate depth of field, provided you know the hyperfocal distance. The hinge distance is usually quite easy to estimate because it involves distances close to the camera. It is harder to visualize just where in the scene the hyperfocal distance is, but even if you make a rough guess, you get a general idea of the extent of the depth of field region. In particular, it helps you avoid overestimating its extent significantly. The hyperfocal distance f^2/Nc is not hard to calculate, but few people are comfortable doing it in their heads. On the other hand, if you take a small table which gives it for the focal lengths you have available for one aperture, it is not hard to estimate it for a different aperture by multiplying by the ratio of the f-numbers.

It is instructive to interpret the split distance by describing what happens in the image plane. To this end, we take Equation (Spl) and multiply it by the magnification $\frac{v}{x_3}$ from Γ to the image plane. We get

$$(Spla) \quad \text{Image of}(x'_2 - x_2) = \frac{Nc}{\sin \phi}$$

$$(Splt) \quad \text{Image of}(x'_2 - x''_2) = 2\frac{Nc}{\sin \phi}$$

Note that this quantity is independent of the distance x_3 in the subject space of the plane Γ from O . So, there is a ‘window’ of constant height in the image plane which moves progressively up in that plane as your attention shifts from background to foreground.

Note that depending on where you look in the DOF wedge, part or all of this window may extend beyond the frame, above it in the foreground and below it in the background. We shall say more about that in Section 8.4.2.

The window is also what you would see in focus on the ground glass after stopping down to that f-number, *provided you are looking at the ground glass with the appropriate magnification*. For, the maximum acceptable CoC c is chosen on the basis of what needs to appear sharp in the final image. The standard choice for that is an 8 x 10 print viewed at 10-12 inches.³⁷ To get the same perspective on a 4 x 5 ground glass, one would have to get proportionately closer, i.e, about 6 inches. Very young people or myopes may be able to do that without aids, but most people will need a 2x loupe. Of course, viewing the ground glass is not the same as viewing a print, and that might blur the image slightly, but in principle, by stopping down to the taking aperture, and looking at the image on the ground glass, what you see in focus should roughly be what you get.³⁸ However, if you view

³⁶It might be noted that the tilt angle would almost never be exactly zero, but calculations such as that above confirm us in our belief that negligible tilts are effectively zero.

³⁷10-12 inches is the normal distance for close viewing. It is generally agreed that viewers are most comfortable viewing a print at a distance equal to its diagonal, which for an 8 x 10 print is about 12 inches. (The diagonal of an 8 x 10 rectangle is 12.8 inches, but there is usually a small margin, making the print a bit smaller.) For larger prints, it is assumed viewers will get proportionately further back. There will always be, of course, viewers—‘grain sniffers’—who will try to get as close as possible, and for them, one would have to adjust c downward.

³⁸Stopping down reduces the brightness of the ground glass image, which makes it harder to judge what is sharp, and most people won’t be able to see much of anything when stopped down below $f/16$ to $f/22$.

the ground glass from further back, you will be effectively increasing the CoC, and thus overestimating what will appear sharp, or, if you use a more powerful loupe, you would be underestimating it. For example, a 4 X loupe in effect reduces the CoC by a factor of two, and the height of the window would be reduced by the same factor. In most circumstances, it is better to underestimate the depth of field visually than to overestimate it.

Note that for small ϕ , we have $\sin \phi \approx \phi$, so we may approximate (Splt) by $2\frac{Nc}{\phi}$. Unfortunately, to use this equation, you have to measure ϕ in radians. To convert from degrees to radians, you divide by 57.3. or very roughly by 60, so a rough approximation would be $120\frac{Nc}{\phi \text{ degrees}}$. For example, if we take $c = 0.1$ mm, for $\phi = 5$ degrees and $N = 22$, we would get $\frac{12 \times 22}{5} \approx 53$ mm. (Using $\sin \phi$ would give ≈ 50.48 mm.) I doubt if many people would be inclined to make even the rough calculation in their heads, but it would be possible to make a short table to refer to in the field.

We now address the upper surface of definition. Assume, as above, x_3 , i.e, the plane Γ parallel to the reference plane, is fixed. Γ intersects Σ is a line L_s parallel to the x_1 -axis. Given a point $P = (x_1, x_2, x_3)$ on that line, let $\overline{P'} = (x_1\overline{x}'_2, x_3)$ be the corresponding point in the intersection of Γ with the upper surface of definition. Then the same reasoning as that employed above applies except that we must introduce the factor $\overline{\kappa}$ evaluated for the ray $-\overrightarrow{OP}$ in the image space

$$\overline{x}'_2 - x_2 = \overline{\kappa} \frac{Nc}{v \sin \phi} x_3$$

Since the right hand side of this equation also depends on \overline{x}'_2 , through $\overline{\kappa}$, to find it exactly we would have to solve a complicated equation. Fortunately, we can avoid that difficulty. Namely, because κ is nonincreasing in the relevant range as j_3 increases—see Figure 30, for example—we have $\overline{\kappa} \geq \kappa$, where the latter is evaluated for the ray $-\overrightarrow{OP}$. Hence, we have the inequality

$$\text{(SplS)} \quad \overline{x}'_2 - x_2 \geq \kappa \frac{Nc}{\sin \phi} \frac{x_3}{v}$$

which will be more than adequate for our needs.

To justify these contentions, consider the image Γ' of the plane Γ in the image space. As indicated in Figure 33, Γ intersects the lens plane Λ at a line L , and Γ' is the plane determined by L and the the image of L_i of L_s . Γ' will intersect the outer surface of definition at a point $\overline{Q'}$, and \overline{x}'_2 is the intersection of the ray determined by $\overrightarrow{O\overline{Q'}}$ with Γ . (As before, the diagram exaggerates the magnitudes significantly.)

Notes. It is also true that $\overline{\kappa}$ is bounded above by the value of κ for the ray OP' where $P' = (x_1, x'_2, x_3)$ is in the upper bounding plane Σ' , so we could use that information to find an upper bound for $\overline{x}'_2 - x_2$, thus constraining it further. In addition, we can obtain exact formulas for the coordinates of the point of intersection \hat{P} of the upper surface of definition with the ray OP . (See Appendix E for further discussion of these matters.) But formula (SplS) will prove adequate for our needs here.

Analogous estimates apply to the lower bounding plane and lower surface of definition. (The reader may be a bit concerned about the estimate $\overline{\kappa} \geq \kappa$ in that

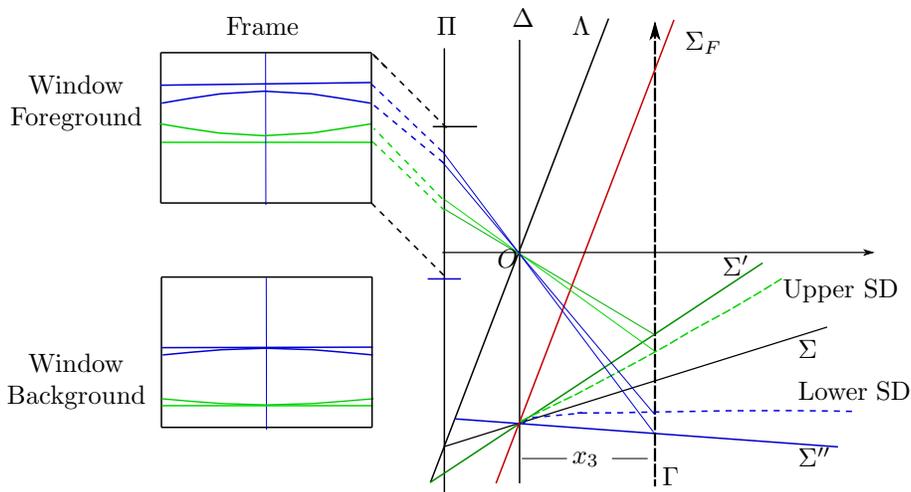


FIGURE 34. Windows for Foreground and Background

Let's look at an example. Suppose we tilt downward from the horizontal 0.25 radians (≈ 14.3 degrees), $v = 130$ mm, $f = 125$ mm, and so $M \approx 0.0077$ mm. Suppose the frame is centered on the central line of the field (the x_2 -axis) in landscape orientation, so that either corner is 60 mm to the side, and its top is 40 mm above the x_1 -axis. Figure 30, with the tickmarks compressed by 1.3, suggests that the frame will lie just inside the contour curve $\kappa = 0.9$. If we stop down to $f/32$, the window the image plane will have height $\frac{2Nc}{\sin \phi} \approx 25$ mm, and surfaces of definition would reduce this by at most 10 percent, i.e., by about 1.25 mm, anywhere in a window, foreground or background. It is questionable whether that would be noticeable, except in very special cases, and in any case stopping down an extra one third of a stop would deal with it. On the other hand, κ as small as 0.7 somewhere in the scene would reduce what is in focus by about 30 percent, which probably would be noticeable. Stopping down a full stop would compensate for that, but it would also require doubling the exposure time, possibly raising difficulties because of subject movement.³⁹

8.4.2. *When is it advantageous to use a tilt or swing?* The usual wisdom is that you should use a tilt if by doing so you can obtain the necessary depth of field from near to far along the exact subject plane, at a larger aperture (smaller N) without restricting the depth of field about that plane so much that you lose significant detail. In considering whether that is feasible, attention is usually drawn to the

³⁹It might be argued that the calculations made in this article are irrelevant because you will automatically incorporate any κ effect as you stop down and observe the result on the ground glass. There are several problems with that argument. First, it is hard under conditions in the field to examine the ground glass image carefully enough not to be surprised by what appears in the final print under normal viewing conditions. Second, in order to deal with the dimming of the image, it may be necessary to use a 4 X or stronger loupe. As noted before, that leads you to overestimate how much you should stop down to get the desired depth of field. Often, you are trying to balance depth of field against exposure time, and you don't want to stop down any more than you really need to. Finally, the more you stop down, the more diffraction becomes an issue.

in this example extends out to infinity, is also of concern. One doesn't usually think about it because typically other elements of the scene—in this example, the ground—entirely block it.

Of course, if we didn't tilt, everything in the frame between the near and far planes would be in focus, and there would be no such concerns. So the question is whether or not we can gain an f-stop advantage by tilting. The f-stop to use without tilting would be determined from the focus spread s by

$$N_1 \approx \frac{s}{2c(M+1)} \approx \frac{s}{2c}$$

where the second approximation holds if we are not in a close-up situation. The height t of the window⁴⁰ on the ground glass would be given by $t = \frac{Nc}{\sin \phi} \approx \frac{Nc}{\phi}$, so the proper f-stop to use for a given t would be

$$N_2 \approx \frac{t}{c}\phi$$

Hence,

$$(ADV) \quad \frac{N_2}{N_1} \approx \frac{t}{s} \frac{\phi}{M+1} \approx \frac{t}{s}\phi$$

with the last approximation holding if we are not in a close-up situation. For tilting to make sense, this ratio should be enough less than one to matter. Let's see how this might work out in the current example. Suppose as usual we take $c = 0.1$ and suppose the focus spread on the rail were 9 mm. That would require stopping down to f/45 without tilting, and, because of subject motion, we might not want to do that. Under our assumptions, t could be 100 mm or larger. Using a 150 mm lens, and supposing the camera were 1.5 meters above the ground, we would have $\phi \approx \frac{150}{1500} = 0.1$ radians. Hence, we would have $\frac{N_2}{N_1} \approx \frac{100}{9}(0.1) \approx 1.1 > 1$ so in this case, tilting would not give us an advantage. Suppose on the other hand that the top of the hill were not so high, say $t = 50$ mm, and although the region between Σ' and Y_1 might extend further, there was nothing in it but empty sky. In that case we would have $\frac{N_2}{N_1} \approx \frac{50}{9}(0.1) \approx .55$. so tilting would allow us to use $N_2 \approx 25$, a gain of almost two stops.

9. ROTATIONS OF THE STANDARD

In this section, we study rotations of the front standard, i.e., the lens plane. (Rotations of the rear standard, are sometime useful, but such movements also change the positions of vanishing points. In any case any rotation of the rear standard is equivalent to a reorientation of the entire camera followed by a rotation of the front standard, so there is no loss of generality in restricting attention to the front standard.)

Usually, a standard may be rotated only about one of two possible axes, one perpendicular to the top and bottom of the standard and the other perpendicular to its sides. (This language assumes that initially the camera is level with the standards perpendicular to the rail, so that top, bottom, and sides make sense, but even in general we can make such distinctions by referring to the rail and the

⁴⁰Any corrections for κ can be made separately, if necessary.

mechanism whereby the standard is attached to it.) The first type of rotation is called a *tilt*, and the second is called a *swing*.

There are some differences among view cameras about how tilt and swing are done. The axis of rotation seldom passes through the principal point, but it can be very close to it, in which case it is called an *axial rotation*. Swings are almost always axial rotations. Tilts, on the other hand may be axial or *base*, meaning that the axis of rotation is below the bottom of the standard and thus far removed from the lens. This creates a potential problem because non-axial rotations will change the position of the principal point. Fortunately, we can compensate for this by a combination of a rise or fall and/or a sideways shift of the standard coupled with a translation of the standard along the rail, in order to return the principal point to its original position. In addition, it is easy to see that if the relevant elements of the scene are sufficiently far from the lens, then the change in the position of the principal point doesn't matter very much. We shall assume in what follows that the principal point stays fixed and all rotation axes pass through it.⁴¹ (The phenomenon called *parallax shift*, is very sensitive to small changes in the center of perspective, which for the lens model we have chosen is identical with the principal point. But we shall see in Section 10.1, for more complex lens models, which better describe how a real lens behaves, the principal point and center of perspective may be different.)

There are a variety of view camera mechanisms. If you perform a tilt first, it may leave the swing axis fixed or it may move it with the standard, and if you swing first, you may or may not move the tilt axis. Such differences can be very important.

Proposition 4. *Any rotation of the lens plane (standard) about an axis through its principal point may be achieved by a combination of a tilt and a swing, in either order.*

Proof. To see this consider the normal vectors \mathbf{N} and \mathbf{N}' to the original position and the desired final position of the lens plane. The intersection of the final plane with the original plane is a line in both and to get what we want we just need to rotate \mathbf{N} to \mathbf{N}' about that line. Choose a rectangular coordinate system such that the tilt axis is the y_1 -axis, the swing axis is the y_2 -axis, and the y_3 -axis is perpendicular to both and hence points along \mathbf{N} . (New notation was chosen in order to avoid confusing these coordinates with x_1, x_2 , and x_3 used in Section 8.3.)

First assume that tilting doesn't affect the swing axis. Project \mathbf{N}' to a vector \mathbf{M} in the y_2, y_3 -plane, and then tilt the lens plane so that \mathbf{N} is moved to \mathbf{N}'' , a multiple of \mathbf{M} . Now swing about the y_2 -axis so that \mathbf{N}'' is moved to \mathbf{N}' as required.

If tilt does move the swing axis, we must proceed slightly differently. Choose an y'_2 -axis in the y_2, y_3 plane which is perpendicular to \mathbf{N}' . (That can be done by solving $(s\mathbf{i}_2 + t\mathbf{N}) \cdot \mathbf{N}' = 0$, where \mathbf{i}_2 is a unit vector along the y_2 -axis, for the ratio $s : t$.) Tilt to move the y_2 -axis to the y'_2 -axis, which will move \mathbf{N} to a vector \mathbf{N}'' in the y_2, y_3 plane which is perpendicular to the y'_2 -axis. Since both \mathbf{N}'' and \mathbf{N}' are now perpendicular to the new swing axis, we can swing the former to the latter by a rotation about that axis.

⁴¹Another advantage of axial tilt is that the points along the centrally place horizontal line on the ground glass don't go very far out of focus. See Equation (FocSh) in Section 6.1 for an estimate of the shift.

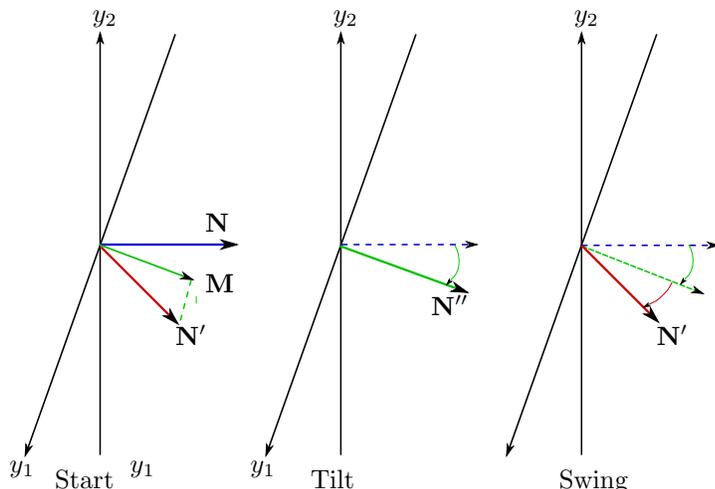


FIGURE 36. Rotating the Normal if Swing Axis Stays Fixed in a Tilt

A similar analysis applies if you want to swing first and then tilt. \square

There is a subtle point about in this procedure that needs explanation. Namely, we know that $SO(3, \mathbb{R})$ is a three-dimensional Lie group since its Lie algebra consists of all real 3×3 skew-symmetric matrices. In general, the best we can do is to decompose a general rotation into three rotations about prescribed axes. So, the above procedure may need to be supplemented by a rotation of the final position of the plane about its normal \mathbf{N}' . Fortunately, rotations about an axis perpendicular to the lens plane have no optical effect, so they don't matter. Thus, the procedure need only give us what we want up to such a rotation. In other words, we are really dealing with the homogeneous space $SO(3, \mathbb{R})/SO(2, \mathbb{R})$, which is two-dimensional.

The rotation about \mathbf{N}' is called *yaw*. As I just noted, it is irrelevant optically, but it can affect the positions of top, bottom and sides of the standard. That can be important for the rear standard which contains the frame. In that case, one usually wants to arrange the order of the operations so that the top and bottom remain horizontal. (The term 'yaw' comes from navigation where we refer to roll, pitch and yaw of a craft such as an airplane or ship, which correspond to tilt, swing and yaw of the standard, in some order, depending on the conventions.)

The problem of the photographer then is to figure out how to use a combination of a tilt and swing, in some order, to get the hinge line to coincide with that of the desired subject plane. Then by moving the rear standard parallel to itself, the subject plane may be rotated into the proper position. The above proposition tells us that this is always possible with an appropriate choice of tilt followed by an appropriate choice of swing, or vice versa.

Unfortunately, the formulas for the respective tilt and swing angles involve trigonometric and inverse trigonometric functions, so calculating the necessary angles in advance is not practical when composing a scene in the field. It is best if the operations can be done using what appears in focus on the ground glass. It can always be done by a sequence of tilt/swing operations, possibly with refocusing, iterated until the photographer is satisfied. But that can be very tedious. It is

the tilt axis, we would be better off swinging first to move the hinge line, parallel to the sides of the standard, until it reached J_h .

There is one possible point of confusion here, which I should clarify. This is complicated to describe, but please bear with me. Let Q' be the intersection of the y_3 -axis, i.e., the original position of the lens axis, with the desired image plane Π , and let Q be its intersection with the desired subject plane. (Note Q could be at infinity or even in the image space, which would mean that Q' is not actually an image of a real point.) Let l_v be the line in the image plane, i.e., the rear standard, through Q' parallel to the y_2 -axis. Note that l_v will be parallel to the sides of the rear standard. Let S be the intersection of l_v with the desired subject plane. Since it lies in the image plane Π , it lies on the Scheimpflug line. Then the line $l_s = SJ_vQ$ in the subject plane is imaged as l_v in the image plane. (But keep in mind that l_s rotates about the current hinge line as we focus the rear standard.) The aim should be first to tilt so the line l_s is in focus on l_v and then to swing without moving it. We could accomplish this, of course, if we could swing about l_s , but that is not a choice available to us. But, if swing fixes J_v as above, then we can readjust the image plane by translating it parallel to itself until the line l_s rotates into the desired position.

