NOTES ON
INFINITE SEQUENCES AND SERIES

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1. Sequences

1.1. **Sequences.** An infinite sequence of real numbers is an ordered unending list of real numbers. E.g.:

$$1, 2, 3, 4, \ldots$$

We represent a generic sequence as $a_1, a_2, a_3, \ldots$, and its n-th as a_n.

In order to define a sequence we must give enough information to find its n-th term. Two ways of doing this are:

1. With a formula. E.g.:

$$a_n = \frac{1}{n}$$
$$a_n = \frac{1}{10^n}$$
$$a_n = \sqrt{3n-7}$$

2. With a recursive definition. E.g.: the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$, in which each term is the sum of the two previous terms:

$$F_1 = 1$$
$$F_2 = 1$$
$$F_{n+1} = F_n + F_{n-1}$$

1.2. **Limit of a Sequence.** We say that a sequence a_n *converges* to a limit L if the difference $|a_n - L|$ can be made as small as we wish by taking n large enough. We write $a_n \to L$, or more formally:

$$\lim_{n\to\infty} a_n = L \,.$$

E.g.:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \,.$$

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If a sequence does not converge we say that it *diverges*. E.g., the following sequences diverge:

$$n = 1, 2, 3, 4, \cdots \to \qquad \text{diverges (to } +\infty)$$

$$(-1)^n = -1, 1, -1, 1, \cdots \to \qquad \text{diverges}$$

1.3. **Limit Laws for Sequences.** If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then:

$$\lim_{n\to\infty} (a_n + b_n) = A + B$$

$$\lim_{n\to\infty} (a_n - b_n) = A - B$$

$$\lim_{n\to\infty} (a_n\, b_n) = A\, B$$

$$\lim_{n\to\infty} (a_n / b_n) = A/B \qquad \text{(provided } B \neq 0)$$

So, a "complicated" limit such as $L = \lim_{n\to} \frac{1+\frac{1}{n}}{3+\frac{1}{10^n}}$ can be computed by replacing smaller parts of it with their limits $1/n \to 0$, $1/10^n \to 0$: $L = \frac{1+0}{3+0} = \frac{1}{3}$.

1.4. **Squeeze Law.** If $a_n \leq c_n \leq b_n$, and a_n and b_n have the same limit: $a_n \to L$, $b_n \to L$, then c_n has also the same limit: $c_n \to L$. This can be used to compute limits such as the following one:

$$\lim_{n\to\infty} \frac{\sin n}{n}\,.$$

In this case we have:

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}\,.$$

Since $-1/n \to 0$ and $1/n \to 0$ then $\frac{\sin n}{n} \to 0$ also.

1.5. **Limits of Functions of Sequences.** If $a_n = f(n)$ for some function f and $\lim_{x\to n} f(x) = L$, then $\lim_{n\to\infty} a_n = L$. This basically allows us to replace limits of sequences with limits of functions. In particular this is useful for using L'Hôpital's rule in computing limits of sequences. E.g:

$$\lim_{n\to\infty} \frac{e^n}{n} = \lim_{x\to\infty} \frac{e^x}{x} = \text{(L'Hôpital's rule)} = \lim_{x\to\infty} \frac{e^x}{1} = \infty\,.$$

1.6. **Bounded Monotonic Sequences.** A monotonic sequence is a sequence that always increases or always decreases. For instance, $1/n$ is a monotonic decreasing sequence, and $n = 1, 2, 3, 4, \dots$ is a monotonic increasing sequence.

A sequence is bounded if its terms never get larger in absolute value than some given constant. For instance $1/n$ is bounded, because $|1/n| < 2$ for every n, but n is unbounded.

