Math B17 - Fall 1999 - Final Exam (solutions)

SOLUTIONS

1. Determine if the following infinite series converge absolutely, converge conditionally or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3}}$

Solution:

We have: $\left|\dfrac{\sin n}{\sqrt{n^3}}\right| \leq \dfrac{1}{\sqrt{n^3}}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ is a p-series with $p = 3/2 > 1$, hence it converges. By comparison $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{\sqrt{n^3}}\right|$ also converges. Hence the given series *converges absolutely.*

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n}$

Solution:

The series is alternating. The absolute value of its nth term $\left|\dfrac{(-1)^n}{\ln n}\right| = \dfrac{1}{\ln n}$ is decreasing and tends to zero, hence the series converges. However $\dfrac{1}{\ln n} \geq \dfrac{1}{n}$, and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln n}$ diverges by comparison. Hence the given series *converges conditionally.*

2. Find the interval of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{(2+x)^n}{n^2\,7^n}$$

Solution:

By the ratio test:

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{(2+x)^{(n+1)}/(n+1)^2\,7^{(n+1)}}{(2+x)^n/n^2\,7^n} \right| \\ &= \lim_{n\to\infty} \frac{|2+x|}{7}\,\frac{n^2}{n^2+1} = \frac{|2+x|}{7}\,.\end{aligned}$$

Hence, the series converges absolutely for $\rho = |2+x|/7 < 1$, i.e., $-9 < x < 5$, or x in $(-9, 5)$. It diverges for $\rho = |2+x|/7 > 1$, i.e., $x < -9$ or $x > 5$. It remains to check convergence at the endpoints.

At $x = 5$ the series is $\sum_{n=1}^{\infty} \frac{(2+5)^n}{n^2\,7^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$, hence it converges. At $x = -9$ the series is $\sum_{n=1}^{\infty} \frac{(2-9)^n}{n^2\,7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which is also convergent.

Hence, the interval of convergence is $[-9, 5]$, including the endpoints.

3. Find a power series for the following function: $f(x) = \int_{t=0}^{x} e^{-\pi t^2}\, dt.$

Solution:

Using the Maclaurin series for e^x we find:

$$e^{-\pi t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^n t^{2n}}{n!} = 1 - \pi t^2 + \frac{\pi^2 t^4}{2!} - \frac{\pi^3 t^6}{3!} + \cdots$$

Integrating termwise we get:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^n x^{2n+1}}{n!\,(2n+1)} = x - \frac{\pi x^3}{3} + \frac{\pi^2 x^5}{2!\,5} - \frac{\pi^3 x^7}{3!\,7} + \cdots$$

4. Solve the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 - x_2 + x_3 + 3\,x_4 = -1 \\ x_1 + x_3 + 2\,x_4 = 0 \end{cases}$$

Solution:

The augmented matrix is:

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 3 & -1 \\ 1 & 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[\text{reduction}]{\text{Gauss-Jordan}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So, the system is equivalent to:

$$\begin{cases} x_1 + x_3 + 2\,x_4 = 0 \\ x_2 - x_4 = 1 \end{cases}$$

Hence, the solution is:

$$x_1 = -x_3 - 2\,x_4$$
$$x_2 = 1 + x_4$$

i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

5. A 3×2 matrix X has the following property:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -3 & 3 \\ 1 & 1 & 4 \end{bmatrix} X = \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Find X.

Solution:

Using Gauss-Jordan reduction on the augmented matrix:

$$\left[\begin{array}{rrr|rr} 1 & 2 & 0 & -1 & 2 \\ -1 & -3 & 3 & 2 & 0 \\ 1 & 1 & 4 & 0 & 5 \end{array}\right] \overset{\text{Gauss-Jordan reduction}}{\Longrightarrow} \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Hence:

$$X = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

An alternative (although longer) solution consists of computing the inverse of the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -3 & 3 \\ 1 & 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 15 & 8 & -6 \\ -7 & -4 & 3 \\ -2 & -1 & 1 \end{bmatrix}$$

and multiplying it by the given matrix:

$$X = \begin{bmatrix} 15 & 8 & -6 \\ -7 & -4 & 3 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

6. Find a basis and the dimension of the column space of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 & -1 \\ 2 & -2 & 3 & 3 & 0 \end{bmatrix}.$$

Solution:

By Gauss reduction we get:

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 & -1 \\ 2 & -2 & 3 & 3 & 0 \end{bmatrix} \overset{\text{Gauss reduction}}{\Longrightarrow} \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivots, so the rank of A is 2, hence the dimension of the column space is 2.

The pivots are on columns 1 and 3, so the columns 1 and 3 of the original matrix form a basis for the column space:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

7. Given the vectors:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

find all possible real numbers x_1, x_2, x_3, such that:

$$x_1\,\mathbf{v_1} + x_2\,\mathbf{v_2} + x_3\,\mathbf{v_3} = \mathbf{0}\,.$$

Solution:

The equation above is equivalent to the following homogeneous system:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \overset{\text{Gauss-Jordan reduction}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, the system is equivalent to:

$$\begin{cases} x_1 \phantom{{}+x_2} + x_3 = 0 \\ \phantom{x_1+{}} x_2 + x_3 = 0 \end{cases}$$

The solution is $x_1 = x_2 = -x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

8. Diagonalize the symmetric matrix $A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$.

Show an *orthogonal* matrix P such that $D = P^t A P$ is diagonal.