On the other hand, if the swing axis follows the standard, since it stays fixed when you swing, its intersection with the *current* position of the image plane, i.e., the current position of S stays fixed. But, since the point J_v will have moved in the reference plane, the line l_s won't stay fixed, and you can't adjust that by refocusing the rear standard. So you must, of necessity, readjust the tilt.⁴²

While the above analysis is helpful, neither the hinge line, nor the Scheimpflug line is directly accessible when viewing the image on the ground glass. Instead, one must concentrate on the cohinge line, which is the image of what we called the tilt horizon. This line is usually in the frame or can be accessed by moving the frame. Refer to Figure 38 which has been rotated by 180 degrees, to put the image right side up, from how it appears on the ground glass. As above, let l_v denote the line through Q' parallel to the sides of the rear standard, and let l_h be the line through Q' parallel to the top and bottom of the standard. Let K_v and K_h be the respective intersections of the cohinge line with l_v and l_h . In addition, suppose we choose an image point Q'' on l_v —it could be Q' —which corresponds to a point l_s in the desired subject plane, and suppose we tilt so as to bring the subject points corresponding to Q'' and K_v simultaneously in focus. If, as above, tilting doesn't move the swing axis, and the point J_v remains fixed in any ensuing swing, then it is not hard to see that the line l_s can be kept in focus along l_v , by translating the rear standard, as we swing. So all we need to do is to swing until the tilt horizon rotates to the desired position. If on the other hand, the tilt axis stays fixed under swings, then we should concentrate instead on the line l_h , the point K_h and an additional point Q''' on l_h . There are several variations of this approach using points not necessarily on the tilt horizon.

Finally, note that ease in focusing on the desired subject plane and yaw with respect to the horizontal are separate issues. There is no simple relationship between

⁴²It may be possible to handle the situation by moving the front standard as well as the rear standard appropriately, without changing the tilt. I haven't been able to settle to my satisfaction exactly what is possible that way. But that would change the position of the principal point relative to the scene, something you usually want to avoid. Since in some cases, that won't make much difference, it might be worth investigating this possibility further.

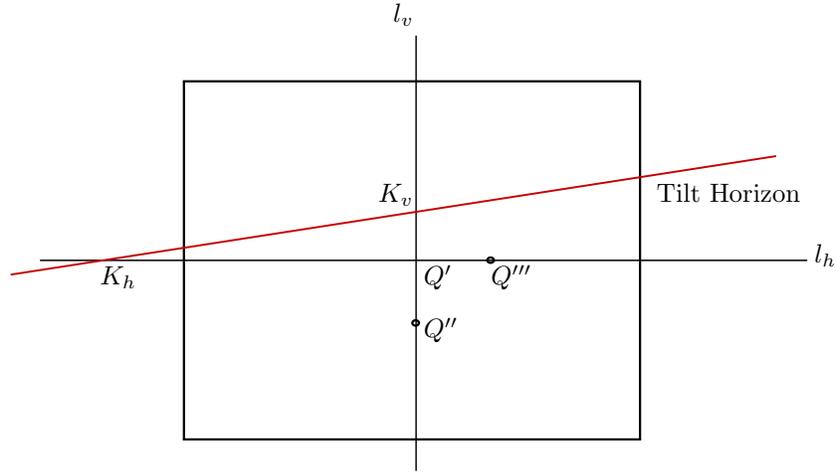


FIGURE 38. View on the Ground Glass

the two. A particular order of operations may be yaw free and behave well optically or it may be yaw free and not behave well optically. All four possible combinations can occur. Unfortunately, the two notions are sometimes mixed up in people's minds.

10. A MORE REALISTIC MODEL FOR A LENS

As noted before, a photographic lens is a complex device. There is a standard model for how such a lens behaves optically, but it more complicated than what we described previously. Instead of a single lens plane, there is a pair of principal planes perpendicular to the lens axis, front and rear. The points where these planes intersect the lens axis are called the front and rear principal points. There are in addition two other points on the lens axis called nodal points, but these only differ from the principal points when there are different media such as air and water on different sides of the lens elements. They can be important in underwater photography, but not for view cameras, so I won't discuss the nodal points further. The principal planes and corresponding principal planes control the formation of images. We shall show how to modify the scheme we set up previously to describe that process.

Let Λ and Λ' be the front and rear principal planes, and let O and O' be the corresponding front and rear principal points. Let \mathcal{V} be a lens transformation with respect to Λ and O , let $T_{OO'}$ denote translation by the vector $\overrightarrow{OO'}$ connecting the principal points, and define $\mathcal{V}' = T_{OO'}\mathcal{V}T_{OO'}^{-1}$. Then \mathcal{V}' is a lens transformation with respect to Λ' and O' . As before there is a front focal plane Σ_F and a rear focal plane Π_F , which are separated respectively from the corresponding principal planes by the focal length f . We define $\tilde{\mathcal{V}} = T_{OO'}\mathcal{V} = \mathcal{V}'T_{OO'}^{-1}$. It is a projective transformation which describes image formation for the complex lens. In terms of light rays, this may be described as follows. A ray proceeds from P to the front principal point O , is translated to O'' on the rear principal plane via $\overrightarrow{OO'}$, and a ray emerges from O' parallel to PO . Another ray parallel to the lens axis passes

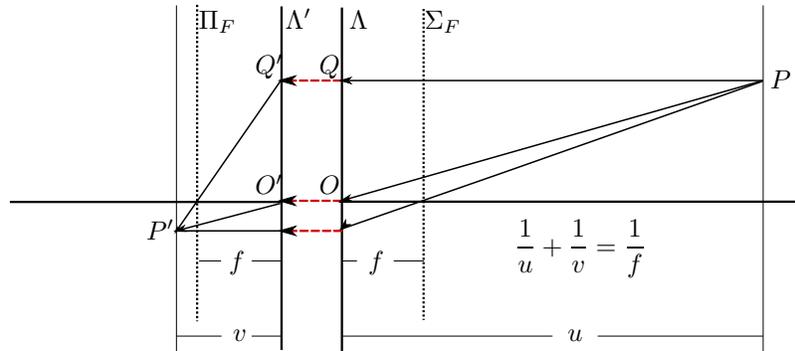


FIGURE 39. Principal Planes & Points, Focal Planes, and Image Formation

through the front principal plane at Q , is translated to Q' on the rear principal plane, emerges from Q' so as to pass through the rear focal point. These two rays then intersect at the image point P' . (Alternately, a ray proceeds from P , passes through the front focal point to a point on the front principal plane, carries through parallel to the lens axis to the rear principal plane and from there to the image point P' .) It is almost (but not quite) as if the region between the two principal planes did not exist.

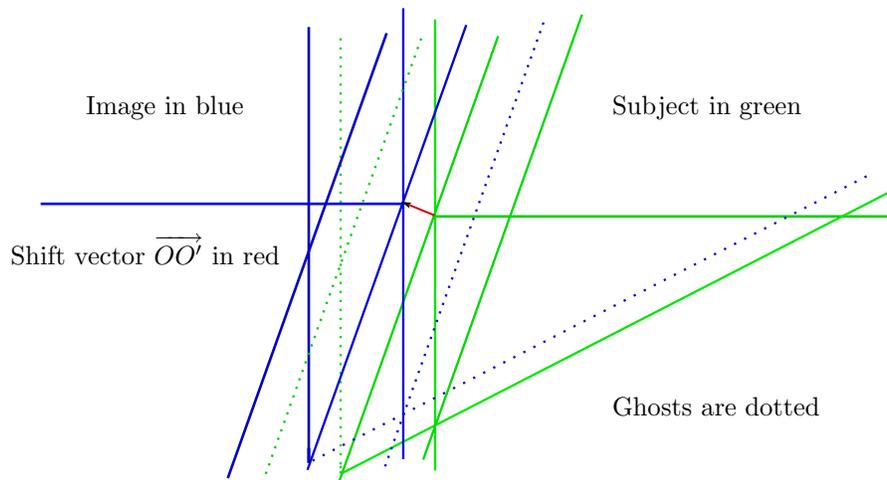


FIGURE 40. Subject, Image, Focal Planes, etc., including Ghosts

There are also 'ghosts' of these planes. For example, the ghost of the rear focal plane is obtained by translating it by $\vec{O'O}$ and that of the rear focal plane by translating it by \vec{OO}' . The antipodal plane of the front focal plane relative to O is the ghost of the rear focal plane, and the antipodal plane of the rear focal plane relative to O' is the ghost of the front principal plane. Similarly given a subject and image plane, each has a ghost. Also, we can define reference planes and hinge lines as before, and they have ghosts. Much of what we derived previously remains

true if we are careful to keep track of the various planes and their ghosts and keep straight what should intersect what and where they should intersect. See Figure 40 where I attempted to distinguish subject from image planes by color and indicated ghosts using dotted lines. I made no attempt to label anything because the diagram is confusing enough as is. But note that when the lens is tilted with respect to the image plane, there will be a shift in the center of the image.

In most cases, the displacement $\overrightarrow{O'O}$ will be small enough not to make a major difference in our previous calculations. But there may be a problem locating just where things are in the case of telephoto lenses, where the principal points may be at some distance along the lens axis from the lens barrel. For example, finding the distance v between the image plane and the reference plane Δ' —the ‘bellows extension’—could be tricky.

10.1. Exit and entrance pupils and parallax. In practice, the principal planes are usually so close together than we may ignore the complications raised above. But there is one effect which can be significant.

The transformation $\tilde{\mathcal{V}}$ tells us how to find, for each subject point, the corresponding image point, but, as noted in Section 8, we have to somehow collect that image in film (or its digital equivalent), so only one image plane, where we place the film, has points that are exactly in focus for a specified subject plane. For an image point P not in that plane image, we considered the circle of confusion where a certain solid cone intersects the film (image) plane. That cone has vertex the image point P and base the aperture. In our earlier model, we assumed the aperture was in the lens plane. In our new model, the physical aperture would be somewhere between the two principal planes, but this aperture is seen through the lens. The image \mathcal{E} of the aperture from the front of the lens is called the entrance pupil, and its image \mathcal{E}' from the back of the lens is called the exit pupil. We have to use the exit pupil rather than the physical aperture to form the solid cone discussed above. If the exit pupil is in the rear principal plane, then everything works as before. (As we shall see, that means the entrance pupil is in the front principal plane.) Unfortunately, the entrance and exit pupils need not be in the principal planes. That is typically true for lenses used for smaller formats such as 35 mm. For most normal view camera lenses, the entrance and exit pupils are usually in the principal planes, or so close that it doesn't matter. But, for lenses of telephoto or inverted telephoto design, which are sometimes used with view cameras, they need not be.

We shall not spend a lot of time here on this subject.⁴³ But we shall outline some basic facts. First, the exit pupil is the image under \mathcal{V} of the entrance pupil, i.e., $\mathcal{E}' = \tilde{\mathcal{V}}(\mathcal{E})$. It follows by calculations similar to those we did previously that we get the lens equation

$$\frac{1}{e} + \frac{1}{e'} = \frac{1}{f}$$

where e is the displacement from the front principal plane along the lens axis of the entrance pupil, e' is the displacement of the exit pupil from the rear principal plane, and f is the focal length. As earlier, e must be taken to be positive to the right and e' positive to the left. But a bit of thought shows how weird the equation

⁴³See the article by Jeff Conrad at www.largeformatphotography.info/articles/DoFinDepth.pdf, which discusses the subject in great detail. Indeed, the material presented here is a slightly modified version of what is in that article.

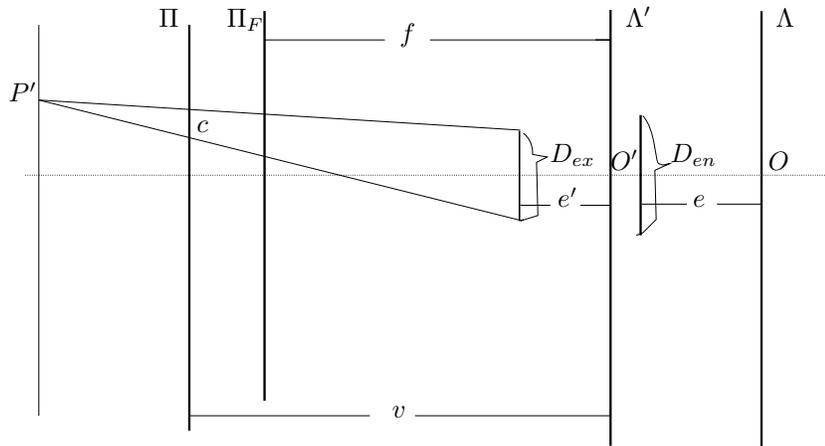


FIGURE 41. Exit and Entrance Pupils

is in this case. Both e and e' are going to be smaller in magnitude than the focal length, so this equation is possible only if e and e' have opposite signs. That usually means that either the entrance pupil or the exit pupil is in the space between the principal planes, and they both lie to the left or they both lie to the right of their respective principal planes.

The ratio of the size of the exit pupil to that of the entrance pupil is called the pupil magnification p . Also, $p = -\frac{e'}{e}$. The pupil magnification is 1 if and only if the entrance and exit pupils are in their respective principal planes, and as we said above, that is usually the case for lenses used with view cameras.

There is one important consequence when $p \neq 1$, and the pupils are not in the principal planes. The principal planes and principal points still determine how subject points are related to their corresponding image points. But, for a point P' not in the exact image plane, i.e., the film plane, we must look at the circle of confusion, i.e., the intersection of that plane with the solid cone with vertex at P' and base the exit pupil. This will certainly have implications for depth of field calculations, which we shall not attempt to address here. But it will also have implications about the apparent position of the out of focus point in the image plane. Namely, as before, if the CoC is small enough it will be indistinguishable from the pinhole image, i.e., the point where the ray from P' to the center of the exit pupil intersects that image plane. If the exit pupil is not in the rear principal plane, that pinhole image will be displaced from where we would place it using the ray $O'P'$, often by enough to be clearly visible in the image. In particular, points in the subject which are on the same ray from O , will have divergent pinhole images, and will no longer line up in the image, i.e., there will be a parallax shift for nearby objects relative to distance objects. That means that relative to the image space, the center of the exit pupil, not the rear principal point, should be considered the center of perspective, and relative to the subject space, the point it comes from, the center of the entrance pupil should be considered the center of perspective. Hence, in applications in which it is important to keep the center of perspective fixed, all

rotations of the lens should be about the center of the entrance pupil. That is specially important in panoramic photography, where one rotates the camera to make exposures of the scene from different angles and then combines the images digitally by making appropriate transformations of those images and merging them. If the center of perspective is not kept fixed in such a rotation, there will be obvious parallax shifts, and it will be impossible, in principle, to merge such elements if they appear in overlapping exposures. The software for combining panoramic images is very sophisticated and may be able deal with such issues in some cases, but everything works much better if the photographer is careful to locate the center of perspective and rotate about it. Special tripod heads or attachments are designed with that in mind.

11. SOURCES OF ERROR

There are several sources of error which may cause results in the field to differ from what geometric optics predicts. We mentioned some before. In this section, we attempt to estimate their magnitudes.

The most obvious of these is the fact geometric optics is only an approximation to the actual behavior of a physical lens. As previously noted, because of lens aberrations, the image of a ‘point’ is not a well defined point. With modern lenses, these aberrations are often so well controlled, that the photographer would be hard put to see any effect. Perhaps the most significant aberration in practice is curvature of field, the fact that the image of a plane is a curved surface rather than another plane. Also, we have to make allowances for the fact that a real lens has some physical extension, as noted in Section 10, and can’t be described as a single ‘principal point’.

Another source of error is possible displacement of the film, where the image is recorded, from the ground glass image used for focusing. This is similar to errors in focusing discussed in Section 11.3, and we will discuss the matter in more detail there. The film’s departure from flatness in the film holder is another source of error, but we won’t discuss it in this article.

11.1. Errors in focusing—Depth of Focus. Another problem is inaccuracy in focusing, which comes about for two reasons. First, there is bound to be some error in positioning the standard on the rail. Most view cameras have a fine focusing knob which through gearing converts large movements of the knob into small displacements on the rail. Such cameras can be positioned to within 0.1 to 0.2 mm or even better. Those without gearing can’t focus as precisely.

More serious is the fact that it is not possible to find a definite point at which the image appears exactly in focus. If you try to focus with the lens at full aperture on some element of the scene, there will be a point on the rail where it just comes into focus and beyond that a point where it just goes out of focus. The spread between those points is called the *depth of focus*. It should be distinguished from depth of field, but the two notions are related, and sometimes it is difficult to disentangle one from the other. Both are based on the same optical effect: a sufficiently small disc in the image plane is not distinguishable from a point

The analysis of depth of focus is very similar to that for depth of field. We will discuss mainly the case of no tilt, but our basic conclusions will remain valid in the general case. Suppose the image of a subject point is at distance v from the principal point, but in attempting to focus on it, you place the ground glass at

distance v_g further from the principal point closer in. In either case, it produces a circle of confusion on the ground glass, and if the CoC is smaller than some threshold \bar{c} the point will appear to be in focus. Refer to Figure 42. It shows two

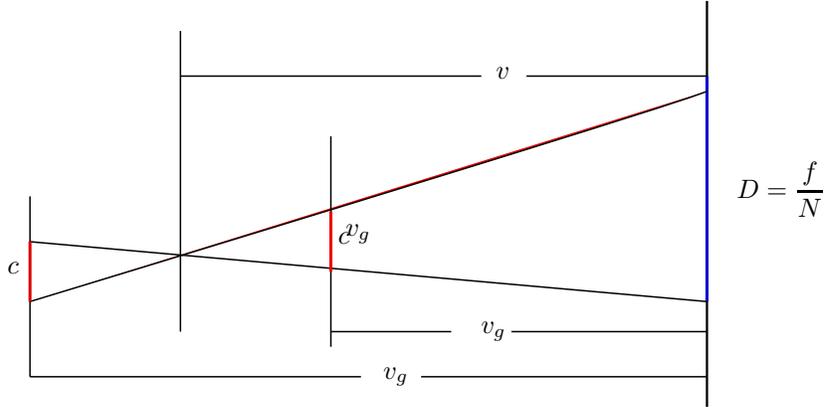


FIGURE 42. Circle of Confusion on Ground Glass

possible positions of the ground glass. Whatever the position, we have by similar triangles

$$\frac{v_g - v}{v} \leq \frac{\bar{c}}{D} = \frac{N\bar{c}}{f}$$

or

$$\frac{v - v_g}{v} \leq \frac{\bar{c}}{D} = \frac{N\bar{c}}{f}$$

where N is the f-number, \bar{c} is the diameter of the maximum acceptable circle of confusion when focusing, and f is the focal length. Thus,

$$|v_g - v| \leq N\bar{c} \frac{v}{f} = N\bar{c}(1 + M)$$

where M is the magnification at v where the exact image is formed.⁴⁴ Taking v_g as far as possible from v in either direction, we obtain

$$\text{Max Focusing Error} = \pm N\bar{c}(1 + M)$$

The depth of focus would be twice this or

$$(\text{DOFoc}) \quad 2N\bar{c}(1 + M)$$

Unless we are in a close-up situation, $M \ll 1$, so a reasonable estimate for the focusing error is $\pm N\bar{c}$. The total depth of focus is twice that or

$$(\text{DOFoc}:\infty) \quad 2N\bar{c}$$

⁴⁴If the lens plane is tilted through angle ϕ with respect to the image plane, we must include the factor $\sec \phi$, but for typical values of ϕ , this is very close to 1.

There is one potential problem with this reasoning. M may be determined from the subject distance u by the formula $M = \frac{1}{u/f - 1}$. But usually, we find instead M_g , the magnification at the position of the ground glass, either by measuring the bellows extension, v_g directly, or by determining the ratio of the image size to the subject size. Using $1 + M_g$ rather than $1 + M$ introduces an error in formula (DOFoc). But, we have

$$|(1 + M) - (1 + M_g)| = |M - M_g| = \frac{|v - v_g|}{u} \leq \frac{Nc v}{f u} = \frac{Nc}{f} M$$

and the quantity on the right is going to be miniscule in any practical case.