A property of any bounded (increasing or decreasing) sequence is that it always has a (finite) limit. This might not seem very useful if what we want is to actually compute the limit, but in some cases it may help. For instance, consider the sequence defined recursively in the following way:

$$\begin{aligned} a_1 &= \sqrt{6}\,, \\ a_{n+1} &= \sqrt{6 + a_n} \qquad (n \geq 1)\,. \end{aligned}$$

It can be shown that a_n it is a bounded monotonic increasing sequence,[1] hence it has some limit A:

$$\lim_{n\to\infty} a_n = A\,.$$

Now, taking limits on both sides of $a_{n+1} = \sqrt{6 + a_n}$ (here we use that the sequence has a limit) we get that $A = \sqrt{6 + A}$, i.e., $A^2 - A - 6 = 0$. Solving this second degree equation we get $A = 3$ or $A = -2$. Since the sequence is positive, the limit cannot be negative, hence it must be $A = 3$:

$$\lim_{n\to\infty} a_n = 3\,.$$

1.7. **Homework Problems.** E & P,[2] 11.2: 9–42, 54, 56.

2. Infinite Series

2.1. **Series.** An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots\,,$$

[1]Proof: We use induction. First note that $0 < a_1 = \sqrt{6} < 3$. By adding 6 and taking square roots we get $\sqrt{6} < \sqrt{6 + a_1} < \sqrt{6+3} = 3$, i.e.: $a_1 < a_2 < 3$. Now assume $a_n < a_{n+1} < 3$ for a given $n \geq 1$ (induction hypothesis). Again, by adding 6 and taking square roots we get $\sqrt{6 + a_n} < \sqrt{6 + a_{n+1}} < 3$, i.e. $a_{n+1} < a_{n+2} < 3$ (induction step). From here we get that $a_n < a_{n+1} < 3$ for every $n \geq 1$, which proves both, a_n is increasing and is bounded by 3.

[2]C.H. Edwards, Jr. & David E. Penney: *Calculus with Analytic Geometry*, 5th edition, Prentice Hall.

where $\{a_n\}$ is a sequence of numbers—sometimes the series starts at $n = 0$ or some other term instead of $n = 1$. Its Nth partial sum is

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N \,.$$

2.2. **Sum of a Series.** The sum

$$S = \sum_{n=1}^{\infty} a_n$$

of a series is defined as the limit of its partial sums

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

if it exists—it this case we say that the series converges. For instance, consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Its partial sum is

$$S_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^N} = 1 - \frac{1}{2^N} \,.$$

Hence, its sum is

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{2^N}\right) = 1 \,,$$

i.e.:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \,.$$

A series may or may not have a sum. For instance, in the following series:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

the sequence of partial sums $S_N = 1, 0, 1, 0, 1, 0, \dots$ diverges, and the series has no sum.

2.3. **Telescopic Series.** Telescopic series are series for which all terms of its partial sum can be canceled except the first and last ones. For instance, consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

Its nth term can be rewritten in the following way:

$$a_n = \frac{1}{n\,(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

Hence, its Nth partial sum becomes:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n\,(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}\,. \end{aligned}$$

Hence:

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1\,.$$

2.4. **Geometric series.** A geometric series $\sum_{n=0}^{\infty} a_n$ is a series in which each term is a fixed multiple of the previous one: $a_{n+1} = r\,a_n$, where r is called the *ratio*. A geometric series can be rewritten in this way:

$$\sum_{n=0}^{\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots\,.$$

If $|r| < 1$ its sum is

$$\sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}\,.$$

Note that a is the first term of the series. If $a \neq 0$ and $|r| \geq 1$, the series diverges.

Examples:

$$\sum_{n=0} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2\,.$$

$$\sum_{n=0} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}\,.$$

Note that in the last example $r = a_{n+1}/a_n = \frac{(-1)^{n+1}/2^{n+1}}{(-1)^n/2^n} = -1/2$.

2.5. **Termwise addition and multiplication of series.** If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} a_n = B$ then

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = A \pm B$$

and

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n = c\, A\,,$$

where c is a constant.

2.6. **The nth Term Test for Divergence.** If a_n does not converge to 0, then $\sum_{n=1}^{\infty} a_n$ does not converge—Note: the reciprocal is not true in general! (a counterexample is the harmonic series; see below.)

For instance, the series $\sum_{n=1}^{\infty} \sin n$ does not converge because $\sin n$ does not converge to 0.

2.7. **The Harmonic Series.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is called **harmonic series**.

