Math B17 - Fall 1999 - Midterm Exam No. 1 (solutions)

SOLUTIONS

1. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+2)}$$

Solution:

We have: $\dfrac{1}{n\,(n+2)} < \dfrac{1}{n^2}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it is a p-series with $p = 2$. Hence, by Comparison Test the given series *converges.*

We can rewrite each term of the series like this:

$$\frac{1}{n\,(n+2)} = \frac{1/2}{n} - \frac{1/2}{n+2}$$

So, the series is telescopic:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n\,(n+2)} &= \sum_{n=1}^{\infty} \left(\frac{1/2}{n} - \frac{1/2}{n+2} \right) \\
&= \frac{1/2}{1} - \frac{1/2}{3} + \frac{1/2}{2} - \frac{1/2}{4} + \frac{1/2}{3} - \frac{1/2}{5} + \frac{1/2}{4} - \frac{1/2}{6} + \cdots \\
&= \frac{1/2}{1} + \frac{1/2}{2} = \frac{3}{4}
\end{aligned}$$

2. Find a power series representation for the hyperbolic sine: $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

Solution:

$$\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
&= \frac{1}{2}\left\{\left(\frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right) - \left(\frac{1}{0!} - \frac{x}{1!} + \frac{x^2}{2!} - \cdots\right)\right\} \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

3. Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

Solution:

We have: $$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0.$$

Since the nth term does not converge to zero, the series diverges.

4. Determine if the following series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n}$$

Solution:

The series is alternating, $\dfrac{1}{\ln n}$ decreases and $\dfrac{1}{\ln n} \to 0$, hence the series converges.

On the other hand $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, by comparison test $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln n}$ diverges.

Hence the given series converges conditionally.

5. Find the interval of convergence of the following series:

$$f(x) = \sum_{n=0}^{\infty} (1 + 2^n)\, x^n$$

Solution:

We use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(1 + 2^{n+1})\, x^{n+1}}{(1 + 2^n)\, x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{(1 + 2^{n+1})}{(1 + 2^n)} \right| = 2\,|x|$$

There is convergence for $\rho < 1$, i.e., $|x| < 1/2$.

At the endpoints we get the series

$$\sum_{n=0}^{\infty} (-1)^n\,(1 + 2^n) \quad \text{and} \quad \sum_{n=0}^{\infty} (1 + 2^n)$$

respectively, whose nth terms do not tend to zero, so they do not converge. Hence, the interval of convergence is:

$$I = (-\tfrac{1}{2}, \tfrac{1}{2}),$$

not including the endpoints.

6. Find a power series for the following integral

$$\int_0^x e^{-t^2}\,dt$$

Solution:

$$e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} = 1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$$

Hence, integrating termwise:

$$\int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)} = x - \frac{x^3}{1!\,3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \cdots$$