Note There is another way to approach the problem. Namely fix the position v_g of the ground glass and ask how far on either side of it an image point can be and still appear in focus. The total range for which that is true is

$$\frac{2Nc(M + 1)}{1 - (Nc/f)^2} = 2Nc(M + 1)(1 + (Nc/f)^2 + (Nc/f)^4 + \dots)$$

which, since $(Nc/f)^2$ is almost always very small, amounts to the same estimate.

One almost always focuses with the lens at maximum aperture. For view camera lenses this can range from $f/4.5$ to about $f/8$ or smaller. The proper choice for \bar{c} depends on a variety of factors. It is related to the value c used for depth of field calculations but is not necessarily the same. In depth of field calculations, one is thinking of a final print viewed at leisure under good lighting. A standard for comparison is an 8 x 10 print viewed at about 12 inches.⁴⁵ The CoC for that is usually taken to be between 0.1 and 0.2 mm, depending on assumptions about who, under what circumstances, is viewing the print. Let's take the value 0.2 mm. (If you prefer a smaller value, it is clear how to modify the calculations.) Since a 4 x 5 negative must be enlarged 2 x to yield an 8 x 10 print, we should reduce the CoC at the level of the film to 0.1 mm.

Let's think a bit about how this correlates with what you see when looking at the ground glass. In comparison to the print, you should put your eye about six inches from the image. Very young people, and some myopic people, with their glasses off, can do that, but most of us can't without a loupe or other aid. So let's assume that for critical focusing you use a loupe, and to start, let's assume it magnifies 2x. Were viewing the ground glass the same thing as viewing the print, then $\bar{c} = 0.1$ mm would be an appropriate choice for estimating depth of focus. A typical aperture would be $f/5.6$, i.e., $N = 5.6$. That would yield a focusing error for distant subjects of about $\pm 5.6 \times 0.1 = \pm 0.56$ mm, and the depth of focus would be twice that or about 1.12 mm.

But, looking at the ground glass under a dark cloth in the field is not the same as viewing a print. Even a well made viewing screen has some texture, which, under sufficient magnification, will interfere with resolving fine detail.⁴⁶ So the proper choice of \bar{c} for you is something you must determine experimentally. You can do that by measuring the distance as above between where a detail just comes

⁴⁵A larger print viewed from proportionately further away appears the same. If your viewers insist on getting very close to a large print, you need to adjust the CoC c accordingly.

⁴⁶For this reason, it is not wise to use high powered magnifiers to focus. Usually one doesn't want to go stronger than 4 to 6 x, and usually 8x is a practical upper limit, to be used only under special circumstances. Even at that level, one may find oneself focusing on viewing screen artifacts rather than on the scene.

into focus and where it just goes out of focus. The focusing CoC can then be determined by using one of the above formulas. You should repeat that several times, with different lenses at different apertures, and take an average to determine the proper value of \bar{c} which works for you.

If you use a stronger loupe than $2\times$, you will be able to look more closely, up to a point, and thereby decrease your focusing error.⁴⁷ Table 1 indicates the results I obtained with my Toho FC-45X camera, with a focusing screen made by Jim Maxwell, and a Rodenstock f/5.6 150 mm lens. Note that the value obtained for

Loupe	Observed depth of focus	Upper bound for \bar{c}
2 X	less than 1.5 mm	0.134 mm
3.6 X	less than 0.7 mm	0.063 mm
7 X	less than 0.3 mm	0.027 mm

TABLE 1. My results focusing on a Maxwell screen with a f/5.6 150 mm lens

\bar{c} with a $2\times$ loupe is not that much larger than the value 0.1 mm that I adopted for depth of field calculations.

As long as you are aware that focusing error is inevitable and you have some idea of its magnitude, there are a variety of things that you can do to reduce it. I'll describe some that I use, but you may come up with better methods that work for you.

First, as noted above, you can use a loupe, as long as its power is not too great.

In addition, you can focus several times, noting in each case where you put the standard, and then focus at the average of those positions. If your errors are random, that may significantly reduce the size of the error. Statistical theory tells us that the error is divided approximately by the square root of the number of measurements. Unfortunately, you have to make quite a few measurements before doing so will have a significant effect. Of course, if you are making a systematic error, then averaging won't help, but in that case, the results should be apparent to you when you examine the pictures you take. You can then try to compensate for that systematic error by focusing either in front or behind where you would otherwise focus, or, by stopping down to extend depth of field.

Many people use the near point/far point method to focus (and also to determine the appropriate f-number for the desired depth of field). I'll describe the method in the case of no tilt. You choose a nearest point you want to be adequately in focus and also a furthest point, noting in each case the position on the rail. You then place the standard halfway between those points. But, if you always approach the near point from the near direction and the far point from the far direction, then your tendency to overrun the correct position may cancel out when you choose the midpoint.

11.2. Where to Focus. In Section 8.1.1, we saw that the correct place to focus on the rail is the harmonic mean v of v' and v'' , the limits determined by the desired depth of field. But, we also said that in most cases this can be approximated

⁴⁷But one must be careful estimating depth of field if you use a stronger loupe. Reducing the effective CoC also reduces the depth of field. Too much depth of field is not usually a problem, but stopping down too far may require longer exposures than subject movement may tolerate.

accurately enough by focusing at the arithmetic mean $\bar{v} = \frac{v' + v''}{2}$. Let's see just how accurate that is. As usual, we have⁴⁸

$$v' = \frac{v}{1 - Nc/f}$$

$$v'' = \frac{v}{1 + Nc/f}$$

so

$$v' + v'' = \frac{v}{1 - Nc/f} + \frac{v}{1 + Nc/f} = \frac{2v}{1 - (Nc/f)^2}$$

$$\text{or } \frac{v' + v''}{2} = \frac{v}{1 - (Nc/f)^2}$$

$$\text{(EF) and } v = \bar{v}(1 - (Nc/f)^2)$$

In most cases $(Nc/f)^2$ is so small that we can ignore the difference. For example, suppose we choose $c = 0.1$ mm, $f = 150$ mm, and $N = 22$. Then $(Nc/f)^2 \approx 0.000215$. In other words, the relative error is a little over 0.02 percent. Except in extreme close-ups, we can assume $\bar{v} \approx f$, so the actual error would be about $0.000215 \times 150 \approx 0.03$ mm. Stopping down to $f/45$ would double this to 0.06 mm, a distance much less than the normal focusing error. The worst case would be a very short focal length lens using a very small aperture in an extreme close-up.⁴⁹ For example, if $f = 65$ mm and $N = 64$, then $(Nc/f)^2 \approx 0.0097$, a relative error of about 1 percent. For $v = 2f = 130$ mm, this would result in an error of about 0.13 mm, which might produce a just barely detectable effect. Of course, at such a small aperture, diffraction would begin to play a role.

So the conclusion is that it is safe to focus halfway between v' and v'' except in extreme close-ups with very short focal length lenses. In such cases, the harmonic mean will be just a trifle short of the arithmetic mean. Even in such a case, it is unlikely that one needs to be so accurate that the difference between v and \bar{v} is important.

11.3. Depth of focus and Depth of Field. At this point, it may be worth discussing how depth of focus affects depth of field. If you are focused slightly off where you think you should be, the total size of the depth of field region won't be significantly affected, but its location will be. It will be shifted in the direction of the error. One way to judge how significant that may be is to compare the depth of focus, which we saw might vary from as little as ± 0.2 mm to as much to ± 0.7 mm, to the focus spread on the rail, which may vary from undetectable up to 2 mm on the low end to as much as 10 mm at the high end. Clearly, the focusing error will be more important when the focus spread is small, so it is in that case that one should try to focus as best as possible, by using a high power loupe and other measures. Given that there is also a limit to how precisely the standard may be placed on the rail, a lower limit for what is attainable might be ± 0.1 to ± 0.2 mm, with the latter being more plausible than the former.

The proper f-stop for a desired depth of field is usually determined, explicitly or implicitly, from the focus spread s . If, you choose the f-number N on the assumption

⁴⁸We can ignore the ' κ effect' since at worst it will reduce the error even more.

⁴⁹Except in close-ups, there is normally little need to use very small apertures with very short focal length lenses.

that the focus position is exactly where it should be, you will have too much depth of field in one direction and not enough in the other, so you should correct for such an error by increasing s by an amount ds to compensate for the shift, which will result in a shift dN in the f-number you get. The relation between the f-number and the focus spread is

$$N \approx \frac{s \cos \phi}{2c(1+M)} \leq \frac{s}{2c}$$

and in any case the N you would choose is proportional to s , so we may conclude

$$\frac{dN}{N} \approx \frac{ds}{s}$$

The associated increase in the f-stop would be

$$\frac{2}{\log 2} \log \left(1 + \frac{dN}{N} \right) \approx 2.89 \frac{dN}{N}$$

where the approximation holds provided $\frac{dN}{N}$ is small. If s is small and ds is large, this can result in a significant change in N . It is easy to see that for larger focus

Focus Spread in mm	f-number
0.9	4.5
1.12	5.6
1.6	8
2.2	11
3.2	16
4.4	22
6.4	32
9.0	45

TABLE 2. Focus Spread and Estimate of f-number using $\frac{s}{2c}$, $c = 0.1$

spreads, we can tolerate a fairly large focusing error. For example, if $s = 6.4$ mm, $ds = 0.7$ mm, $\frac{dN}{N} \approx 0.11$ which would amount to about one third of a stop correction. With a focus spread of 0.9 mm or less, we would need a fairly powerful loupe even to detect it, and even if we could reduce ds to 0.2 mm, the needed correction to the f-stop would be about $2.89 \times 0.22 \approx 0.64$ or two thirds of a stop.

Fortunately, when the focus spread is small, we may want to stop down several stops in any case. But in cases where that is inadvisable, it is specially important to focus as carefully as possible. For larger focus spreads, it might be prudent to stop down up an additional one third to one half stop.

A shift of the film plane from the focusing plane on the ground glass will have a similar effect. It will shift the entire depth of field in one direction. Fortunately, it is a systematic error which the photographer can measure and either correct by adjusting the position of the ground glass or compensate for.

The best way to measure such a shift is to focus as carefully as possible on a scale⁵⁰ at a known distance from the lens and making a 45 degree angle with the line

⁵⁰Marks on the scale should be placed close enough so that in the image, their separation will be comparable in magnitude to the CoC. Otherwise, the drop in resolution may not be detectable.

of sight, take a picture, and see how far off the focus is. Focusing should be done with a high power loupe. The subject distance u should be close enough—perhaps two to three meters—to be able to calculate the magnification $M = \frac{f}{u-f}$ reasonably accurately. Ideally, the process should be repeated several times, perhaps using different lenses, and different target distances. The subject distance should be measured from the front principal point, which for most lenses is close to the front of the lensboard.⁵¹

From the lens equation, it is easy to see that the displacement du in subject distance u is related to the corresponding displacement dv in image distance v by

$$(Sh) \quad dv = -M^2 du$$

You should remember to correct for the angle between the target scale and the line of sight, e.g., for a 45 degree angle, du would be about 70 percent of the distance along the scale as indicated in the print.

It is impossible to eliminate such a shift entirely, and in some cases it may be as large as 0.1 to 0.2 mm, i.e., comparable in magnitude to normal focusing and positioning errors.

Of course, systematic errors, whatever their cause, will become apparent with experience. If you can't track them down and correct them, you can just compensate when focusing or by stopping down a bit more.

11.4. Leveling the Camera. For certain scenes, it is important that the film plane be plumb. This is particularly important to prevent convergence of verticals in architectural photography. Leveling is usually done with a level, which depends on centering a bubble between two reference lines. Checking with my carpenter's level, I found this could produce an error of about 2–4 mm over a distance of 600 mm, which could result in the rear standard being out of plumb by as much as $\psi = 0.005$ radians. Will such a small error be noticeable?

Figure 43 shows the effect of tilting the back on convergence of verticals. In the diagram on the left, Π and Λ indicate the standards of the out of plumb camera, where the angle ψ is greatly exaggerated. Γ indicates a plane through the lens parallel to a vertical building facade. The diagram on the right shows the images of the sides of the building converging to the vanishing point P_∞ . $w_1 + w_2$ is the width of the image of the building at the center of the field, and χ is the angle the indicated side makes with the center line. We have

$$\tan \chi = \frac{w_1}{v \cot \psi} = \frac{w_1}{v} \tan \psi$$

But because the angles are so small, we may use $\tan \phi \approx \psi$ and $\tan \chi \approx \chi$, which leads to

$$\chi \approx \frac{w_1}{v} \psi$$

In most cases, $v \approx f$ and w_1 is bounded by the dimensions of the frame. For plausible values of f and w_1 , $\frac{w_1}{f}$ is not much larger than 1, and is usually somewhat smaller. So χ is, at worst, the same order of magnitude as ψ , and usually it is smaller. For $\psi \approx 0.005$ radians, the departure from the vertical over 100 mm in the image would be at worst about 0.5 mm, i.e., it would be undetectable to the unaided eye.

⁵¹Avoid telephoto or wide angle lenses unless you know the position of the principal point.

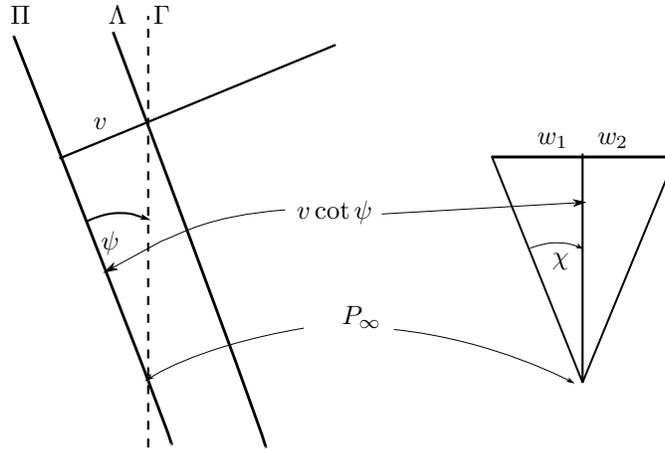


FIGURE 43. Effect on Convergence of Verticals of Tilted Camera

Being out of plumb also slightly affects focus on the building facade because the image of the facade would be tilted by a very small angle ξ with respect to the rear standard. Let S be the distance at which plane of the building facade intersects the lens plane and, by Scheimpflug's Rule, the image of that plane. Then

$$S = u \tan \psi = v \tan \xi \quad \text{whence}$$

$$\tan \xi = \frac{v}{u} \tan \psi = M \tan \psi \quad \text{and}$$

$$\xi \approx M\psi$$

Say $M = 0.01$, $\psi = 0.005$ radians, so $\xi \approx 5 \times 10^{-5}$. Over a distance of 120 mm, that would move the image of the facade about 6×10^{-3} mm. But, even at $f/4.5$, an image point can be as far as about $Nc = 4.5 \times 0.1 = 0.45$ mm on either side of the film plane and still produce a circle of confusion not distinguishable from a point. Hence, this effect is truly so insignificant that it need not concern anyone.

11.5. Alignment of the Standards. The above discussion assumes that the standards are exactly parallel, but in practice, that is unlikely to be the case. The lens plane will make a very small angle ϕ with respect to the film plane. There are a couple of methods for checking parallelism. First, one can check the distance between the standards at each of the four corners. Without using sophisticated equipment, a plausible estimate for the error in such a measurement is about ± 0.5 mm. An estimate for the corresponding tilt angle is obtained by dividing that by the distance from the center of the lens board to a corner. Depending on the design of the camera, that could range between 50 mm and 100 mm. The corresponding tilt would be between 0.01 and 0.005 radians. Alternately, one might use a level, in which case, as noted in Section 11.4, a tilt as large as 0.005 radians might not be detectable. So 0.005 radians is the value we shall use in the remainder of this discussion.

A small tilt ϕ will move the hinge line from infinity to hinge distance $J = \frac{f}{\sin \phi} \approx \frac{f}{\phi}$. For $f = 150$, $\phi = 0.005$, this would be about 30 meters, and it might be significantly shorter for a shorter focal length lens. The corresponding angle the subject plane makes with the vertical would be $\gamma = \arctan \frac{u}{J}$, where u is the distance to the focus point along the line of sight. The value of u would of course depend on the scene, but a rough estimate for a typical scene might be the hyperfocal distance at $f/22$. That would be about 10 meters for a 150 mm lens. So instead of being zero, as would be the case if the standards were exactly parallel, $\gamma \approx 18$ degrees.

Figure 44 shows the presumed region in focus were the standards exactly parallel and the actual region resulting from a slight unintended tilt. The tilt is greatly exaggerated, for purposes of illustration, in comparison to what it would be in reality. *Don't be misled by the orientation in the diagram. The hinge line resulting from an unintended tilt, together with the associated subject planes, could be anywhere.*

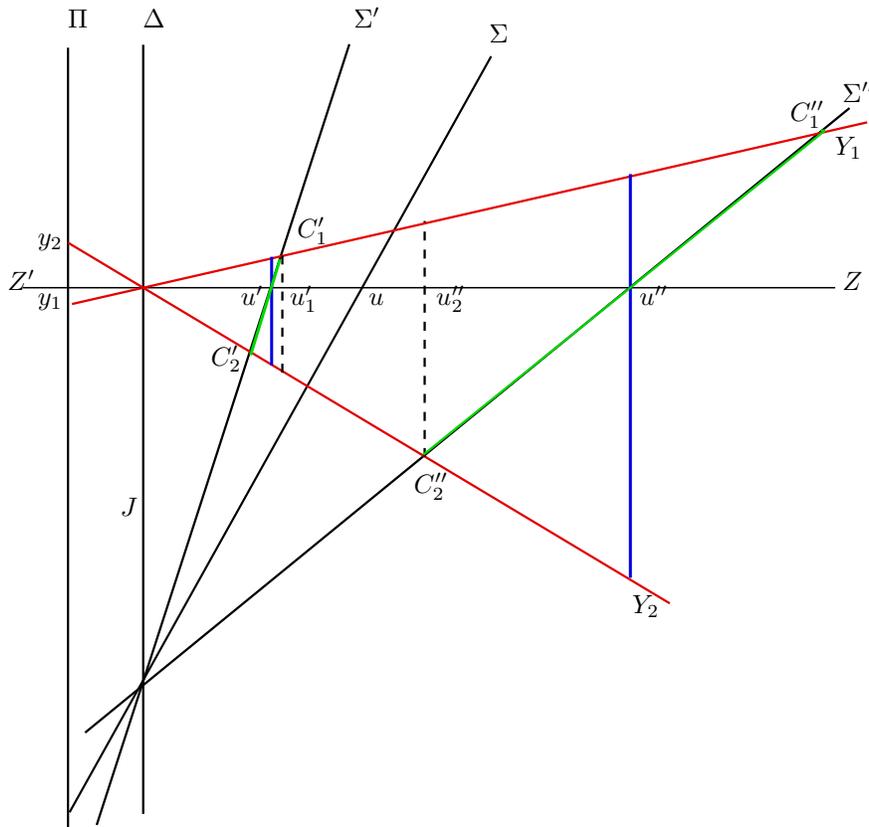


FIGURE 44. Comparison of Depth of Field Regions

We need only consider subject points such that the corresponding image points lie in the frame, and the red lines indicate planes Y_1 and Y_2 , which set the limits on the depth of field imposed by that restriction. Since the tilt axis will generally be skew to the sides of the frame, those planes should be assumed to be parallel to the tilt axis and to pass through opposite corners of the frame. y_1 and y_2 denote the distances at which those planes intersect Π , where those distances are measured from the plane, denoted $Z'Z$ in the diagram, determined by the tilt axis and the line of sight, the positive direction being that away from the Scheimpflug line. (So in the diagram y_1 would be negative.) The blue lines indicate the near and far limits of depth of field under the assumption that the standards are parallel, and the green lines the limits arising because of the tilt. The actual depth of field overlaps the presumed depth of field, being greater or less, depending on the position of the subject point in relation to the plane $Z'Z$. Note that the effect of the inadvertent tilt appears to be much more pronounced in the background than in the foreground, but this may be misleading because of compression of distant points as seen on the ground glass.