The harmonic series diverges. This can be proven graphically, by looking at the graph of the function $f(x) = 1/x$ (fig. 1). The terms of the harmonic series are the areas of the rectangles. Their sum is greater than the area under the graph of $f(x) = 1/x$ between 1 and ∞, which can be computed with the following improper integral:

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{x}\, dx = \lim_{M\to\infty} [\ln x]_1^M = \lim_{M\to\infty} (\ln M - \ln 1) = \infty\,.$$

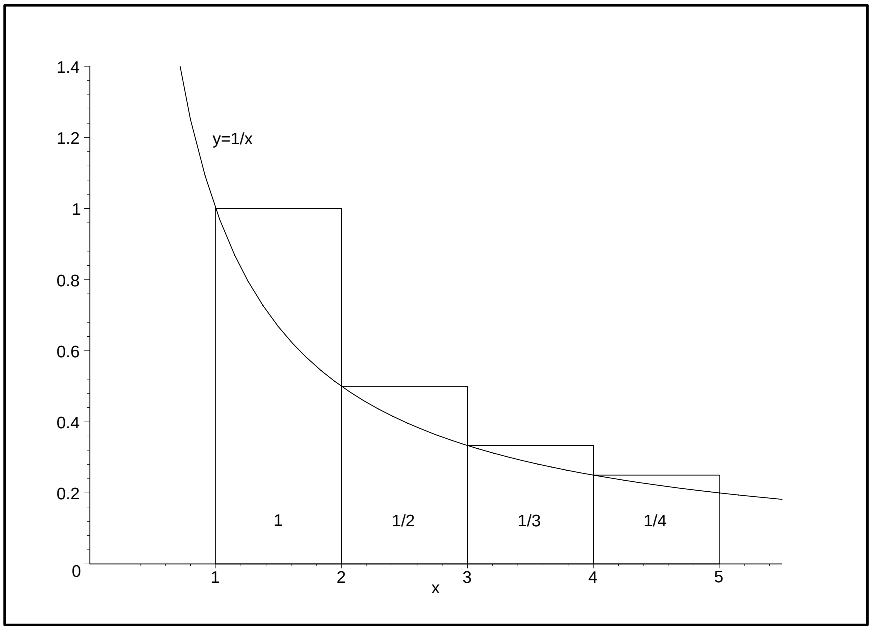

FIGURE 1. The harmonic series

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \,.$$

2.8. **Series that are Eventually the Same.** If $a_n = b_n$ for every n large enough, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. In other words, the convergence or divergence of a series depends only on its "tail" $\sum_{n=k}^{\infty} a_n$.

2.9. **Homework.** E & P, 11.3: 1–37, 49, 50, 64. Maple Worksheet: WS1 (due 10/7/99).

3. Taylor Series and Taylor Polynomials

3.1. **Linear Approximation of a Function.** Assume that we want to compute the value of a function such as $\sin 0.1$ or $\ln 1.1$. Evaluating polynomials is easy, it can be accomplished by a sequence of computations involving only arithmetic operations $(+, -, \times)$. For instance, if $f(x) = x^2 + 7x + 2$ then $f(3) = 3^2 + 7 \times 3 + 2 = 32$. For other functions in general we may not be able to evaluate them exactly, but we can do it approximately. A first approximation consists of substituting the graph of the function by a tangent line (fig. 2).