(Hint: all eigenvalues of A are multiple of 3.)

Solution:

The matrix P will be of the form $P = [\,\mathbf{u_1}\ \mathbf{u_2}\ \mathbf{u_3}\,]$, where $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is an orthonormal basis for $\mathbb{R}^3$ made of eigenvectors for A.

The eigenvalues of A are the roots of its characteristic polynomial:

$$\det(A - \lambda I) = -\lambda^3 + 12\,\lambda^2 - 45\,\lambda + 54 = -(\lambda - 6)\,(\lambda - 3)^2\,,$$

i.e., $\lambda = 6$ and $\lambda = 3$ (double).

For $\lambda = 6$ we must solve $(A - 6\,I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $x_1 = x_3$, $x_2 = -x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

so we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as the first eigenvector.

For $\lambda = 3$ we must solve $(A - 3I)\,\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = x_2 - x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

so we take $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ as the two remaining eigenvectors.

The vector $\mathbf{v_1}$ is orthogonal to $\mathbf{v_2}$ and $\mathbf{v_3}$, but $\mathbf{v_2}$ and $\mathbf{v_3}$ are not orthogonal, so we must apply the Gram-Schmidt process to $\{\mathbf{v_2}, \mathbf{v_3}\}$:

$$\mathbf{v_2'} = \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v_3'} = \mathbf{v_3} - \frac{\mathbf{v_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Next, we normalize $\mathbf{v_1}$, $\mathbf{v_2'}$ and $\mathbf{v_3'}$:

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{|\mathbf{v_1}|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{u_2} = \frac{\mathbf{v_2'}}{|\mathbf{v_2'}|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u_3} = \frac{\mathbf{v_3'}}{|\mathbf{v_3'}|} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Hence:

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Finally, the diagonal form of A is:

$$D = P^t \, A \, P = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

9. Find the principal axes and classify the central conic:

$$x^2 + y^2 - 8\,xy = 15$$

Solution:

The conic can be represented as $\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 15$, where $A = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$.

We must diagonalize A as $D = P^t AP$ for some *orthogonal* matrix $P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}$, where $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A.

The eigenvalues of A are the roots of the characteristic polynomial:

$$\det(A - \lambda\, I) = \det \begin{bmatrix} 1-\lambda & -4 \\ -4 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\,\lambda - 15 = (\lambda - 5)\,(\lambda + 3)$$

The eigenvalues are $\lambda = -3$ and $\lambda = 5$.

For $\lambda = -3$ we must solve $\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvector.

For $\lambda = 5$ we must solve $\begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = -x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that $\mathbf{v_1}$ and $\mathbf{v_2}$ are already orthogonal, so all we need is to normalize them: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\mathbf{v_1}$, $\mathbf{u_2} = \frac{1}{\sqrt{2}}\mathbf{v_2}$. The matrix for the change of basis is:

$$P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In the new basis the conic is $\begin{bmatrix} x' & y' \end{bmatrix} D \begin{bmatrix} x' \\ y' \end{bmatrix} = 15$, where

$$D = P^t\,A\,P = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix},$$

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.:

$$\begin{cases} x' = \frac{1}{\sqrt{2}}\,(x+y) \\ y' = \frac{1}{\sqrt{2}}\,(-x+y) \end{cases}$$

Hence the conic is $-3\,x'^2 + 5\,y'^2 = 15$, or equivalently: $-\dfrac{x'^2}{5} + \dfrac{y'^2}{3} = 1$, which is an *hyperbola*. Its principal axes are given by the basic vectors

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note: An alternative solution is $\dfrac{x'^2}{3} - \dfrac{y'^2}{5} = 1$, and

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

10. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 2xy$ given the constrain $g(x, y) = x^2 + y^2 = 1$.

Solution:

We must solve $\nabla f(x, y) = \lambda \nabla\, g(x, y)$, $g(x, y) = 1$, i.e.:

$$\begin{cases} & 2y & = & \lambda\,(2x) \\ 2x & & = & \lambda\,(2y) \\ x^2 & + \quad y^2 & = & 1 \end{cases}$$

which is equivalent to $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} = 1$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. So the problem consists of finding eigenvectors of length 1 for the matrix A.

First we find the eigenvalues for A:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1),$$

hence the eigenvalues are $\lambda = \pm 1$.

For $\lambda = -1$ we solve $(A + I)\,\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$. The solutions of length 1 are $\mathbf{x_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $-\mathbf{x_1}$.

For $\lambda = 1$ we solve $(A - I)\,\mathbf{x} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$. The solutions of length 1 are $\mathbf{x_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $-\mathbf{x_2}$.

Hence, the extreme values of $f(x, y)$ are:

$$\begin{aligned} f(\mathbf{x_1}) &= f(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = -1 \\ f(-\mathbf{x_1}) &= f(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}) = -1 \\ f(\mathbf{x_2}) &= f(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = 1 \\ f(-\mathbf{x_2}) &= f(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}) = 1\,. \end{aligned}$$