The depth of field will of course depend on the relative aperture (f-number) N and the CoC c . Note that c is based on what we hope to achieve in the final print, not on the \bar{c} , the focusing CoC. As noted in Section 11.1, the latter depends strongly on the characteristics of the focusing screen and the power of the loupe.

Σ' and Σ'' intersect $Z'Z$ at distances u' and u'' from the reference plane, which is also the presumed lens plane. As we shall see, these are essentially the same as the values determined in Section 8.4.1 under the assumption that the standards (and hence Σ' , Σ , and Σ'') are parallel. To gauge the effect of the unintended tilt, we need to find the distances, in increasing order, u'_2, u'_1, u''_2 , and u''_1 at the points C'_2, C'_1, C''_2 , and C''_1 at which Σ' and Σ'' cross Y_1 and Y_2 .

If the alignment were perfect, we would expect that everything between two planes parallel to the image plane Π would be in focus, so it is useful to consider what happens in such planes at each of these points. There would be essentially nothing in focus captured by the frame for such a plane at either of the extremes u'_2 (C_2) and u''_1 (C'_1). At each of u' and u'' , the limits which a depth of field table would predict, part of of such a plane captured by the frame would be in focus. Finally, at either u'_1 (C'_1) or u''_2 , everything captured by the frame would be in focus, or at least the figure suggests that would be the case.

To see why the figure is accurate in this regard, recall from Section 8.4.1 that for each distance in the subject space, there is a window in Π showing what is in focus at that distance. That window is of constant height $\frac{Nc}{\sin \phi} \approx \frac{Nc}{\phi}$, but it moves in the image plane Π as we move from foreground to background.⁵² Taking $c = 0.1$ mm, $N = 22$ and $\phi = 0.005$ radians yields about 880 mm for that height. So if the window starts either at C'_1 or C''_2 , the height of the window will be more than enough to cover the frame. Even if we opened up as far as $f/5.6$, we would still have room to spare.

It follows that if we want to be sure we have identified two planes between which everything captured in the frame is in focus, we should use u'_1 and u''_2 rather than u' and u'' as the near and far limits of depth of field.

We now derive formulas for the quantities described above.

⁵²As above, we can ignore the κ effect.

The relevant formulas describing the planes Σ , Σ' , and Σ'' were derived in Section 8.4.1⁵³. We have

$$\begin{aligned} x_2 &= \frac{J}{u}x_3 - J = J\left(\frac{x_3}{u} - 1\right) \\ x'_2 &= x_2 + \frac{J \cos \phi}{H(M+1)}x_3 = J\left(\frac{x_3}{u} - 1\right) + \frac{J \cos \phi}{H(M+1)}x_3 \\ x''_2 &= x_2 - \frac{J \cos \phi}{H(M+1)}x_3 = J\left(\frac{x_3}{u} - 1\right) - \frac{J \cos \phi}{H(M+1)}x_3 \end{aligned}$$

where x_3 denotes the distance to the reference plane, x_2, x'_2 , and x''_2 denote the distances from the plane $Z'Z$ to Σ, Σ' , and Σ'' at x_3 , $H = \frac{f^2}{Nc}$ is the approximation to the hyperfocal distance introduced earlier, and M is the magnification at distance u .⁵⁴ Hence,

$$(SP) \quad \frac{x'_2}{x_3} = J\left(\frac{1}{u} - \frac{1}{x_3}\right) + \frac{J \cos \phi}{H(M+1)}$$

$$(MP) \quad \frac{x''_2}{x_3} = J\left(\frac{1}{u} - \frac{1}{x_3}\right) - \frac{J \cos \phi}{H(M+1)}$$

We shall do the calculation for u'_1 . The calculations for the other quantities are similar. Refer to Figure 45. Using similar triangles, and putting $x_3 = u'_1$ in equation

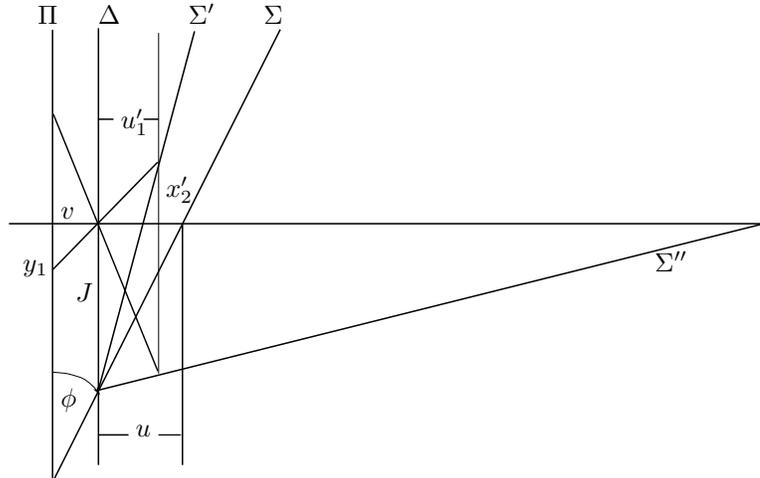


FIGURE 45. Effect of Small Tilt on Depth of Field

(SP), we have

$$-\frac{y_1}{v} = \frac{x'_2}{u'_1} = J\left(\frac{1}{u} - \frac{1}{u'_1}\right) + \frac{J \cos \phi}{H(M+1)}$$

⁵³We may ignore the ' κ effect' on the circle of confusion because ϕ is so small.

⁵⁴Because of the slight tilt, M differs insignificantly from the value that would apply if the standards were parallel. See the end of this section for the details.

The minus sign on the left reflects the fact that y_1 and x'_2 necessarily have opposite signs. Solving for u'_1 , we obtain

$$\begin{aligned}\frac{1}{u'_1} &= \frac{1}{u} + \frac{\cos \phi}{H(M+1)} + \frac{y_1}{vJ} \\ &= \frac{vJH(M+1) + vJu \cos \phi + y_1uH(M+1)}{vJuH(M+1)}\end{aligned}$$

Hence,

$$\begin{aligned}u'_1 &= \frac{vJuH(M+1)}{vJH(M+1) + vJu \cos \phi + y_1uH(M+1)} \\ &= \frac{uH}{H + u\left(\frac{\cos \phi}{M+1} + \frac{y_1H}{vJ}\right)}\end{aligned}$$

But, expanding and using $v = f(M+1) \sec \phi$, we obtain

$$\frac{y_1H}{vJ} = \frac{y_1f^2 \sin \phi}{Nc} = \frac{f y_1 \sin \phi}{v N c} = \frac{f}{f(M+1) \sec \phi} \frac{y_1 \sin \phi}{N c} = \frac{\cos \phi}{M+1} \frac{y_1 \sin \phi}{N c}$$

In addition, it is not hard to show, using the lens equation⁵⁵, that

$$\frac{u \cos \phi}{M+1} = u \cos \phi - f$$

Hence,

$$(C'_1) \quad u'_1 = \frac{uH}{H + \frac{u \cos \phi}{M+1}(1 + \eta_1)} = \frac{uH}{H + (u \cos \phi - f)(1 + \eta_1)}$$

$$\text{where } \eta_1 = \frac{y_1 \sin \phi}{N c} \approx \frac{y_1 \phi}{N c}$$

the approximation holding because $\sin \phi \approx \phi$.

Similarly, using equation (MP), we have for the other intermediate value

$$(C''_2) \quad u''_2 = \frac{uH}{H - (u \cos \phi - f)(1 - \eta_2)}$$

$$\text{where } \eta_2 = \frac{y_2 \sin \phi}{N c} \approx \frac{y_2 \phi}{N c}$$

Note that in the most common situation, $y_1 < 0$ and $y_2 > 0$, so $\eta_1 < 0$ and $\eta_2 > 0$ and in both cases, something would be subtracted from 1 in the parentheses. If the frame is shifted far enough, y_1 and y_2 could have the same sign, but that will only rarely be the case. Note also that $y_2 - y_1$ is the total extent of the frame which, for 4 x 5 format, will be between 100 and 150 mm, depending on the orientation of the frame with respect to the tilt axis. In the most common case, $y_2 - y_1 = |y_1| + |y_2|$, which will help establish bounds for η_1 and η_2 .

⁵⁵From $\frac{1}{u} + \frac{1}{v} = \frac{\cos \phi}{f}$, we get $M+1 = \frac{v \cos \phi}{f}$ so

$$\frac{u \cos \phi}{M+1} = \frac{u \cos \phi}{v \cos \phi / f} = fu \frac{1}{v} = fu \left(\frac{\cos \phi}{f} - \frac{1}{u} \right) = u \cos \phi - f$$

The formulas for the two extreme values are

$$(C'_2) \quad u'_2 = \frac{uH}{H + (u \cos \phi - f)(1 + \eta_2)}$$

$$(C''_1) \quad u''_1 = \frac{uH}{H - (u \cos \phi - f)(1 - \eta_1)}$$

If we set $\eta_1 = \eta_2 = 0$ in the above equations, we get we get the distances at which Σ' and Σ'' cross the plane $Z'Z$.

$$(T') \quad u' = \frac{uH}{H + (u \cos \phi - f)}$$

$$(T'') \quad u'' = \frac{uH}{H - (u \cos \phi - f)}$$

These formulas should be compared to those derived in Section 8.2 under the assumptions that the standards are exactly parallel. They are the same except that $u \cos \phi - f$ replaces $u - f$. Since ϕ is so small, $\cos \phi$ is essentially equal to 1, and the results would be indistinguishable in practice.

How the above formulas relate to what one would do in practice is a bit subtle. They certainly indicate what would happen were one to focus on one specific target point at a known distance u and then try to use depth of field tables to estimate how to set the aperture so that all interesting detail is in focus in the final print. Let's estimate η_1 and η_2 for a typical scene to see what might happen in that case. Take $\phi = 0.005$ radians, $N = 22$, $c = 0.1$ mm, and $y_1 = y_2 \approx 50$ mm. This yields $-\eta_1 = \eta_2 \approx 0.11$. Then with $H = 10$ meters and $u = 8$ meters, we would obtain in the foreground $u'_1 \approx 4.67$ meters and $u' \approx 4.44$ meters, a difference of about 5 percent from the expected value. But that large an effect would only be seen at one corner of the frame. We would actually gain depth of field at opposite corner. Similarly, in the background, $u''_2 \approx 27.8$ meters, and $u'' \approx 40$ meters for a difference of about 33 percent from the expected value, a much larger shift.

But, as noted above, because of the compression of the metric in the distance, as seen on the ground glass, the above figures are misleading. There is another way to look at it, which gives a better idea of what we will see on the ground glass and in the final print. Divide both numerator and denominator of equation (C'_1) by $1 + \eta_1$, and put $H'_1 = \frac{H}{1 + \eta_1}$. We get

$$(D'_1) \quad u'_1 = \frac{uH'_1}{H'_1 + (u \cos \phi - f)}$$

But $H'_1 = \frac{f^2}{Nc(1 + \eta_1)}$ is just the (approximation) to the hyperfocal distance we would obtain for f-number $N(1 + \eta_1)$. If we put that value for the f-number in our depth of field calculator, rather than the value N that we are actually using, the calculator will give u'_1 for the near limit. Similarly, if we use $N(1 - \eta_2)$ instead of N , the calculator will give us u''_2 for the far limit. Alternately, provided $y_1 < 0$, and $y_2 > 0$ —by far the most common case—if we set the aperture at $\frac{N}{1 - \eta}$, where η is the larger of $|\eta_1|$ and $|\eta_2|$, we will get near and far limits on the depth of field encompassing u' and u'' , the values we would have got using N , were the standards

exactly parallel. Thus

$$(ST) \quad -2 \frac{\log(1 - \eta)}{\log 2}$$

is an estimate for the amount we need to stop down to compensate for an inadvertent tilt. In the above example, $1 - \eta = 0.89$, and amount we need to stop down is about one third of a stop.

What happens if you don't use depth of field tables? There are two possible approaches which are commonly used. First, you can just stop down and try to visually estimate the depth of field on the basis of what you see on the ground glass. Alternately, you can use the near point/far point method: focus on the rail halfway between where specific points in Σ' and Σ'' (presumed parallel to Π) come into focus, and set the aperture based on the focus spread and the chosen CoC c . Or, you can use some combination of the two approaches.

The major advantage of the purely visual approach is that you can examine the entire frame, both foreground and background to make sure all interesting detail is adequately in focus, so the precise shape of the depth of field region is not relevant. But the approach also has some drawbacks, the most obvious being that the image may be too dim at plausible taking apertures to see much of anything. In addition, if you use a powerful loupe in order to deal with dimness, what you see will be determined by \bar{c} , the focusing CoC, which depends on the power of the loupe. The net effect is likely to be that you will stop down too far, by up to two stops. That can be a problem if subject movement is an issue because of longer exposure time.

Suppose instead you use the near point/far point method. The near and far points, together with the position of the hinge line resulting from the unintended tilt, will determine the planes Σ' and Σ'' . The proper aperture is given in terms of the focus spread s by the formula $\frac{s}{2c(1+M)} \cos \phi$. But, ignoring the term $\cos \phi \approx 1$, you get the same answer you would get under the assumption of perfect alignment. At that aperture, if the near point were chosen at C'_1 and the far point at C''_2 , then everything between the planes through those points, as indicated in Figure 44, would be in focus in the final print. In order to pick those points out, you need to know which plane is Y_1 and which is Y_2 .

Here is one way to do it. At each corner of the frame, focus at infinity or as close to infinity as possible. Look at Figure 46. If you focus so that the subject plane Σ_1 is parallel to Y_1 , the depth of field wedge, at the focusing aperture, will cut across the region between Y_1 and Y_2 , so there will be lots of detail along the diagonal which is in focus all the way up to the opposite corner. On the other hand, if you focus such that so the subject plane Σ_2 is parallel to Y_2 , the wedge will look very different. The 'upper' bounding plane may intersect Y_1 at great distance, but the 'lower' bounding plane won't intersect it anywhere in the the subject space. As a result, little, if anything, will remain in focus as you look on the diagonal towards the other corner..

There may be a similar but less pronounced effect for the other pair of corners, but the proper choice is that for which the effect is most pronounced. Of course, it may turn out that there is no detectable difference for either pair of corners, but that just means that the inadvertent tilt is too small to worry about and the standards are exactly parallel as best as can be determined.

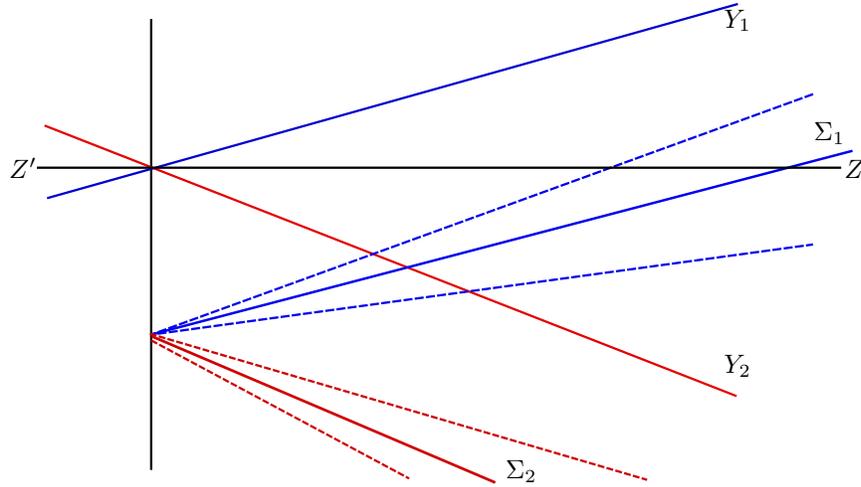


FIGURE 46. The Difference between focusing at ∞ at Y_1 and Y_2

To summarize, focus at infinity at each of the four corners and note the corner for which the greatest amount remains in focus along the diagonal. That is the corner in which you should look for the near point, and the opposite corner is the one where you should look for your far point. If you can't tell the difference, then your standards are aligned well enough and it doesn't matter.

The above analyses also suggest some ways to reduce the magnitude of the error, perhaps to the point that you can ignore it entirely. One should of course do one's best to align the standards. It may be possible thereby to reduce ϕ to as little as 0.001–0.002 radians. Also, stopping down will reduce the magnitude of the error in two ways. First it will reduce η_1 and η_2 , but, in addition, it will increase J , decrease H , and most likely decrease u , the distance at which you focus, because of increased depth of field. On the other hand, being fussier about fine detail by choosing a smaller value for c , paradoxically, will actually make matters worse because of the quotient $\frac{\phi}{c}$ will have a smaller denominator.

Aside. There are two possible expressions for the magnification at distance u

$$M = \frac{f}{u - f}$$

$$M_\phi = \frac{f}{u \cos \phi - f}$$

where the first is calculated on the assumptions that the standards are exactly parallel and the second incorporates the tilt. Dividing the first by the second and doing some algebra results in the relation

$$M_\phi = M \frac{1}{1 - (1 - \cos \phi)/(1 - f/u)}$$

But $1 - \cos \phi \approx \phi^2/2 \leq \frac{0.005^2}{2} = 1.25 \times 10^{-5}$. Even in close-ups $u \geq 2f$ so $1 - f/u > 1/2$ so at worst that increases the relative error to 2.5×10^{-5} or 0.0025 percent. Thus M_ϕ is so close to M that we can ignore the difference.

11.6. Error in Setting the Tilt. Let's now consider the case of an intentional tilt, the purpose of which is either to set an explicit exact subject plane Σ , or, more often, a desired depth of field between bounding planes Σ' and Σ'' . A crucial parameter is the tilt angle⁵⁶ ϕ or equivalently the hinge distance $J = \frac{f}{\sin \phi}$.

If our only aim is to establish an exact subject plane, then the error in setting ϕ is essentially irrelevant. If the desired plane looks in focus on the ground glass then there is an excellent chance it will look in focus in the final print. That would be true even if we didn't use a particularly powerful loupe since the taking aperture is likely to be several stops smaller than the focusing aperture, thus increasing the depth of field about the exact subject plane. On the other hand, an error in the tilt angle may in some cases result in a significant shift in the position of the hinge line. We need to determine if that will result in a significant change in the expected depth of field, and that is the aim of the analysis below.

Recall Wheeler's Rule in Section 6.1.

$$(WR) \quad \sin \phi = \frac{1}{1 + M} \frac{s}{t}$$

where s is the focus spread between the images of two points in the desired subject plane Σ , t is the vertical⁵⁷ distance between those points, when each is in focus, on the ground glass, and M is the magnification at the distance u at which the Σ crosses the normal line to the standards. *The measurements of s and t and the value of M are those that hold before any tilt is applied.* It is easy to estimate the error in ϕ from formula (WR), and it is clear how it depends on the error in measuring the focus spread. But the rule is seldom used in practice, except possibly to establish an initial guess for the tilt. Instead one tries, visually, to adjust the tilt so that the desired subject plane is entirely in focus. If two selected points in that plane are not simultaneously in focus, there will be some focus spread between them as one focuses first on one and then on the other. When there is no discernible focus spread, the tilt angle is presumed correct. Unfortunately, because of depth of focus considerations, a very small focus spread can't be distinguished from no focus spread, and that will lead to an error in ϕ , J , and the position of the resulting subject plane.

In Figure 47, we show the relation between the subject plane Σ and the image plane Π , when the latter is close to being vertical but not quite there. ϕ is the angle the lens plane makes with the vertical⁵⁸, and β is the angle the image plane makes with the vertical.

Note that we have chosen the orientations for ϕ and β opposite to one another, the former being positive in the counterclockwise direction and the latter in the clockwise direction. The tilt will be exactly right when $\beta = 0$, but in practice it will be some small angle. As before, we will have

$$\tan \beta = \frac{s}{t}$$

where s (possibly negative) is the focus spread between two image points and t is their vertical separation on the ground glass, when each is in focus. If s is smaller

⁵⁶In most cases, we don't actually need to know what that angle is, but it still plays a role.

⁵⁷The usual disclaimer applies. The language must be changed if the tilt axis is not horizontal.

⁵⁸This assumes a pure tilt about a horizontal axis. In general, we would measure ϕ with respect to the line in the original position of the lens plane perpendicular to the tilt axis.