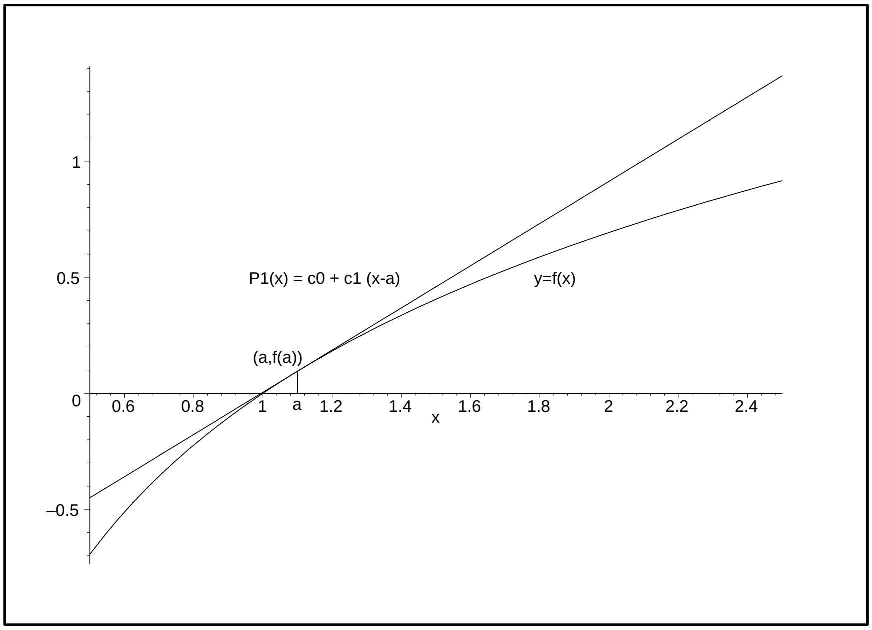

FIGURE 2. Linear approximation

The equation of the tangent line at $x = a$ is

$$P_1(x) = c_0 + c_1(x - a)\,.$$

Also, the following conditions must be met:

$$\begin{aligned} P_1(a) &= f(a) \\ P_1'(a) &= f'(a) \end{aligned}$$

i.e., the tangent line must pass through the point $(a, f(a))$, and its slope should be the derivative of $f(x)$ at $x = a$. From here we get:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \end{aligned}$$

hence the tangent line and first approximation of $f(x)$ at $x = a$ is

$$P_1(x) = f(a) + f'(a)\,(x - a)\,.$$

$P_1(x)$ is called the 1st-degree **Taylor polynomial** of $f(x)$ at $x = a$.

For instance, if $f(x) = \ln x$ then its first-degree Taylor polynomial at $x = 1$ is

$$P_1(x) = \ln 1 + \frac{1}{1}\,(x - 1) = x - 1\,.$$

Hence

$$\ln x \approx x - 1$$

for x close to 1. In particular $\ln 1.1 \approx 1.1 - 1 = 0.1$. Compare to the actual value $\ln 1.1 = 0.095310179\dots$.

3.2. **Higher Degree Polynomial Approximations.** The linear (first degree) polynomial approximation might be enough for many practical purposes, but sometimes we need a better approximation. This can be accomplished by using higher degree polynomials.

The nth degree Taylor polynomial of a function $f(x)$ at $x = a$ is a polynomial of the form:

$$\begin{aligned} P_n(x) &= c_0 + c_1\,(x-a) + c_2\,(x-a)^2 + \cdots + c_n\,(x-a)^n \\ &= \sum_{k=0}^{n} c_k\,(x-a)^k \end{aligned}$$

whose value and derivatives up to the nth are equal to those of $f(x)$ at $x = a$, i.e.:

$$\begin{aligned} P_n(a) &= f(a) \\ P_n'(a) &= f'(a) \\ P_n''(a) &= f''(a) \\ &\dots \\ P_n^{(n}(a) &= f^{(n}(a) \end{aligned}$$

Solving those equations we get that $c_k = f^{(k}(a)/k!$, hence the nth-degree Taylor polynomial of $f(x)$ at $x = a$ is

$$\begin{aligned} P_n(x) &= \frac{f(a)}{0!} + \frac{f'(a)}{1!}\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \cdots + \frac{f^{(n}(a)}{n!}\,(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k}(a)}{k!}\,(x-a)^k\,. \end{aligned}$$

For instance, for $f(x) = \ln(x)$ and $a = 1$ we get

$$P_2(x) = (x-1) - \frac{1}{2}\,(x-1)^2\,,$$

hence, $\ln 1.1 \approx (1.1-1) - \frac{1}{2}\,(1.1-1)^2 = 0.095$, closer to the actual value $\ln 1,1 = 0.095310179\dots$ than the first degree approximation computed above.

3.3. **Taylor series.** The next logical step is to use an "infinite"-degree polynomial, i.e., a series of the form:

$$\sum_{k=0}^{\infty} \frac{f^{(k}(a)}{k!}\,(x-a)^k\,.$$

This series is called the **Taylor series** of $f(x)$ at $x = a$. If $a = 0$ then the Taylor series is called **Maclaurin series**.

A function $f(x)$ such that

1. its Taylor series converges, and
2. $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k}(a)}{k!}\,(x-a)^k\,,$

is called *analytic*. A few examples of analytic functions are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} \qquad (|x| < 1)$$

$$\sin x = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad (|x| < 1)$$

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad (|x| < 1)$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha\,(\alpha-1)\,(\alpha-2) \overset{(n \text{ factors})}{\cdots} (\alpha-n+1)}{n!}$.

As an example, the Taylor series for e^x can be used for computing the number e:

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.718281828\ldots$$

3.4. **Homework.** E & P, 11.4: 11–20, 23–25, 29–40.

4. The Integral Test.

In section 2.7 we found that the harmonic series diverges by comparing its sum with an integral (see fig. 1). This technique is called **integral test**. More generally we have that the following holds:

4.1. **Integral Test.** Suppose $\sum_{n=1}^{\infty} a_n$ is a positive-term series (i.e., $a_n > 0$ for every $n = 1, 2, 3, \dots$). Also assume that $f(x)$ is a positive valued, decreasing, continuous function for $x \geq 1$ such that $a_n = f(n)$ for every $n = 1, 2, 3 \dots$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the improper integral $\displaystyle\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.

So, in order to test the series for convergence it is enough to test the corresponding integral for convergence, which in many cases is easier.

The next one is a typical example of application of the integral test.

4.2. **p-Series.** A p-series is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,,$$

where $p > 0$ is a fix exponent.

The case $p = 1$ is the harmonic series, which was shown to be divergent in section 2.7. On the other hand, if $p \neq 1$:

$$\int_1^M \frac{1}{x^p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^M = \frac{1}{1-p}\,\left(M^{1-p} - 1\right)\,.$$

As $M \to \infty$ the limit of the above expression converges for $p > 1$, and diverges for $p \leq 1$. Hence, the integral test shows that the p-series converges for $p > 1$, and diverges for $p \leq 1$.

4.3. **Homework.** E & P, 11.5: 1–30, 35–38.

5. Comparison Test for Positive-Term Series

The comparison test allows us to test a series for convergence by comparing it to another series for which convergence is easier to test. It says the following:

5.1. **Comparison Test.** Suppose $\sum a_n$ and $\sum b_n$ are positive-term series such that $a_n \leq b_n$ for every n. Then:

1. If $\sum a_n$ diverges then $\sum b_n$ diverges.
2. If $\sum b_n$ converges then $\sum a_n$ converges.

For instance, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} .$$

Since $1/\sqrt{n^3+1} < 1/\sqrt{n^3} = 1/n^{3/2}$, the convergence of the given series can be derived from the convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} .$$

This is a p-series with $p = 3/2 > 1$, hence it converges, and so does the given series.

In some cases the ordinary comparison test is hard to apply, but the following is easier to use:

5.2. **Limit Comparison Test.** Suppose $\sum a_n$ and $\sum b_n$ are positive-term series, and the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is not zero nor infinity: $0 < L < +\infty$. Then either both series converge or both diverge.

For instance, look at the following series:

$$\sum_{n=1}^{\infty} \frac{2+\sqrt{3n}}{\sqrt{n^3+n+1}} .$$

We can test its convergence by limit comparison with the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} .$$

In fact:

$$\lim_{n\to\infty} \frac{(2+\sqrt{3n})/\sqrt{n^3+n+1}}{1/n} = \sqrt{3} ,$$

which is not zero nor infinity. Since the harmonic series diverges, we conclude that the given series also diverges.