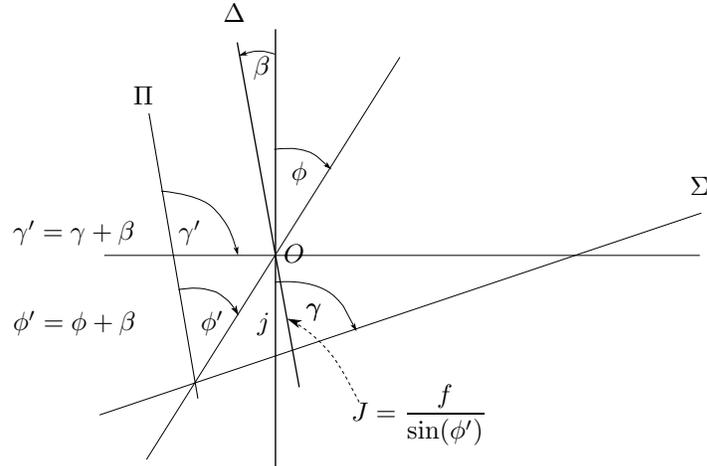


FIGURE 47. Tilt Angle Dependence

than the depth of focus, then we won't be able to distinguish it from zero. By our previous discussion, depth of focus depends on a variety of factors, and is roughly inversely proportional to the power of the loupe used in focusing. For $4\times$ power, it is unlikely that we can be sure of the position of each point to better than 0.25 mm, so an upper bound to the uncertainty in s is about 0.5 mm. One usually tries to choose the points being compared as far apart as practical, so let's take $t = 100$ mm. In that case,

$$\beta \approx \tan \beta \approx \frac{0.5}{100} = 0.005 \text{ radians}$$

On the other hand, using a $2\times$ loupe and taking $t = 50$ mm, yields four times that or $\beta \approx 0.02$ radians. Of course, in either case, the errors in measuring the positions of the two points might be in opposite directions and exactly cancel, but we can never be sure of that.

We now determine how ϕ is related to β . The vertical distance j from O to the subject plane Σ is fixed by the subject plane, and so is the angle γ that the subject plane makes with the vertical. As indicated in the diagram, the angle the image plane Π makes with the lens plane is given by $\phi' = \phi + \beta$, and the angle it makes with the vertical is given by $\gamma' = \gamma + \beta$. As usual, the reference plane Δ is parallel to the image plane, so the angle it makes with the vertical is also γ' and the angle it makes with the lens plane is also ϕ' . Hence, the hinge distance is $J = \frac{f}{\sin(\phi')} = \frac{f}{\sin(\phi + \beta)}$. Consider the small triangle bounded by j and J . The angle opposite j is the supplement of γ' and that opposite J is the angle γ . Hence, by the law of sines and the fact that supplements have the same sine, we get

$$\frac{j}{\sin(\gamma + \beta)} = \frac{J}{\sin \gamma} = \frac{f}{\sin(\phi + \beta) \sin \gamma}$$

$$(R) \quad j \sin(\phi + \beta) \sin \gamma = f \sin(\gamma + \beta)$$

Differentiate both sides of (R) with respect to ϕ , keeping in mind that j, J , and γ are fixed. We obtain

$$j \cos(\phi + \beta) \sin \gamma \left(\frac{d\phi}{d\beta} + 1 \right) = f \cos(\gamma + \beta)$$

which we can solve to obtain

$$\begin{aligned} \frac{d\phi}{d\beta} + 1 &= \frac{f \cos(\gamma + \beta)}{j \cos(\phi + \beta) \sin \gamma} \\ \frac{d\phi}{d\beta} &= -1 + \frac{f \cos(\gamma + \beta)}{j \cos(\phi + \beta) \sin \gamma} \\ \text{(TR)} \quad &= -1 + \frac{\sin \phi_0 \cos(\gamma + \beta)}{\cos(\phi + \beta) \sin \gamma} \end{aligned}$$

where we use the fact that $\frac{f}{j} = \sin \phi_0$, where ϕ_0 is the tilt angle for which the image plane is vertical and $\beta = 0$.

Let's look carefully at the expression on the right of equation (TR). We already saw that a reasonable estimate for β is 0.005, and ϕ will not usually be larger than 0.2, so $\cos(\phi + \beta) \approx 1$, and $\sin \phi_0 \approx 0.2$. The subject plane doesn't usually tilt very strongly up or down, so it is safe to suppose $1.5 < \gamma < 1.65$ and the same can be said of $\gamma + \beta$. So $-0.08 < \cos(\gamma + \beta) < 0.08$. Putting that all together gives a rough estimate

$$\frac{d\phi}{d\beta} \approx -1 \pm 0.016$$

within the relevant range. That tells that over the relevant range, the relationship between ϕ and β is very close to linear, with slope very close to -1. In particular, when we try to set the tilt angle ϕ to ϕ_0 , the value with $\beta = 0$, the difference $d\phi = \phi - \phi_0$ is approximately equal to β .

Note however, that the image will be recorded in the plane $\tilde{\Pi}$ of rear standard, which is vertical, not in Π . We can presume the standard has been placed to split the slight focus spread evenly, as indicated in Figure 48. That will displace the subject plane slightly from Σ to $\tilde{\Sigma}$. The figure does not show the tilt angle ϕ_0 which we really wanted, but it appears implicitly as $j = \frac{f}{\sin \phi_0}$, the distance at which Σ crosses the reference plane. Also, we saw above that $d\phi = \phi - \phi_0$ is approximately equal to the negative of β , the angle Π makes with the vertical. In the figure, β is positive, $\phi < \phi_0$, whence $J > j$, and $\tilde{\Sigma}$ passes under A and above B . If $\beta < 0$, and $\phi > \phi_0$, then $\tilde{\Sigma}$ passes over A and under B .

Corresponding to $d\phi$, we have, using $J = \frac{f}{\sin \phi}$,

$$\begin{aligned} dJ &= -\frac{f}{\sin^2 \phi} \cos \phi d\phi \\ \text{(R}\phi) \quad \frac{dJ}{J} &= -\frac{\cos \phi}{\sin \phi} d\phi = -\frac{d\phi}{\tan \phi} \approx -\frac{d\phi}{\phi} \end{aligned}$$

Note that $d\phi \approx \beta$, being determined by depth of focus considerations, is essentially independent of ϕ , which means that for small values of ϕ , the *relative error* $\frac{d\phi}{\phi}$ could be pretty large. So we should stop to think a bit about how small ϕ might be in practice.

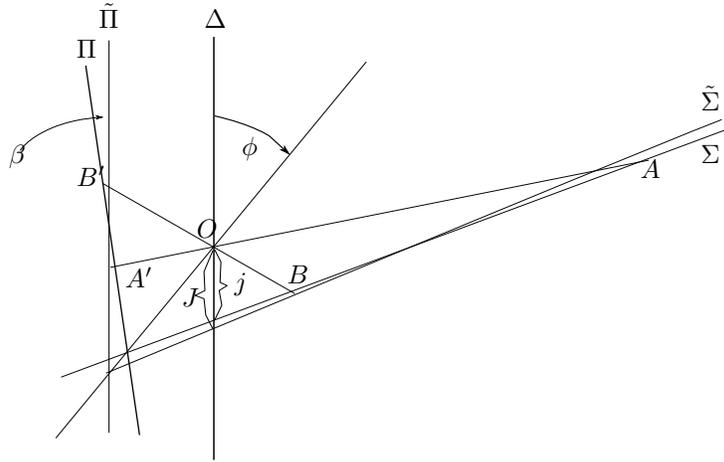


FIGURE 48. Subject Plane Error

Refer again to Wheeler's rule, i.e., formula (WH). If, with the standards parallel, s is less than the depth of focus, you can't set a tilt on the basis of what you see on the ground glass. With the depth of focus between 0.5 mm and 1.0 mm, and t between 50 mm and 100 mm, as we saw above, that means that the minimum possible value for $\phi \approx \tan \phi$ could range between 0.005 and 0.02 radians. I've done some experiments which suggest that even those estimates are optimistic in practice. An informal online survey suggested that tilt angles less than 2 degrees or 0.035 radians are rare. So let's assume $\phi \geq 0.03$ radians except in special circumstances. Clearly, in such cases, we would do our best to minimize $d\phi$ so let's assume $d\phi \approx 0.005$ radians, so the relative error could be as large as 1/6 or about 15 percent. On the other hand, for a moderate tilt of 0.1 radians, even if we were a bit less careful and $d\phi \approx .01$ radians, then the relative error would be only 10 percent. By being careful, it could be reduced to 5 percent.

We are now in a position to see if these errors will affect the expected depth of field, or equivalently, the f-stop needed to obtain some desired depth of field.

There are no tables which give the depth of field T above and below the exact plane of focus in terms of the parameters, but we could rely on the formula (SplM) from Section 8.4.1

$$(SplM) \quad T = 2 \frac{J}{H} \frac{\cos \phi}{M+1} x_3 = \frac{Nc}{f \tan \phi} (M+1) x_3$$

with the total being twice that. As we saw, the relative change in M is miniscule, and, except in close-ups, M is small enough to disregard in any case. So we have

$$\frac{dT}{T} = \sec^2 \phi \frac{d\phi}{\tan \phi} + \frac{dN}{N}$$

If we set this to zero on the assumption that N has been adjusted to compensate for the change in ϕ , we obtain

$$\frac{dN}{N} = -\sec^2 \phi \frac{d\phi}{\tan \phi} = -\sec \phi \frac{d\phi}{\sin \phi} \approx \frac{d\phi}{\phi}$$

In number of stops of fractions thereof, this amounts to

$$\frac{2}{\log 2} \log \left(1 + \frac{d\phi}{\phi} \right) \approx 2.89 \frac{d\phi}{\phi}$$

According to the above estimates, $\frac{d\phi}{\phi}$ can range from 0.05 to 0.17. So the needed correction to the aperture might range between one sixth to one half a stop, with the former being more likely in typical situations.

The above calculation would be useful only if you knew the actual value of T that you want at some specific distance x_3 . But that is unlikely to be the case. Instead, you usually identify target points on either side of the exact subject plane that you want to be in focus and then try to adjust the wedge so that it includes those points. There are two basic approaches to doing that *after establishing the tilt angle by choosing an exact subject plane*: (i) stopping down and visually determining what is in focus or (ii) calculating the f-number based on the focus spread between the target points.

The first approach has the advantage of being entirely visual, and you can also adjust the position of the exact subject plane at the same time. It won't matter if there is an error in setting ϕ because you can see what is in focus. But you do have to be careful that your focusing CoC is close to the CoC which is appropriate for figuring depth of field in the print. Otherwise, you may end up either with too little or too much depth of field in the end result. The latter, in particular, can be a problem if you want to limit depth of field or if subject motion becomes apparent because of increased exposure time. It also has the disadvantage that it only works if you can still see fine detail as you stop down. Many people may have difficulty seeing well at $f/16$ and few can see much of anything beyond $f/22$. Using a powerful loupe helps, but in effect reduces the focusing CoC, which, as noted above, may lead you to stop down too much.

Just how an error in ϕ affects depth of field when using the focus spread method is more complex. It may depend on the specific scene. Consider a scene such as a flat field of wildflowers receding into the distance with nothing above the ground in the foreground, a tree in the middle ground, and perhaps some mountains in the far distance. Figure 49 illustrates such a scene schematically. Refer back to

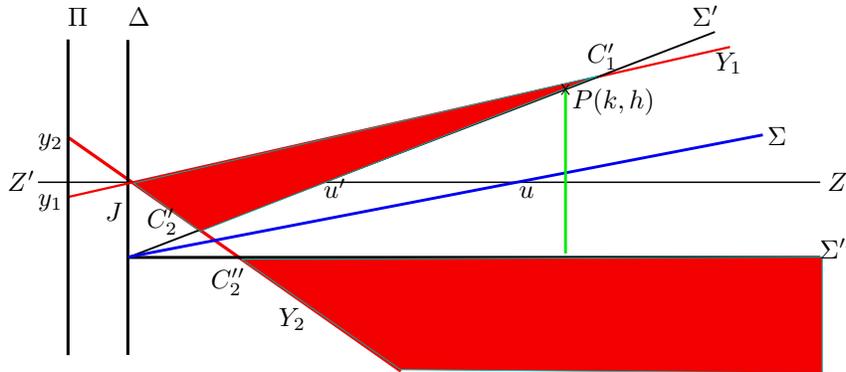


FIGURE 49. Example

Figure 44, and note the differences. Σ'' has been chosen to lie along the ground,

and Σ' to pass through $P(k, h)$ at the top of the tree, close C_1' ⁵⁹. The image of C_2'' in the foreground would be at the top of the frame on the ground glass, and the image of high point $P(k, h)$, which is presumed to be below the images of the mountain tops, is close to the bottom of the frame. Note that although the scene almost fills the frame, two regions (marked in red), one between Σ' and Y_1 and the other between Σ'' and Y_2 could produce images which are in the frame but outside the depth of field. That could be a problem were there anything of interest in those regions which was not obscured by other parts of the scene. In this case, it is assumed that there is nothing of interest in the first region and that the second region is underground.⁶⁰

How would you go about setting up the camera for this scene. First, you would almost certainly use the plane Σ'' to determine the tilt angle, since it would contain lots of detail, and most likely you could minimize the error $d\phi$ by using it. You would then note the position on the rail, refocus on the point P and note its position on the rail. Next, you would measure the focus spread S in order to find the f-stop, using

$$(NEq) \quad N = \frac{S}{2c} \frac{\cos \phi}{M+1} \approx \frac{S}{2c(M+1)} \approx \frac{S}{2c}$$

Finally, you would place the standard halfway between the two reference points on the rail in order to set the position of the exact subject plane Σ .

Note that all of this will be done with the actual tilt, whatever it is, and that in turn will determine the focus spread S . If you went to the trouble of independently measuring ϕ and used the first expression in formula (NEq), then any error in that measurement would make a difference in the calculated value for N , as noted above, and you would have to compensate for it. But you are highly unlikely to do anything like that. Instead you would use the second or third expressions which do not involve ϕ . As a result you might overestimate slightly how much you need to stop down, and in general that would be innocuous.

More serious are the errors which result when focusing either on the low point or the high point, or, in placing the standard halfway between them. But that was addressed in Section 11.3, and the tilt does not significantly change the conclusions drawn there. If the spread S is large, then the necessary correction to the aperture will probably be less than one third of a stop. But since the point of using a tilt is stop down less than we would without tilting—at the price of dealing with the wedg—the focus spread may be relatively small. If so, you may want to stop down an extra stop or more to be sure. And, as noted before, you should try to determine the positions of the reference points on the rail as carefully as possible by using a loupe and the other measures discussed before.

There is a related question which is of passing interest, but is unlikely to affect what you do. Suppose you go through the process more than once. Since the tilt angle will vary, so also will the resulting focus spread, and hence the value of N you end up with. Let's see by how much.

⁵⁹The intersection C_2' of Y_2 with Σ' would be in the image space, which in effect means that the depth of field extends to infinity.

⁶⁰Note also that there are two regions, that bounded by Y_2, Σ' and Σ'' and that between Y_1 and Σ' , which are in the depth of field but outside the frame and hence irrelevant.

By similar triangles—not drawn in figure 49—we have

$$\begin{aligned}\frac{k+J}{k} &= \frac{J}{u'} \\ u' &= \frac{J}{J+k} \quad \text{so} \\ du' &= \frac{(J+k)dJ - JdJ}{(J+k)^2} = \frac{k dJ}{(J+k)^2}\end{aligned}$$

Hence

$$\frac{du'}{u'} = \frac{k}{J+k} \frac{dJ}{J} = -\frac{k}{J+k} \frac{d\phi}{\tan \phi} \approx -\frac{k}{J+k} \frac{d\phi}{\phi}$$

Let's see how much effect this has on the position of the standard.

$$\begin{aligned}v' &= \frac{f u'}{u' \cos \phi - f} \\ dv' &= \frac{(u' \cos \phi - f) f /, du' - f u' (du' \cos \phi - u' \sin \phi d\phi)}{(u' \cos \phi - f)^2} \\ &= \frac{-f^2 du' + f(u')^2 \sin \phi d\phi}{(u' \cos \phi - f)^2}\end{aligned}$$

Hence

$$(DR\phi) \quad \frac{dv'}{v'} = -\frac{v'}{u'} \frac{du'}{u'} + \frac{v'}{f} \sin \phi d\phi = -M' \frac{du'}{u'} + (M' + 1) \frac{d\phi}{\phi}$$

where M' is the magnification at the distance u' at which Σ' crosses the reference normal. A similar calculation gives the relative error in v'' in terms of M'' , the corresponding magnification at u'' where Σ'' crosses the reference normal. M' (and similarly M'') will be very small except in close-ups, and by the above $\frac{du'}{u'}$ is the same order of magnitude as $\frac{d\phi}{\phi}$. Also, $\sin \phi d\phi$ is miniscule. Hence, the change in the focus spread due to $d\phi$ will be much less than the error in measuring it, and hence can be ignored.

As noted above, this calculation, while confidence building, is not really relevant.

11.7. Conclusion. The above discussions suggest that it would be prudent to stop down one additional stop or more to compensate for the cumulative errors discussed above as well as for the ' κ ' effect discussed previously. In critical situations where that is not feasible, one should be as careful as possible in focusing, maintaining alignment, and performing movements.

APPENDIX A. RELATION TO DESARGUES THEOREM

We say two triangles are perspective with each other from a point, if the three lines connecting corresponding vertices intersect in that point. In effect the second triangle is obtained from the first triangle by the perspectivity from that point. We say that two triangles are perspective with respect to each other from a line if corresponding sides are coplanar and the three points of intersection formed thereby

all lie on that line. Desargues Theorem⁶¹ says that if two triangles are perspective from a point then they are also perspective from a line. Desargues Theorem is virtually a tautology in $\mathbb{P}^3(\mathbb{R})$ if the two triangles don't lie in the same plane. (See Figure 50. The line l containing $R, S,$ and T is the intersection of the planes containing $\triangle ABC$ and $\triangle A'B'C'$.)

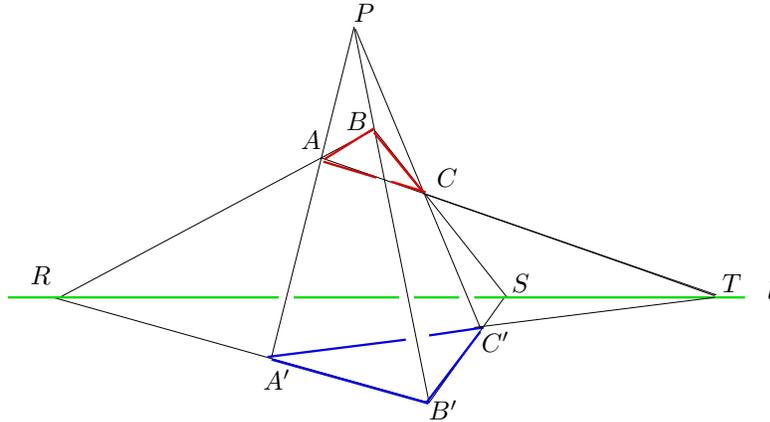


FIGURE 50. Desargues Theorem in Space

To prove it when the triangles lie in the same plane, you must construct an appropriate third triangle, not in that plane, which is perspective with each of the original two triangles from appropriate points.⁶² The dual of Desargues Theorem says that if two triangles are perspective from a line then they are also perspective from a point. (Figure 50 also makes clear why it is true in space. The point P is the intersection of the three planes determined by the corresponding sides, which, by assumption, are coplanar.)

We shall derive the Scheimpflug Rule from the law of optics that says that the image of any point P in the subject plane Σ is formed as follows. (See Figure 52 in Appendix B.) Take a ray from P parallel to the lens axis—hence perpendicular to the lens plane—to its intersection Q with the lens plane; refract it so it passes through the rear focal point F ; then the lines QF and PO are coplanar, and their intersection is the image point P' .

Refer to Figure 51. In the diagram, O is the principal point, F is the rear focal point, P_1 and P_2 are arbitrary points in the subject plane Σ , Q_1 and Q_2 the corresponding projections on the lens plane Λ , and P'_1 and P'_2 the corresponding image points. Consider the triangles $\triangle P_1OP_2$ and $\triangle Q_1FQ_2$. By construction, the lines $OF, P_1Q_1,$ and P_2Q_2 connecting corresponding vertices are parallel, and

⁶¹Some may be tempted to place an apostrophe at the end of 'Desargues'. That would be appropriate if it were a plural noun, but Girard Desargues (1591 - 1661) was definitely one person. It would be correct to use 'Desargues's Theorem' or 'the Desargues Theorem' instead, but either seems awkward, so, as is common practice, we shall ignore the problem and omit the apostrophe entirely.

⁶²See any good book on projective geometry, e.g., *Foundations of Projective Geometry* by Robin Hartshorne, for further discussion.

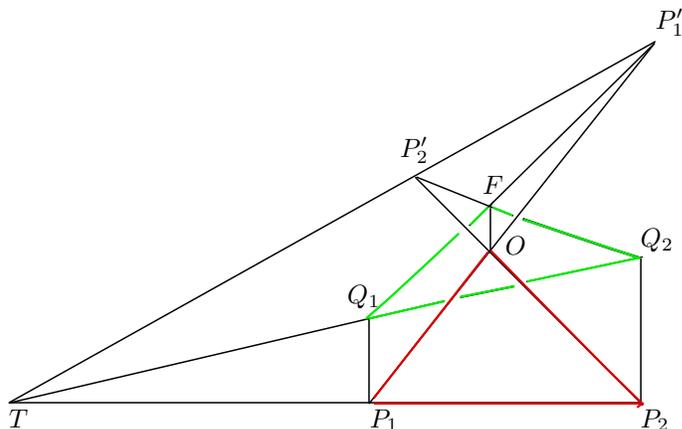


FIGURE 51. Derivation of the Scheimpflug Rule from Optical Rules

hence they intersect in a common point at infinity. By Desargues Theorem⁶³, the intersections of corresponding sides $P'_1 = OP_1 \cap FQ_1$, $P'_2 = OP_2 \cap FQ_2$, and $T = P_1P_2 \cap Q_1Q_2$ are collinear. Since P_1P_2 is in the subject plane, and Q_1Q_2 is in the lens plane, T is in their intersection. Since it also lies on $P'_1P'_2$, it is in the image plane. Since P_1 and P_2 were arbitrary points of the subject plane, it follows that T could be any point on the intersection of the lens plane and the subject plane, and so that line is also contained in the image plane, and the argument is complete.

Note that for the above arguments to work physically, the points P_1 and P_2 in the subject plane must be chosen close enough so that the points Q_1 and Q_2 are in the lens opening. If not, no rays would actually emerge from them on the other side of the lens plane. There is no problem in accomplishing that, so the proof is valid *provided we agree that the lens map takes planes into planes*. But it is in fact true in general, i.e., given any point P in the subject plane, once we know the properties of the lens map, including the Scheimpflug Rule, we saw that we can prove the rule stated above for image formation holds, even if the lens plane is opaque at Q . Of course, from the perspective of a mathematician who is used to such things, it is simpler and more elegant just to assume the rules (1), (2) and (3), all of which are quite plausible, and derive any optical principles we need from them. Others may find it more appealing physically to derive Scheimpflug's law from optical principles.

As we noted at the beginning of this article, real lenses don't actually obey the laws of geometric optics because of lens defects, but they come pretty close because of advances in lens design. In principle, the approximations are only supposed to be valid for rays close to the lens axis, but as we have noted previously, they work much more generally. It would be an interesting exercise to determine how well

⁶³Note that this uses only Desargues Theorem in the case where the triangles are not coplanar. In any event, as noted above, it is inherent in the geometry, so it would hold in any case. So statements that imply that the Scheimpflug Rule is a consequence of Desargues Theorem are misleading. It is actually a consequence of the rules of geometric optics, in fact equivalent to them, as we have shown.

the conclusions we have reached in this article hold up for real lenses. But that is something, I am not qualified to do, so I leave it for experts in lens design.

APPENDIX B. A SIMPLE GEOMETRIC DERIVATION OF THE LENS EQUATION

I've tried to relate the lens equation to the properties of the lens map \mathcal{V} using cross ratios, since I feel this illustrates the underlying structure of the theory better. But it is possible to derive the lens equation by a simple geometric argument, if we just use two basic principles of geometric optics, no matter how they were derived. These are (see Figure 52)

(A) The line connecting a subject point P to the corresponding image point P' passes through the principal point O .

(B) A line from the subject point P , parallel to the lens axis is refracted at Q in the lens plane so as to pass through the focal point F and thereafter to the image point P' .

These rules allow us to determine the image point P' if we are given the subject point P or vice-versa.

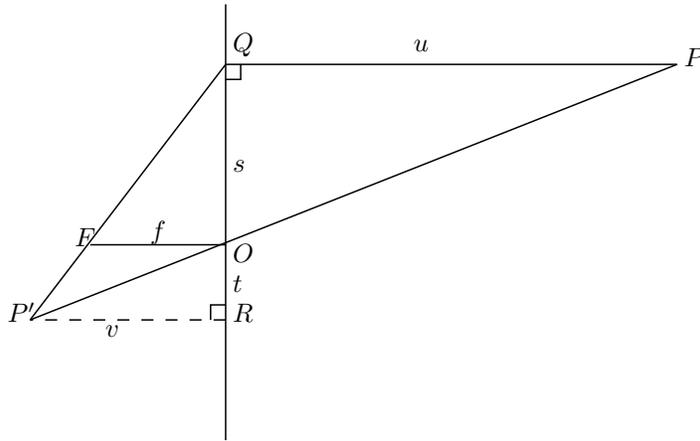


FIGURE 52. Similar Triangles

Look again at the figure. The similar triangles $P'FO$ and $P'QP$ give us

$$\frac{f}{u} = \frac{t}{s+t}$$

Similarly, the similar triangles QFO and $QP'R$ and give us

$$\frac{f}{v} = \frac{s}{s+t}.$$

Hence,

$$\frac{f}{u} + \frac{f}{v} = \frac{t}{s+t} + \frac{s}{s+t} = 1$$

from which the lens equation follows.

APPENDIX C. MAPPING THE LENS PLANE TO THE REFERENCE PLANE

In this section we derive a coordinate description of the projection through P' mapping points in the lens plane Λ to points in the reference plane Δ . We use the coordinate systems and notation introduced in Section 8.3. See, in particular Figure 20.

A projective transformation \mathcal{P} from one projective plane to another is determined by a 3 x 3 matrix

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

which is unique up to multiplication by a non-zero scalar. The transformation is described using homogeneous coordinates by multiplying by a 3 x 1 column vector \mathbf{X} , representing a point in Λ , on the left by A , to obtain a 3 x 1 column vector \mathbf{x} , representing its image in Δ . We choose the notation, so that $X_0 = 0$ and $x_0 = 0$ represent the lines at infinity in each plane. In terms of the affine coordinates (X_1, X_2) in Λ and (x_1, x_2) in Δ , the transformation is given by linear fractional transformations

$$(LF) \quad x_1 = \frac{a_{10} + a_{11}X_1 + a_{12}X_2}{a_{00} + a_{01}X_1 + a_{02}X_2} \quad x_2 = \frac{a_{20} + a_{21}X_1 + a_{22}X_2}{a_{00} + a_{01}X_1 + a_{02}X_2}$$

To determine the 9 matrix entries a_{ij} up to a nonzero multiple, we need 8 relations. These can be provided by finding the images of 4 points, no three of which are collinear, or any equivalent information.

We first use the fact that all points on the common $x_1(X_1)$ -axis are fixed. The fact that $(0, 0)$ is fixed tells us that $a_{10} = a_{20} = 0$. Then, the fact that $(1, 0)$ is fixed tells us that $a_{11} = a_{00} + a_{01}$ and $a_{21} = 0$. Thus, \mathcal{P} is described by

$$(X2) \quad x_1 = \frac{a_{11}X_1 + a_{12}X_2}{a_{00} + a_{01}X_1 + a_{02}X_2} \quad x_2 = \frac{a_{21}X_1 + a_{22}X_2}{a_{00} + a_{01}X_1 + a_{02}X_2}.$$

We next consider the point at infinity on the X_2 -axis. In Figure 53, the line $P'P_\infty$ is parallel to the X_2 -axis, so P_∞ is the projection of the point at infinity on that axis in Δ . It is clear that P_∞ has coordinates $(p_1, p_2 + p_3 \cot \phi)$.

Putting this information in Equation (X2) yields

$$p_1 = \frac{a_{12}\infty}{a_{00} + a_{02}\infty} = \frac{a_{12}}{a_{00}/\infty + a_{02}} = \frac{a_{12}}{a_{02}}$$

and

$$p_2 + p_3 \cot \phi = \frac{a_{20} + a_{22}\infty}{a_{00} + a_{02}\infty} = \frac{a_{22}}{a_{02}}$$

so $a_{12} = p_1 a_{02}$ and $a_{22} = (p_2 + p_3 \cot \phi) a_{02}$.

Finally using Figure 54, we see that $Q_\infty = (p_1, -p_3 \csc \phi)$ in Λ maps to the point at infinity on the x_2 -axis. Hence,

$$0 = \frac{a_{11}p_1 - a_{12}p_3 \csc \phi}{a_{00} + a_{01}p_1 - a_{02}p_3 \csc \phi} \quad \infty = \frac{a_{21}p_1 - a_{22}p_3 \csc \phi}{a_{00} + a_{01}p_1 - a_{02}p_3 \csc \phi}$$

so

$$a_{11}p_1 = a_{12}p_3 \csc \phi \quad a_{00} + a_{01}p_1 = a_{02}p_3 \csc \phi$$

Now put $a_{02} = 1$. From the above relations, we see that $a_{12} = p_1$, $a_{22} = p_2 + p_3 \csc \phi$, $a_{11}p_1 = p_1 p_3 \csc \phi$, and $a_{00} + a_{01}p_1 = p_3 \csc \phi$. Combining the last equation

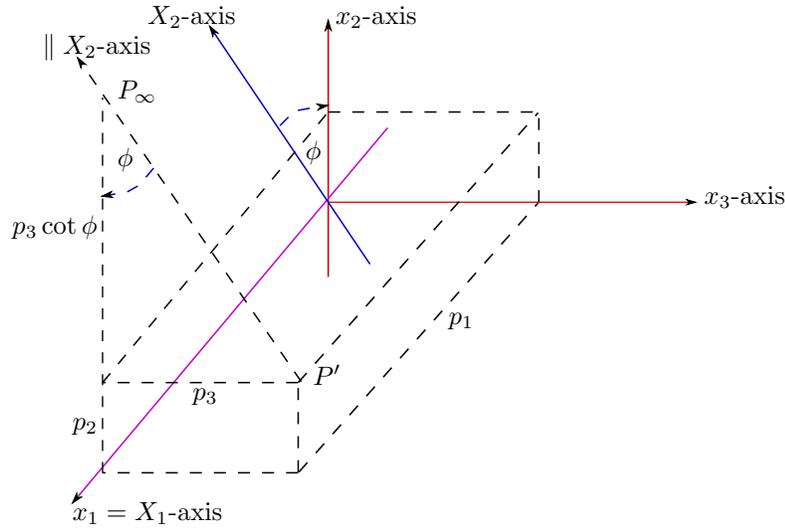


FIGURE 53. Finding P_∞

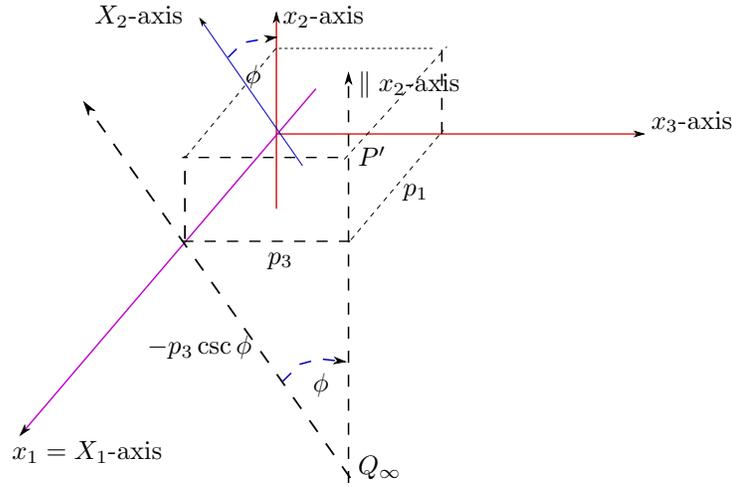


FIGURE 54. Finding Q_∞

with $a_{11} = a_{00} + a_{01}$, yields $a_{01} = 0$ and $a_{11} = a_{00} = p_3 \csc \phi$. It follows that \mathcal{P} has the form

$$x_1 = \frac{p_3 \csc \phi X_1 + p_1 X_2}{p_3 \csc \phi + X_2} \quad x_2 = \frac{(p_2 + p_3 \cot \phi) X_2}{p_3 \csc \phi + X_2}.$$

If we multiply through both numerator and denominator by $\sin \phi$ in each equation, we get

$$x_1 = \frac{p_3 X_1 + p_1 \sin \phi X_2}{p_3 + \sin \phi X_2} \quad x_2 = \frac{(p_2 \sin \phi + p_3 \cos \phi) X_2}{p_3 + \sin \phi X_2}.$$

If we now put $X_1 = R \cos \theta$ and $X_2 = R \sin \theta$, we get the parametric representation described in Section 8.3.

APPENDIX D. APPROXIMATIONS

In this section we analyze in greater detail the approximations we have used. We shall consider the case for the 4 x 5 inch format, but the results generally hold more generally since most things tend to scale appropriately with the format size. For example, for 8 x 10 inch format, the dimensions of the camera would be about twice as large as would be the acceptable diameter of the circle of confusion and focal lengths of lenses.⁶⁴

D.1. How close can the projection point P' be to the reference plane?

We now look at estimates for $\epsilon = \frac{f \sin \phi}{2Np_3}$. It depends on p_3 , which is the distance of the projection point P' to the reference plane. This will clearly occur on the inner surface defined in equations (SF), by $v'' = \frac{v}{1 + \kappa Nc/f}$ for rays such that $\kappa = 1$, so we may take $p_3 = v'' = \frac{v}{1 + Nc/f}$, which in turn depends on v the distance of the image plane to the reference plane. Look at Figure 55. As in equation (GLE), we have

$$\frac{1}{v} + \frac{1}{J/m} = \frac{\cos \phi}{f}$$

so using $J = \frac{f}{\sin \phi}$, we obtain

$$\frac{1}{v} = \frac{\cos \phi - m \sin \phi}{f}$$

⁶⁴Not everything simply scales up. For example, if you double both the focal length and the CoC in the formula $\frac{f^2}{2Nc}$, you end up doubling the hyperfocal distance, which means roughly that you halve the depth of field. Similarly, for the same subject distance, magnification M will differ according to format, even if one compensates by modifying the degree of enlargement of the resulting image.

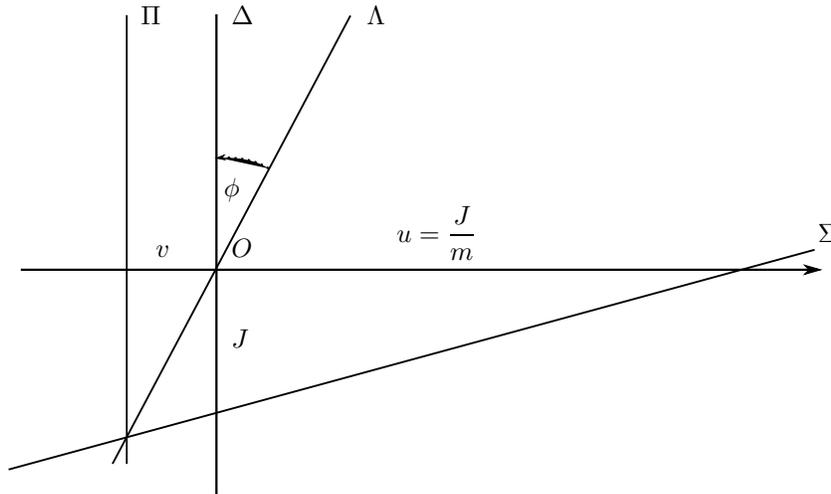


FIGURE 55. Estimating v from the Slope m of the Subject Plane

so

$$(Lv) \quad v = \frac{f}{\cos \phi - m \sin \phi}.$$

Note that m can be negative, which means that the subject plane slopes downward, away from the lens. (It means that u is negative, and the line of sight intersects the subject plane *behind the lens* in the image space. But the algebra still gives the right answer for v .) That situation might arise, for example, if one were on a hill and wanted the subject plane to follow the downhill slope. But in such a case, it is rather unlikely that angle would exceed 45 degrees. So it is safe to posit that $m \geq -1$. Under that assumption, we get

$$v \geq \frac{f}{\cos \phi + \sin \phi} = \frac{f \sec \phi}{1 + \tan \phi}.$$

and

$$p_3 = v'' \geq \frac{f \sec \phi}{(1 + \tan \phi)(1 + \frac{Nc}{f})}$$

Putting this in the formula for ϵ , we obtain

$$\epsilon \leq \frac{\sin \phi \cos \phi}{2N} (1 + \tan \phi) \left(1 + \frac{Nc}{f}\right)$$

or

$$(E1) \quad \epsilon \leq \frac{\sin(2\phi)(1 + \tan \phi)}{4} \left(\frac{1}{N} + \frac{c}{f}\right).$$

A reasonable upper bound for ϕ as noted before is about 0.25 radians. A typical value for the acceptable circle of confusion in 4 x 5 photography is $c = 0.1$, and a typical lower bound for the f-number encountered in practice might be $N = 8$. Finally, while in 4 x 5 inch photography, one might occasionally encounter a focal length of 50 mm or less, a reasonable lower bound is $f = 65$ mm. Those values give $\epsilon \leq 0.01904$, which represents an error of about 2 percent. Even taking extreme values such as $\phi = 0.5$ radians (about 29 degrees), $N = 5.6$, and $f = 50$ mm, only increases the bound to 0.059. A more typical scene might have $\phi = 0.15$ radians (about 8.6 degrees), $N = 22$, and $f = 135$ mm. Those values decrease the upper bound to 0.004.

It should be noted however that there are other limitations which may place a significantly smaller lower bound on ϵ . Limits on rise, fall, and shift may force the rear standard further back to prevent too much of the region between v' and v'' overlapping the optically prohibited zone between the lens plane and the rear focal plane. (No such point can be an image of a real subject point.)

Another factor is the fact that every lens has a limited coverage. That means that there is right circular cone centered on the principal point O with axis the lens axis, outside of which lens aberrations preclude any possibility of sharp images. Even if the camera allows for sufficiently large movements, one would avoid placing the frame so that it extends outside this cone, and that will further limit how close P' can be to the reference plane.

As a result, in practice we can usually do a lot better than $\epsilon \leq 0.02$. The cases where this might not be the case are lenses with very wide coverage, unusually large rises, falls, or shifts, scenes where the subject plane is strongly tilted, or scenes requiring, for some reason, very large tilt angles.