5.3. **Rearrangement and grouping.** In a positive-term series rearranging and regrouping its terms does not alter its sum. Note that this is not true in general for other series. For instance, the following series is divergent:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots .$$

However, if we regroup its terms in the following way, we get a convergent series with zero sum:

$$(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0\,.$$

A different regrouping still give us a different sum:

$$1+(-1+1)+(-1+1)+(-1+1)+\cdots = 1+0+0+0+\cdots = 1\,.$$

However, if the terms are all positive, then we can rearrange and regroup them without any fear of changing the sum. So, for instance:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}\frac{1}{2^n} = \frac{1/2}{1-1/2} = 1\,.$$

Now, if we regroup its terms in the following way, we get:

$$\left(\frac{1}{2}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{16}\right)+\cdots = \frac{3}{4}+\frac{3}{16}+\cdots = \sum_{n=1} 3\,\frac{1}{4^n} = \frac{3/4}{1-1/4} = 1\,,$$

so, we get the same sum.

5.4. **Homework.** E & P, 11.6: 1–36.

6. Alternating Series and Absolute Convergence

An *alternating series* is a series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

where $a_n > 0$ for every n. Examples: the *alternating harmonic* series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\,;$$

an alternating geometric series:

$$\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots\,.$$

6.1. **Alternating Series Test.** If an alternating series verifies:

1. a_n it is decreasing: $a_n \geq a_n > 0$ for every n, and
2. the nth term tends to zero: $\lim_{n\to\infty} a_n = 0$,

then the series converges.

So, in this particular case the "reciprocal" of the nth term test holds.

E.g.:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} .$$

6.2. **Alternating Series Remainder Estimate.** Let S be the sum of an alternating series:

$$S = \sum_{n=0}^{\infty} (-1)^{n+1} a_n ,$$

and S_N its Nth partial sum:

$$S_N = \sum_{n=0}^{N} (-1)^{n+1} a_n .$$

Then the difference $R_N = S - S_N$ (remainder) between the sum of the series and that of the Nth partial sum has the same sign as the following term of the series $(-1)^{n+2}\, a_{n+1}$, and

$$0 \leq |R_N| < a_{n+1} .$$

In particular, S is always between two consecutive partial sums.

For instance, we know that

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} ,$$

but how close do we get to π by adding, say, one hundred terms of that series? Answer: its sum π is between

$$4 \sum_{n=0}^{99} \frac{(-1)^{n+1}}{2n+1} = 3.131592904\ldots$$

and

$$4 \sum_{n=0}^{100} \frac{(-1)^{n+1}}{2n+1} = 3.151493401\ldots$$

Compare to the actual value $\pi = 3.141592654\dots$.

6.3. **Absolute Convergence.** A general series $\sum a_n$ is said to be *absolutely convergent* if the series of absolute values of its terms $\sum |a_n|$ is convergent.

We have that a series can be:

1. Convergent and absolutely convergent, e.g:
$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$
2. Convergent but not absolutely convergent—in this case the series is called **conditionally convergent**—, e.g:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
3. Not convergent nor absolutely convergent, e.g:
$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$$

However, a series cannot be absolutely convergent and not convergent, because absolute convergence implies convergence:
$$\text{absolute convergent} \implies \text{convergent}$$

Example: Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converge? Answer: Look at the series of absolute values:
$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$
By comparison test, it converges (the right hand side is a p-series with $p > 1$), hence the given series is absolutely convergent, which implies that it is indeed convergent.

6.4. **Ratio Test.** Suppose that the limit $\displaystyle\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is infinity. Then

1. If $\rho < 1 \implies \sum a_n$ converges absolutely.
2. If $\rho > 1 \implies \sum a_n$ diverges.
3. If $\rho = 1 \implies$ the ratio test is inconclusive.

As a rule of thumb, for geometric series $\rho = |r|$ (the ratio), and the conclusion of the ratio test is analogous to the one for geometric series, i.e., the series converges for $|r| < 1$ and diverges for $|r| > 1$.