D.2. **Estimating κ .** To get a more precise estimate of κ , we find the semi-major axis of the limiting reference ellipse by optimizing $x_1^2 + x_2^2$ as a function of θ . We start by deriving a convenient expression for $r^2 = x_1^2 + x_2^2$, the square of the distance from O to a point on the reference ellipse. Using Equations (L1) and (L2) in Section 8.3, we have

$$r^2 = x_1^2 + x_2^2 = \frac{1}{j_3^2}[(j_2 \cos \phi + j_1 \sin \phi \sin \theta)^2 + (j_2 \sin \phi + j_3 \cos \phi)^2 \sin^2 \theta]$$

where for convenience we have taken $R = 1$, i.e., we take the radius of the aperture as the unit of length. (We can put R back in afterwards where necessary.) It turns out to be more convenient to work with the quantity

$$j_3^2(r^2 - 1) = (j_2 \cos \phi + j_1 \sin \phi \sin \theta)^2 + (j_2 \sin \phi + j_3 \cos \phi)^2 \sin^2 \theta.$$

After a couple of pages of algebra and trigonometry, we find this simplifies to

$$j_3^2(r^2 - 1) = \sin \phi \{[(1 - 2j_3^2) \sin \phi + 2j_2j_3 \cos \phi] \sin^2 \theta + 2j_1j_3 \sin \theta \cos \theta\}$$

or

$$r^2 - 1 = \frac{\sin \phi}{j_3^2} \left\{ [(1 - 2j_3^2) \sin \phi + 2j_2j_3 \cos \phi] \frac{1 - \cos 2\theta}{2} + 2j_1j_3 \frac{\sin 2\theta}{2} \right\}$$

Put

$$(DEN) \quad A = -[(1 - 2j_3^2) \sin \phi + 2j_2j_3 \cos \phi] = (2j_3^2 - 1) \sin \phi - 2j_2j_3 \cos \phi$$

$$(NUM) \quad B = 2j_1j_3$$

so

$$(RT) \quad r^2 - 1 = \frac{\sin \phi}{2j_3^2} \{-A(1 - \cos 2\theta) + B \sin 2\theta\}$$

Differentiating with respect to θ , and setting the derivative to zero yields

$$-A \sin 2\theta + B \cos 2\theta = 0$$

or

$$(TAN) \quad \tan 2\theta = \frac{B}{A} = \frac{2j_1j_3}{(1 - 2j_3^2) \sin \phi - 2j_2j_3 \cos \phi}$$

This condition determines the values of θ for which the maximum and minimum values of $r^2 - 1$ occur, but the reader should keep in mind that we may replace the numerator and denominator by their negatives, since that leaves $\tan 2\theta$ unchanged. The maximum, the semi-major axis, occurs where $r^2 - 1 > 0$ and the minimum, the semi-minor axis, where $r^2 - 1 < 0$.

Using some simple trigonometry, we get

$$(SIN) \quad S = \sin 2\theta = \pm \frac{B}{H}$$

$$(COS) \quad C = \cos 2\theta = \pm \frac{A}{H}$$

where

$$(HYP) \quad H = \sqrt{B^2 + A^2}$$

where, we must choose the \pm consistently in both equations, as noted above. Let us choose the $+$ signs. If, as we shall see, $r^2 - 1$ ends up positive, we made the

right choice. Putting these values back in the formula RT, we obtain the following formula for a , the semi-major axis.

$$\begin{aligned} a^2 - 1 &= \frac{\sin \phi}{2j_3^2} \left\{ -A \left(1 - \frac{A}{H}\right) + \frac{B^2}{H} \right\} \\ &= \frac{\sin \phi}{2j_3^2 H} \{-AH + A^2 + B^2\} \\ &= \frac{\sin \phi}{2j_3^2 H} \{-AH + H^2\} \end{aligned}$$

$$(MAX) \quad a^2 - 1 = \frac{\sin \phi}{2j_3^2} (H - A)$$

Since $H - A \geq 0$, we made the correct choice. Finally, we get for the semi-major axis a

$$(SMA) \quad a = \sqrt{1 + \frac{(H - A) \sin \phi}{2j_3^2}}$$

$$(KE) \quad \kappa = \frac{1}{a} = \frac{1}{a} = \frac{1}{\sqrt{1 + \frac{(H - A) \sin \phi}{2j_3^2}}}$$

In general, we must multiply the expression in (SMA) by R , the radius of the aperture, but since we would also have to put R in the numerator of the expression in (KE), we would get the same result in for κ .

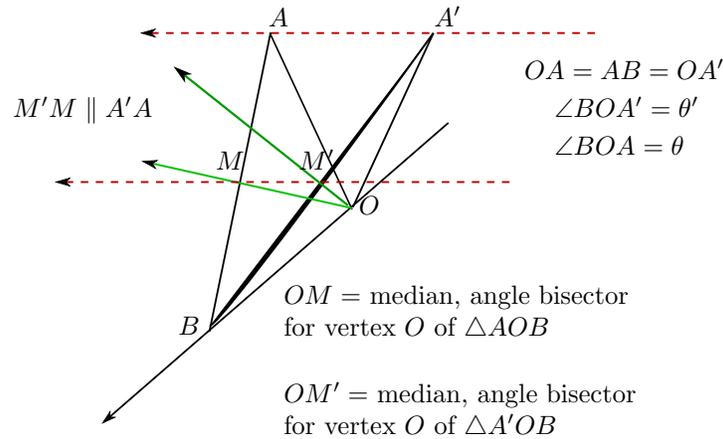


FIGURE 56. Consistent Half Angles

Note. There is another more geometric way to derive the expression for $a^2 - 1$. Namely, if you set $r^2 = 1$ in (RT), you obtain

$$0 = -A(1 - \cos 2\theta) + B \sin 2\theta = -2A \sin^2 \theta + 2B \sin \theta \cos \theta$$

whence

$$\sin \theta = 0 \quad \text{or} \quad \tan \theta = \frac{B}{A}$$

The first corresponds to the point $(\pm 1, 0)$ (generally $(\pm R, 0)$), and the second determines which values the parameter θ takes so that the corresponding points on the ellipse are at the same distance from O . On a non-circular ellipse, there are generally four points whose distance from the center has any given value between the semi-minor and semi-major axes, and the major and minor axes bisect the two angles made by the two chords through O connecting them. That tells us that the angle θ' in the reference plane that the second chord makes with the x_1 -axis is twice that made by the major axis with the x_1 axis. In general the relation between the angles θ' and θ is rather complicated (see Section 8.3, equations (T1) and (T2), and it is not consistent with halving angles. But as Figure 56 indicates, it is in this case.

It should be noted further that the above calculations still leave some ambiguity about the positions of the major and minor axes. Equation (TAN) restricts us to two possible values of $2\theta_0$ between 0 and 2π , and hence two possible values of θ_0 between 0 and π . Also, θ_0 is the parameter value in the lens plane, and the above calculations provide no guidance about how the corresponding angle θ'_0 , in the reference plane, is related to it. Fortunately, we have other means to determine the orientations of major and minor axes, as explained under item (e) in Section 8.3.

We can now derive a simplified expression for $F(\theta)$. Let $\theta = \theta_0$ at the semi-major axis. Then $A = H \cos(2\theta_0)$, and $B = H \sin(2\theta_0)$. Hence,

$$\begin{aligned} F(\theta) &= \frac{\sin \phi}{2j_3^2} [-A + A \cos(2\theta) + B \sin(2\theta)] \\ &= \frac{\sin \phi}{2j_3^2} [-A + H \cos(2\theta_0) \cos(2\theta) + H \sin(2\theta_0) \sin(2\theta)] \\ \text{(GT)} \quad F(\theta) &= \frac{\sin \phi}{2j_3^2} [-A + H \cos(2(\theta - \theta_0))] \end{aligned}$$

which shows clearly how $F(\theta)$ varies near the maximum.

Having attended to the algebra, we are now ready to make estimates for κ . Note that, were we to write out A, B , and $H = \sqrt{A^2 + B^2}$, in equation (KE), we would obtain a rather complex expression for κ in terms of $\mathbf{j} = (j_1, j_2, j_3 = \sqrt{1 - j_1^2 - j_2^2})$ and ϕ . But, set up in this way, it is fairly simple to draw graphs and do calculations using a symbolic manipulation program. The graphs in Section 8.3.2 were obtained using this formulation with Maple.

In Section 8.3, I asserted

Proposition 5. (a) *If $0 < \kappa_0 < 1$, then the projection from the sphere $|\mathbf{j}| = 1$ to the image plane, of the contour curve defined by $\kappa = \kappa_0$, is an ellipse, centered at the Scheimpflug point $(0, -S)$, ($S = v \cot \phi = (1 + M)J$, where the x_2 -axis intersects the Scheimpflug line. Its semi-major axis and semi-minor axes are given by*

$$\text{(Mn)} \quad d_2 = S \frac{\sec \phi}{\kappa_0} \quad d_1 = d_2 \sqrt{1 - \kappa_0^2}$$

In particular, the ellipse crosses the positive x_2 -axis at

$$S \left(\frac{\sec \phi - \kappa_0}{\kappa_0} \right).$$

(b) *If $\kappa_0 = 1$, the projection is the line segment on the x_2 -axis from $-S(\sec \phi + 1)$ to $S(\sec \phi - 1)$*

(c) For κ_0 fixed, the set of all such contour ellipses forms an elliptical cone centered on the line through O in the lens plane with the plane perpendicular to the x_1 -axis (the tilt axis).

Proof. ⁶⁵ (a) $\kappa = \kappa_0$ where $0 < \kappa_0 \leq 1$ yields

$$(C0) \quad a^2 - 1 = \frac{\sin \phi}{2j_3^2}(H - A) = \frac{1}{\kappa_0^2} - 1 = \frac{1 - \kappa_0^2}{\kappa_0^2} = K$$

$$H = A + \frac{2K}{\sin \phi} j_3^2 = A + \overline{K} j_3^2$$

$$B^2 + A^2 = H^2 = A^2 + 2\overline{K} j_3^2 A + \overline{K}^2 j_3^4$$

$$(C1) \quad B^2 = 2\overline{K} j_3^2 A + \overline{K}^2 j_3^4$$

But, $B^2 = (2j_1 j_3)^2 = 4j_1^2 j_3^2$, so equation (C1) has a common factor of j_3^2 , which we may assume is non-zero, so we get

$$4j_1^2 = 2\overline{K} A + \overline{K}^2 j_3^2$$

$$(C2) \quad 4j_1^2 = 2\overline{K}[(2j_3^2 - 1) \sin \phi - 2j_2 j_3 \cos \phi] + \overline{K}^2 j_3^2$$

If we put $j_1 = \frac{x_1}{p}$, $j_2 = \frac{x_2}{p}$, $j_3 = \frac{v}{p}$, where $p^2 = x_1^2 + x_2^2 + v^2$, we obtain

$$2j_3^2 - 1 = \frac{2v^2 - p^2}{p^2} = \frac{v^2 - x_1^2 - x_2^2}{p^2}$$

Putting all this in (C2), we obtain

$$4 \frac{x_1^2}{p^2} = 2\overline{K} \left(\frac{v^2 - x_1^2 - x_2^2}{p^2} \sin \phi - 2 \frac{x_2 v}{p^2} \cos \phi \right) + \overline{K}^2 \frac{v^2}{p^2}$$

$$4x_1^2 = 2\overline{K}[(v^2 - x_1^2 - x_2^2) \sin \phi - 2x_2 v \cos \phi] + \overline{K}^2 v^2$$

$$(C3) \quad (4 + 2\overline{K} \sin \phi)x_1^2 + 2\overline{K} \sin \phi(x_2^2 + 2v \cot \phi x_2) = (4\overline{K} \sin^2 \phi + 4\overline{K}^2)v^2$$

But $v \cot \phi = S$, $2\overline{K} = 4 \frac{K}{\sin^2 \phi}$, $2\overline{K} \sin \phi = 4K$, and $\overline{K}^2 = \frac{4K^2}{\sin^2 \phi}$, so

$$(1 + K)x_1^2 + K(x_2^2 + 2Sx_2) = \left(K + \frac{K^2}{\sin^2 \phi} \right) S^2 \tan^2 \phi$$

$$(1 + K)x_1^2 + K(x_2 + S)^2 = (K \tan^2 \phi + K^2 \sec^2 \phi) S^2 + K S^2$$

$$(C4) \quad (1 + K)x_1^2 + K(x_2 + S)^2 = K(1 + K) \sec^2 \phi S^2$$

This is an ellipse with center at $(0, -S)$ provided $K \neq 0, \kappa_0 \neq 1$. In that case, we also have

$$\left(\frac{d_1}{d_2} \right)^2 = \frac{K}{1 + K} = 1 - \kappa_0^2 \quad \text{and}$$

$$d_2^2 = \frac{K(1 + K) \sec^2 \phi S^2}{K} = (1 + K) \sec^2 \phi S^2 = \frac{S^2 \sec^2 \phi}{\kappa_0^2}$$

as claimed.

⁶⁵It is not too hard to see that the points on rays satisfying $\kappa = \kappa_0$ satisfy a quadratic equation, so since all those rays pass through O , they form a degenerate quadric, in fact, a cone. But I don't see an obvious way to derive the additional facts about its axis and shape. So, the only alternative is brute force calculation.

(b) If $K = 0$, all equation (C4) tell us is that $x_1 = 0$. But if we go back to equation (C0), we see that $K = 0$ implies that $H = \sqrt{A^2 + B^2} = A$, which can only happen if $B = 0$ and $A \geq 0$. But $B = 2j_1j_3 = 0$ tells us that $j_1 = 0$, so $x_1 = 0$. Expressing the second condition in terms of x_2 and v yields

$$(v^2 - x_2^2) \sin \phi - 2x_2v \cos \phi \geq 0.$$

Since this is a continuous function of x_2 and since it is positive for $x_2 = 0$, we need only find where it vanishes. But that reduces to the quadratic equation

$$x_2^2 + 2x_2v \cot \phi - v^2 = x_2^2 + 2x_2S - S^2 \tan^2 \phi = 0.$$

Solving this gives to the two limits $-S(1 + \sec \phi)$ and $S(-1 + \sec \phi)$.

Remark. It is worth pointing out that if we put $x_2 = v \sin \psi$, where ψ is the angle the ray makes with the x_3 -axis, then this amounts to saying that ψ ranges from $-\frac{\pi}{2} + \frac{\phi}{2}$ at the lower end to $\frac{\phi}{2}$ at the upper end.

(c) This is clear from the formulas since $S = v \cot \phi$ is proportional to v , and all the Scheimpflug points lie on the indicated line. \square

It remains to place some limits on how skew the ray \mathbf{j} can be, which will set plausible lower bounds for κ_0 . This is dependent on the maximum possible shift E of a point in the image plane from its intersection with the x_3 -axis. This will be subject—as in D.1—to reasonable assumptions about the structure of the camera and likely scenes.

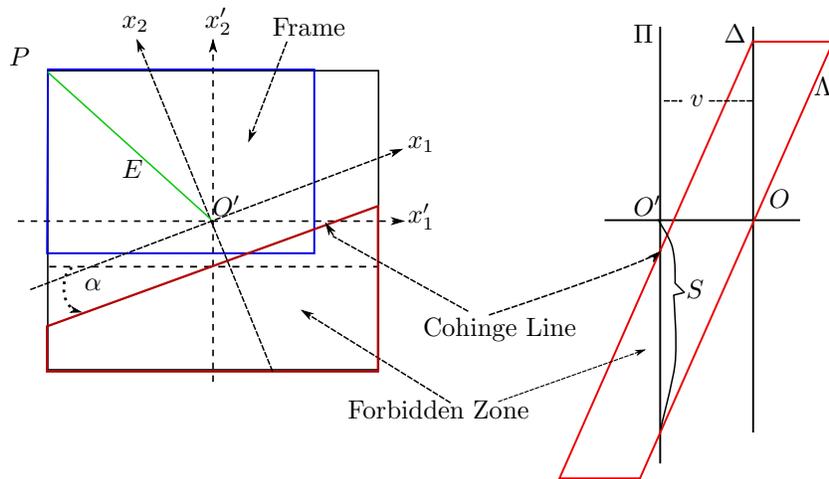


FIGURE 57. Possible Shifts

See Figure 57. The diagram on the left shows the rectangular region within which the frame may be moved, as it would appear from the vantage point of the lens. (The directions of the x_1 and x_2 - axes are chosen consistently with what we did earlier in the image space.)⁶⁶ The forbidden zone is outlined in red, and the

⁶⁶If the standard starts out with sides vertical and horizontal, and the rotation of the standard is closer to being a tilt, then the x_1 -axis is more horizontal. On the other hand, if the rotation is closer to being a swing, then the x_1 -axis is more vertical.

frame is outlined in blue. Let α be the smaller of the two angles that the edges of the frame make with the (tilt) x_1 axis, which, you recall is parallel to the hinge and Scheimpflug lines. Note that there are two coordinate systems in play: the *tilt* system (x_1, x_2) we have been using and a second *frame* system (x'_1, x'_2) , with axes parallel to the edges of the frame, and both are centered at O' where the x_3 axis intersects the image plane. We label the primed axes so that the positive primed axes are closest to the corresponding unprimed axes. With our assumptions, it is clear that $-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}$. But since κ is symmetric about the x_2 -axis, it is clear that there is no loss in generality in assuming $0 \leq \alpha \leq \frac{\pi}{4}$. Note that in Figure 57, we allowed the frame to dip partly below the cohinge line, which you recall is the image of the tilt horizon. No point in the scene will come to exact focus at such a point, but, if it is still below the upper tilt horizon, i.e., the image point is above the corresponding image line, it may still be in adequate focus. If it is also above the upper tilt horizon image, it can't possibly be in focus. That by itself may still not be fatal provided there is nothing of interest at such points. Thus, often such points in the scene will be in the open sky, where there is nothing that needs to be in focus. Of course, for such points, we don't care about the value of κ , which measures how much the upper subject surface of definition departs from its bounding plane. With that in mind, we can just imagine that the frame has been truncated below to eliminate any points that need not be in adequate focus. With that convention, Figure 57 is an accurate picture of what we need to study.

In Figure 58, we show the frame in relation to the contour ellipses, in the case $v = 100$ mm, $\phi = 0.25$ radians. $\alpha = 0.15$ radians (about 8.6 degrees). The frame in landscape orientation with dimensions 120 x 96 mm, and its upper left corner is at $(-90, 60)$ in the frame coordinates. It is clear from the geometry that κ is

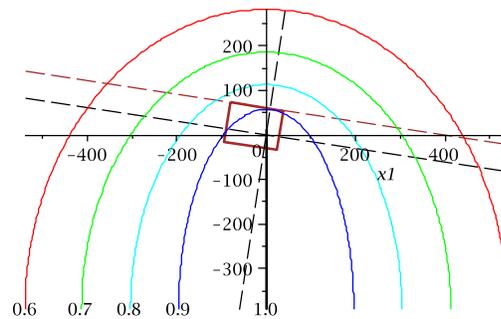


FIGURE 58. The Frame Relative to the Contour Ellipses, for $\phi \approx 14.3$ degrees, $\alpha \approx 8.6$ degrees, and $v = 100$ mm.

minimized on the frame at one of the upper corners. But there does not seem to be any simple rule for determining which one. In the case $\alpha = 0$, the top of the frame is parallel to the x_1 -axis, and κ decreases symmetrically from its value at the x_2 axis, so the corner furthest from that axis will minimize κ . For $\alpha > 0$, the situation

appears to be more complicated. I have not been able to solve the analytic problem completely, but Maple examples provide some insight.