Example: For the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ we have

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2/2^{n+1}}{n^2/2^n} = \lim_{n\to\infty} \frac{(n+1)^2}{2\,n^2} = \frac{1}{2} < 1\,,$$

hence, it converges absolutely.[3]

6.5. **Root Test.** In some cases in which the ratio test is unable to provide an answer, the root test may help. It says the following: Suppose that the limit $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists or is infinity. Then

1. If $\rho < 1 \implies \sum a_n$ converges absolutely.
2. If $\rho > 1 \implies \sum a_n$ diverges.
3. If $\rho = 1 \implies$ the root test is inconclusive.

Example: Consider the following series $\sum_{n=1}^{\infty} \frac{1}{2^{n+\sin n}}$. For this series the ratio test cannot be used, because

$$\frac{a_{n+1}}{a_n} = 2^{-1+\sin n - \sin(n+1)} = 2^{-1-2\sin\frac{1}{2}\cos(n+\frac{1}{2})}$$

which has no limit. However, the root tests shows that the series is absolutely convergent:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{n+\sin n}}} = \lim_{n\to\infty} \frac{1}{2^{1+\sin n/n}} = \frac{1}{2} < 1\,.$$

6.6. **Homework.** E & P, 11.7: 1–42.

7. Power Series

A power series is a sort of infinite polynomial of the form

$$\sum_{n=0}^{\infty} a_n\, x^n$$

Taylor series are particular cases of power series. E.g.:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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[3] The sum is exactly $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots \qquad (|x| < 1)$$

7.1. **Convergence of a Power Series.** The first question we must answer about a powers series is "for what values of x does the series converge?" This can be answered with the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{R},$$

where

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

(If the limit is zero then we take $R = \infty$. If it is infinity then $R = 0$.)

We know that the series converges absolutely if $\rho < 1$, i.e., $\frac{|x|}{R} < 1$, and diverges if $\rho > 1$, i.e., $\frac{|x|}{R} > 1$. Hence the series converges absolutely for $|x| < R$, i.e., on the interval $(-R, R)$, and diverges for $|x| > R$. The number R is called the *radius of convergence* of the series.

It remains to determine if the series converges at the endpoints $x = R$ and $x = -R$. This can be answered by substituting $x = R$ and $x = -R$ in the power series and studying the resulting series. After doing that, we will find that the *interval of convergence* is one of the following:

$$(-R, R), \quad [-R, R), \quad (-R, R] \quad \text{or} \quad [-R, R]$$

If $R = \infty$ then the interval of convergence is the whole real line $\mathbb{R}$.

Example: consider the Maclaurin series for $\ln(1 + x)$:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} .$$

Using the ratio test we get:

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| .$$

Hence the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. It remains to study the endpoints $x = \pm 1$.

For $x = 1$ the power series becomes

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n},$$

which is the alternating harmonic series. We know that it converges.

For $x = -1$ the power series becomes

$$-\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the (usual) harmonic series. We know that it diverges.

Hence the interval of convergence is $(-1, 1]$.

7.2. **Power Series in Powers of $(\mathbf{x} - \mathbf{c})$.** The same techniques can be applied to power series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Using the ratio test we get

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| = |x-c| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-c|}{R},$$

where

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Hence the series converges absolutely if $\frac{|x-c|}{R} < 1$ and diverges if $\frac{|x-c|}{R} > 1$. In other words, it converges absolutely for $|x-c| < R$, i.e., on the interval $(c-R, c+R)$, and diverges for $|x-c| > R$. As above, the endpoints $x = c-R$ and $x = c+R$ must be tested separately.

So the radius of convergence is as before, but the interval of convergence is centered at $x = c$ instead of $x = 0$.

7.3. **The Binomial Series.** The binomial series is the Maclaurin series of the function $f(x) = (1+x)^{\alpha}$. It can be computed in the usual way:

$$\begin{array}{rclcrcl}
f(x) & = & (1+x)^{\alpha} & \quad & f(0) & = & 1 \\
f'(x) & = & \alpha\,(1+x)^{(\alpha-1)} & & f'(0) & = & \alpha \\
f''(x) & = & \alpha\,(\alpha-1)\,(1+x)^{(\alpha-2)} & & f''(0) & = & \alpha\,(\alpha-1) \\
 & \cdots & & & & \cdots &
\end{array}$$

$$f^{(n}(x) = \alpha\,(\alpha-1)\dots(\alpha-n+1)\,(1+x)^{(\alpha-2)}$$
$$\dots$$
$$f^{(n}(0) = \alpha\,(\alpha-1)\dots(\alpha-n)$$
$$\dots$$

Hence:

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1} x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \quad (|x|<1)$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha\,(\alpha-1)\,(\alpha-2)^{(n \text{ factors})}\cdots(\alpha-n+1)}{n!}$.