Let (E_1, E_2) be its coordinates with respect to the frame (primed) system of an upper corner P of the frame as in Figure 57. The values of E_1 and E_2 depend on the dimensions of the frame and how it has been oriented. Those values will be limited by possible shifts of the standards. In the diagram $E_2 \leq E_1$, but that depends on the orientation of the frame—portrait or landscape—, and the reverse could be true. P has coordinates (p_1, p_2) with respect to the tilt (unprimed) system where

$$\begin{aligned} p_1 &= E_1 \cos \alpha - E_2 \sin \alpha \\ p_2 &= E_1 \sin \alpha + E_2 \cos \alpha \end{aligned}$$

The coordinates of P in the (x_1, x_2, x_3) coordinate system will be (p_1, p_2, v) , and the ray OP will be determined (as in the proof of Proposition 5) by

$$(JE) \quad j_1 = \frac{p_1}{p}, \quad j_2 = \frac{p_2}{p}, \quad \text{and} \quad j_3 = \frac{v}{p}$$

where

$$p = \sqrt{p_1^2 + p_2^2 + v^2} = \sqrt{E^2 + v^2}, \quad E^2 = E_1^2 + E_2^2$$

Look at the dashed line in Figure 58 determined by the top of the frame. It is not hard to see that its equation is $x_2 = E_2 \sec \alpha - x_1 \tan \alpha$, and we may graph κ along that line. See Figure 59. The graph rises to a maximum, which appears to be at $x_1 = E_2 \tan \alpha$, or close by, but I see no reason why that should be the case. It falls off close to linearly for much of the domain on either side of the maximum, but the fall-off on the left appears to be steeper than that on the right. Maple plots with different values of the parameters produce similar results. So it does appear that the corner to the left, i.e., that opposite the x_2 -axis from the x_1' axis, will yield the smaller value of κ . So the conclusion from the graphical data seems to be the following. *Look at all potential positions of the frame allowed by the structure of the camera, and consider the two furthest from the Scheimpflug point. Choose the one opposite the central axis from the line perpendicular to the tilt axis, and calculate κ_0 there. That is the worst possible case for the camera. Conversely, given what you consider an acceptable value of κ_0 , for any given E_2 and alpha, find the two points where the line $x_2 = E_1 \sec \alpha - x_1 \tan \alpha$ intersects that contour ellipse. If the separation between those points is large enough to accommodate the frame, the two points define the limits of shift to either side which are consistent with that value of κ_0 . Otherwise, choose a smaller κ_0 and redo the calculation.*

We shall now derive another expression for κ which will allow us to see how sensitive it is to the position of the frame. We start with the equation of a typical contour ellipse as given in Proposition 5

$$\begin{aligned} \frac{x_1^2}{d_1^2} + \frac{(x_2 + S)^2}{d_2^2} &= 1 \\ \frac{\kappa^2 x_1^2}{(1 - \kappa^2) S^2 \sec^2 \phi} + \frac{\kappa^2 (x_2 + S)^2}{S^2 \sec^2 \phi} &= 1 \end{aligned}$$

Note that we dropped the subscript from κ . Read this way, these equations can be read as a relation between κ , x_1 , and x_2 for fixed ϕ and S . If κ is fixed, we get the appropriate contour ellipse. But if we specify (x_1, x_2) as, for example, at a corner

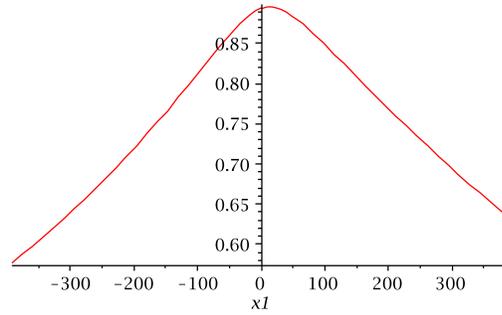


FIGURE 59. κ along the line $x_2 = E_2 \sec \alpha - x_1 \tan \alpha$, $E_2 = 60$ mm, $\alpha = 8.6$ degrees, and for $v = 100$ mm and $\phi = 14.3$ degrees.

of the frame, then the equation may be used to find κ in terms of it. Continuing, we get

$$\begin{aligned} \kappa^2 \frac{x_1^2}{S^2} + \kappa^2 (1 - \kappa^2) \frac{(x_2 + S)^2}{S^2} &= (1 - \kappa^2) \sec^2 \phi \\ \kappa^4 \frac{(x_2 + S)^2}{S^2} - \kappa^2 \left(\frac{x_1^2 + (x_2 + S)^2}{S^2} + \sec^2 \phi \right) + \sec^2 \phi &= 0 \\ \text{(BQ)} \quad \kappa^4 \frac{s_2^2}{S^2} - \kappa^2 \left(\frac{s^2}{S^2} + \sec^2 \phi \right) + \sec^2 \phi &= 0 \end{aligned}$$

where $s_2 = x_2 + S$ is the distance of the Scheimpflug point $(0, -S)$ to $(0, x_2)$, and $s = \sqrt{x_1^2 + s_2^2}$ is its distance of the point (x_1, x_2) . Put $\bar{x}_1 = \frac{x_1}{S}$, $\bar{x}_2 = \frac{x_2}{S}$, $\bar{s}_2 = \frac{s_2}{S} = (1 + \bar{x}_2)$ and $\bar{s} = \frac{s}{S} = \sqrt{\bar{x}_1^2 + (1 + \bar{x}_2)^2}$. Note that \bar{s}_2 and \bar{s} are close to 1 provided x_1 and x_2 are small relative to S . By the quadratic equation formula,

$$\text{(QSa)} \quad \kappa^2 = \frac{\bar{s}^2 + \sec^2 \phi \pm \sqrt{(\bar{s}^2 + \sec^2 \phi)^2 - 4\bar{s}_2^2 \sec^2 \phi}}{2\bar{s}_2^2}$$

But, by Schwartz's Inequality

$$\bar{s}^2 + \sec^2 \phi \geq 2\bar{s} \sec \phi \geq 2\bar{s}_2 \sec \phi > 2\bar{s}_2$$

which implies first that the quantity under the square root is positive, and secondarily that the first term in the numerator divided by the denominator is greater than 1. Were we to add anything to it, we would get something larger than one, which is not possible since $\kappa^2 \leq 1$. Hence, the correct formula is

$$\begin{aligned} \kappa^2 &= \frac{\bar{s}^2 + \sec^2 \phi - \sqrt{(\bar{s}^2 + \sec^2 \phi)^2 - 4\bar{s}_2^2 \sec^2 \phi}}{2\bar{s}_2^2} \\ &= \frac{\bar{s}^2 + \sec^2 \phi - \sqrt{(\bar{s}^2 - \sec^2 \phi)^2 + 4(\bar{s}^2 - \bar{s}_2^2) \sec^2 \phi}}{2\bar{s}_2^2} \end{aligned}$$

so finally⁶⁷

$$\text{(Q}\kappa) \quad \kappa = \sqrt{\frac{\bar{x}_1^2 + (1 + \bar{x}_2)^2 + \sec^2 \phi - \sqrt{(\bar{x}_1^2 + (1 + \bar{x}_2)^2 - \sec^2 \phi)^2 + 4\bar{x}_1^2 \sec^2 \phi}}{2(1 + \bar{x}_2)^2}}$$

⁶⁷There is a problem for $\bar{s}_2 = 0$ in which case the expression becomes indeterminate. We can either apply L'Hospital's Rule or just solve equation (BQ), which becomes linear for $\bar{s}_2 = 0$.

This expression allows us to see how κ , behaves as a function of the parameters. It is not obvious from the formulas, but, for fixed ϕ , it is a decreasing function of the \bar{x}_1 , and \bar{x}_2 when both are small compared to 1. This is clear from Maple plots such as the one in Figure 60.⁶⁸ It is also increasing as a function of ϕ , if the other

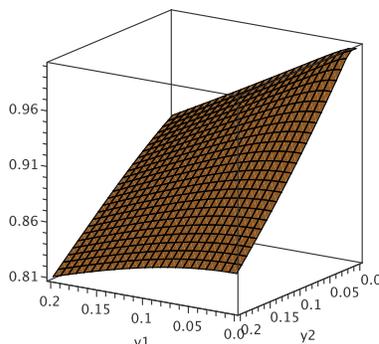


FIGURE 60. κ as a function of $y_1 = \bar{x}_1, y_2 = \bar{x}_2$ for $\phi = 0.15$ radians

two variables are fixed. But recall that ϕ is essentially pre-determined by the scene, since $\sin \phi = \frac{f}{J}$. f is determined by the desired field of view and what is available to the photographer, and J is determined by the position of the desired subject plane. Thus, ϕ may vary as the photographer composes the scene, but within a very narrow range.⁶⁹

To proceed further we must make some estimates of the parameters.

We start by considering $S = J(1 + M)$, which is approximately J as long as M is small. It is not much smaller than J , in any event, unless the subject plane tilts strongly away from the reference normal, which is rather unlikely. A reasonable lower bound for J would be 1 meter = 1,000 mm.⁷⁰ For example, for a pure tilt downward, the camera is at eye level, which can range from about 1.5 to 1.75 meters, depending on the height of the photographer, and how far he or she bends over. The subject plane usually passes well below the lens, even below the ground, so the 1 meter bound is plausible. In cases of pure swing, or even skew tilt axis, the language would be different, but the conclusions would be similar.

Consider next the tilt angle ϕ . We have been taking $\phi = 0.25$ radians, but this is a trifle extreme, so let's assume ϕ is at most 0.15 radians (about 8.6 degrees instead). Then $1 < \sec \phi < 1.012$, and is essentially fixed by the scene.

Finally, let's assume that $x_1, x_2 \leq 100$ mm. In the case of a pure tilt, that would mean the frame might be largely above the horizon image, even when in portrait mode, and possible well off to one side. Similarly for a pure swing, with

⁶⁸I've also checked it by finding the partial derivatives with respect to $x = \bar{x}_1^2, y = (1 + \bar{x}_2)^2$, and $z = \sec^2 \phi$ which are monotonic in the corresponding parameters. But the algebra is a bit messy and need not be included here given the availability of plots.

⁶⁹Note also that for fixed f , changing ϕ entails changing J and S , and so also \bar{x}_1 and \bar{x}_2 , so the actual dependence on ϕ might be fairly complex. Fortunately we need not worry about that, because, as noted above, ϕ is essentially constant.

⁷⁰The most important exception to this would be close-up or table-top photography, which we largely ignore here.

an appropriate change of language. In the extreme case where the tilt axis makes a 45 degree angle with the edges of the frame, this would still place a corner of the frame two thirds of its diameter (≈ 150 mm) above the horizon image, and far off to one side, which would be a highly unlikely situation, to say the least.

Since the right hand side of (Q κ) is decreasing, we can determine a lower bound by evaluating it for $\bar{x}_1 = \bar{x}_2 = \frac{100}{1000} = 0.1$. We get $\kappa^2 > 0.81$ from which we conclude that $\kappa > 0.9$, and, to compensate, that would require a correction of at most three tenths of a stop, which should be acceptable. Also, the positions of the frames described above are unusual, and, in any event, the worst estimates would occur only in one or two corners. For most of the frame κ would be closer to 1. Note also that even in extremely unlikely scenarios the correction is likely to be at worst something like three quarters of a stop.

D.3. How Displacement from the Limiting Ellipse Changes Things. The calculations in the previous sections are based on using the limiting reference ellipse defined by $\mathbf{x} = (x_1, x_2)$ where, as in Section 8.3 with $R = 1$,

$$(L1) \quad x_1 = \frac{j_3 \cos \theta + j_1 \sin \phi \sin \theta}{j_3}$$

$$(L2) \quad x_2 = \frac{(j_2 \sin \phi + j_3 \cos \phi) \sin \theta}{j_3}$$

But, as we saw previously, the actual reference ellipse is given by

$$\bar{\mathbf{x}} = \mathbf{x} \frac{1}{1 + \epsilon \sin \theta}$$

which is perturbed from the limiting ellipse, for each value of θ , by the indicated factor. We also saw that the center of the perturbed ellipse was at $\mathbf{k} = -\mathbf{q} \frac{\epsilon}{1 - \epsilon^2}$ where $\mathbf{q} = \mathbf{x}(\pi/2) = (q_1, q_2)$ and

$$(Q1) \quad q_1 = \frac{j_1 \sin \phi}{j_3}$$

$$(Q2) \quad q_2 = \frac{j_2 \sin \phi + j_3 \cos \phi}{j_3}$$

So, for the displacement from the center of the perturbed ellipse, we have

$$(PEa) \quad \mathbf{y} = \bar{\mathbf{x}} - \mathbf{k} \quad \text{i.e.,} \\ \mathbf{y} = \frac{\mathbf{x}}{1 + \epsilon \sin \theta} + \frac{\epsilon \mathbf{q}}{1 - \epsilon^2}$$

There are two triangle inequalities. The more familiar asserts that the third side of a triangle is less than or equal to the sum of the other two sides, but it is also true that the third side is greater than or equal to the larger of the two other sides less the smaller. These yield

$$(TEa) \quad |\mathbf{y}| \leq \frac{|\mathbf{x}|}{1 + \epsilon \sin \theta} + \frac{\epsilon |\mathbf{q}|}{1 - \epsilon^2}$$

$$(TEb) \quad |\mathbf{y}| \geq \frac{|\mathbf{x}|}{1 + \epsilon \sin \theta} - \frac{\epsilon |\mathbf{q}|}{1 - \epsilon^2}$$

Our analysis will depend primarily on inequality (TEa), but first we must digress and use inequality (TEb). Since $\mathbf{x} \geq |\mathbf{q}|$, and $\sin \theta \leq 1$, if we ignore terms of order

ϵ^2 and higher⁷¹, we see that a good estimate for the lower bound on $|\mathbf{y}|$ is $|\mathbf{x}|(1-2\epsilon)$. For the moment we shall draw just one conclusion from this estimate, namely, that

$$(F) \quad |\mathbf{y}| \text{ attains a maximum for } 0 \leq \theta \leq \pi$$

Indeed, if both maximum values of $|\mathbf{y}|$ occurred for $\pi < \theta < 2\pi$, since $1 - 2\epsilon > 0$, the same would have to be true for $|\mathbf{x}|$, which is obviously false.

We now investigate how equation (PEa) affects the conclusions drawn previously about Problem (1) where we used the limiting reference ellipse. We reasoned that CoCs for points along the same ray through O in the outer and inner planes would match provided the focus was set at the harmonic mean. Thus, we have two values of ϵ to consider

$$\epsilon' = \frac{f \sin \phi}{2Nv'} \quad \text{and} \quad \epsilon'' = \frac{f \sin \phi}{2Nv''}$$

where v' and v'' are respectively the distances of the outer and inner image planes from the reference plane. (Note that $v' > v''$.) Let \mathbf{y}' and \mathbf{y}'' be the corresponding quantities for the two perturbed ellipses.⁷² From (PEa) we have

$$\begin{aligned} |\mathbf{y}' - \mathbf{y}''| &= |\mathbf{x}| \left| \frac{1}{1 + \epsilon' \sin \theta} - \frac{1}{1 + \epsilon'' \sin \theta} \right| \\ &= |\mathbf{x}| \left(\frac{(\epsilon'' - \epsilon') \sin \theta}{1 + (\epsilon' + \epsilon'') \sin \theta + \epsilon' \epsilon'' \sin^2 \theta} \right) \\ &\leq |\mathbf{x}| |(\epsilon'' - \epsilon') \sin \theta| \leq |\mathbf{x}| |\epsilon'' - \epsilon'| \end{aligned}$$

since by (F), we may assume without loss of generality that $\sin \theta \geq 0$.

But

$$\begin{aligned} |\epsilon'' - \epsilon'| &= \frac{f \sin \phi}{2N} \left| \frac{1}{v''} - \frac{1}{v'} \right| \\ &= \frac{f \sin \phi}{2N} \frac{|v' - v''|}{v'v''} \\ &= \frac{f \sin \phi}{Nv} \frac{|v' - v''|}{v' + v''} \quad \text{so} \end{aligned}$$

$$(PEc) \quad |\mathbf{y}' - \mathbf{y}''| \leq 2\epsilon \frac{|v' - v''|}{v' + v''} |\mathbf{x}|$$

where $\epsilon = \frac{f \sin \phi}{2Nv}$ is the value of ϵ for the harmonic mean v of v' and v'' .

The quantity $v' - v''$ is what we previously called the focus spread, and we used it to derive estimates for the f-number needed to produce the desired depth of field. A rough estimate for the focus spread is $2Nc$. Lenses used in 4 x 5 photography seldom can be stopped down beyond $f/64$, and such a small aperture would raise diffraction issues, even for a perfect lens. Taking $c = 0.1$ mm, and $N = 45$ yields a focus spread of 9 mm. So let's assume the focus spread is not larger than 10 mm. If the inner image distance is approximately the focal length, and we assume the latter is larger than 50 mm, that means the outer image distance is larger than 60

⁷¹We saw earlier that $\epsilon < 0.02$, so $\epsilon^2 < 0.0004$. If we incorporated those terms the effect would clearly be negligible. The skeptical reader can test this contention by working it all out, but we shall avoid such unnecessary complications here.

⁷²This notation may conflict with the usual notation $\mathbf{y}' = \frac{d\mathbf{y}}{d\theta}$, but the context should make it clear which is intended.

mm, and hence $\frac{v' - v''}{v' + v''} < \frac{10}{60} = 1/6$. It follows that a reasonable approximation for the upper bound for $|\mathbf{y}' - \mathbf{y}''|$ is $\frac{\epsilon|\mathbf{x}|}{3}$. If we take 0.02 as the upper bound for ϵ , we see that the failure to match is less than 0.67 percent. Usually, it would be much less than that. So, if we place the image plane at the harmonic mean, we would be making a slight error, but it would be so small that it would be undetectable in normal practice.

Next, we address the effect of the perturbation on our estimate of κ , as needed in analysis of Problem (2). From inequality (TEa), and using $|\mathbf{q}| \leq |\mathbf{x}|$ as well as $\sin \theta \geq 0$, we obtain

$$|\mathbf{y}| \leq |\mathbf{x}| \left(1 + \frac{2\epsilon}{1 - \epsilon^2} \right)$$

So, if a_ϵ is the semi-major axis of the perturbed ellipse, and κ_ϵ its reciprocal, we have

$$a_\epsilon \leq a \left(\frac{1 - \epsilon^2 + 2\epsilon}{1 - \epsilon^2} \right)$$

and

$$\kappa_\epsilon \geq \kappa \frac{1 - \epsilon^2}{1 + 2\epsilon - \epsilon^2} \approx \kappa(1 - 2\epsilon)$$

where in the last estimate we ignore terms of order ϵ^2 or smaller. Taking 0.02 as the upper bound for ϵ suggests that a reasonable estimate for a lower bound for κ_ϵ is 0.94κ . Interpreted as a necessary change in the f-number to compensate, that would amount to about 0.12 stops. It would typically be considerably less.

So the general conclusion is that using the limiting ellipse instead of the actual perturbed ellipse whatever the problem under consideration, has such a small effect, that in practice it would be virtually undetectable. In particular, given the other approximations inherent in the use of geometric optics, errors in focusing, and the difficulty in setting the standards or the f-number exactly as we want them, it is not usually worth worrying about.

Final Note A much more difficult problem is determining where the semi-major axis of the perturbed ellipse is in comparison with that of the limiting ellipse. You do this by relating the quantity $\mathbf{y} \cdot \frac{d\mathbf{y}}{d\theta}$ to the corresponding dot product for \mathbf{x} , which we can calculate from the formula we derived previously for $\mathbf{x} \cdot \mathbf{x}$. It turns out that the θ displacement, while small, may be considerably larger than the quantities we considered above. Fortunately, it doesn't matter in the analysis.

APPENDIX E. EXACT CALCULATION FOR THE UPPER SURFACE OF DEFINITION

Refer to Figure 33 in Section 8.4.1. The problem is to determine the coordinates of the point \overline{P}' on the upper surfaces of definition in the subject space in terms of the corresponding point \overline{Q}' on outer surface of definition in the image space. The latter is given, as usual, by $v' = \frac{v}{1 - N\epsilon\kappa/f}$ where $\kappa = \kappa(\mathbf{j}, \phi)$. The approach is to use the lens equation $\frac{1}{s} + \frac{1}{t} = \frac{\cos \zeta}{f}$ where s and t represent the respective distances of \overline{Q}' and \overline{P}' along the ray determined by \mathbf{j} and ζ is the angle that the ray makes with the lens plane. It is a little tricky using this formula. We take the vector \mathbf{j}

pointing into the image space, and what we get for t tells us the distance, again in the image space to a corresponding point. But then we must take the *antipodal point* in the subject space to get $\overline{P'}$. It is not hard to determine s and $\cos \zeta$ from the known data. I omit the calculations, but the final result I got was that $\overline{P'}$ is the point antipodal to $t\mathbf{j}$ where

$$(LE) \quad t = \frac{(1 + M)f}{(1 + M)j_2 \sin \phi + (M \cos \phi + Nc\kappa(\mathbf{j}, \phi)/f)j_3}$$

I have not rechecked my calculations, so I don't really vouch for the formula. I recommend that any reader who is interested enough in the result redo the calculations according to the scheme outlined above.

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