Note that if $\alpha = m$ a positive integer, then $\binom{m}{n} = 0$ for $n > m$, so the series becomes a finite sum identical to the binomial expansion:

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n$$

Other examples:

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 + \cdots$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n = 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \cdots$$

7.4. **Differentiation and Integration of Power Series.** If a function $f(x)$ can be represented as a power series:

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n$$

with radius of convergence R, then:

1. It is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}\,.$$

2. It is integrable on $(-R, R)$ and

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n\, x^{n+1}}{n+1}\,.$$

As an application, we use this result for computing the power series for the arctangent $f(x) = \tan^{-1} x$. In fact, we have:

$$f'(t) = \frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n t^{2n} .$$

Integrating termwise we get:

$$f(x) = \int_0^x f'(t)\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} .$$

Hence:

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots .$$

7.5. **Homework.** E & P, 11.8: 1–12. Maple Worksheet: WS2 (due 10/14/99).

8. Power Series Computations

8.1. Adding and Multiplying Series.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)\, x^n$$

and

$$f(x)\, g(x) = \sum_{n=0}^{\infty} c_n\, x^n$$

where

$$c_n = a_0\, b_n + a_1\, b_{n-1} + \cdots a_n\, b_0 = \sum_{k=0}^{n} a_k\, b_{n-k}$$

As an example, we can compute the first few terms of the power series for $\tan x = a_0 + a_1 x + a_2 x^2 + \cdots$, using the power series for $\sin x = x - x^3/3! + x^5/5! - \cdots$ and $\cos x = 1 - x^2/2! + x^4/4! - \cdots$, and the relation $\sin x = \tan x \, \cos x$:

$$\overbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}^{\sin x} = \overbrace{\left(a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots\right)}^{\tan x} \overbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}^{\cos x}$$
$$= a_0 + a_1\, x + \left(-\frac{a_0}{2} + a_2\right) x^2 + \left(-\frac{a_1}{2} + a_3\right) x^3 + \cdots$$

Identifying coefficients we get:

$$\begin{cases} a_0 & & & & = & 0 \\ & a_1 & & & = & 1 \\ -\frac{1}{2}a_0 & & +a_2 & & = & 0 \\ & -\frac{1}{2}a_1 & & +a_3 & = & -\frac{1}{6} \\ & & & & \cdots & \end{cases}$$

Solving that system of equations we get $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 1/3, \ldots$, hence:

$$\tan x = x + \frac{x^3}{3} + \cdots$$

8.2. **Computing Limits with Power Series.** Since power series are continuous in their interval of convergence, we have that if

$$f(x) = \sum_{n=0}^{\infty} a_n\,(x-c)^n = a_0 + a_1\,x + a_2\,x^2 + \cdots$$

then

$$\lim_{x\to c} f(x) = f(c) = a_0 + 0 + 0 + \cdots = a_0\,.$$

This can be applied to computing limits of indeterminate forms $f(x)/g(x)$ by substituting power series for $f(x)$ and $g(x)$. Example:

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos x}{x\,\sin x} &= \lim_{x\to 0} \frac{1-(1-x^2/2+x^4/24-\cdots)}{x\,(x-x^3/6+\cdots)} \\ &= \lim_{x\to 0} \frac{x^2/2-x^4/24+\cdots}{x^2-x^4/6+\cdots} \\ &= \lim_{x\to 0} \frac{1/2-x^2/24+\cdots}{1-x^2/6+\cdots} \\ &= \frac{1}{2} \end{aligned}$$

8.3. **Homework.** E & P, 11.9: 23–28.