Math B17 - Spring 1999 - Midterm Exam No. 1 - Lerma (answers)

ANSWERS

1. Determine whether the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty} \sin^n 1$$

Answer:

It is a geometric series of ratio $r = \sin 1$. Since $|r| < 1$, the series converges. The sum is

$$\sum_{n=1}^{\infty} \sin^n 1 = \frac{\sin 1}{1 - \sin 1}$$

2. Find the Taylor series of $\ln x$ at $a = 1$.

Answer:

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

3. Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$$

Answer:

We have that: $$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} > \sum_{n=1}^{\infty} \frac{1}{n},$$

so by the comparison test, the series *diverges* (recall that the harmonic series diverges).

4. Determine if the following series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=0}^{\infty} \frac{(-10)^n}{n!}$$

Answer:

We use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10^{n+1}/(n+1)!}{10^n/n!} = \lim_{n\to\infty} \frac{10}{n+1} = 0 < 1,$$

hence the series converges absolutely.

5. Find a power series for the following function and find its radius of convergence: $f(x) = \dfrac{x}{1-x}$

Answer:

Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, we have:

$$\frac{x}{1-x} = x \sum_{n=0}^{\infty} x^n = x\,(1 + x + x^2 + x^3 + \cdots) =$$

$$x + x^2 + x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} x^n$$

In order to find its radius of convergence, we use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x|.$$

There is convergence for $\rho < 1$, i.e., $|x| < 1$. At the endpoints we get the series

$$\sum_{n=1}^{\infty} (-1)^n \quad \text{and} \quad \sum_{n=1}^{\infty} 1^n$$

respectively, whose n-th terms do not tend to zero, so they do not converge. Hence, the interval of convergence is:

$$I = (-1, 1),$$

not including the endpoints.

6. Use power series to evaluate the following limit:

$$\lim_{x\to 1} \frac{\ln(x^2)}{x-1} =$$

Answer:

$$\begin{aligned}\lim_{x\to 1} \frac{\ln(x^2)}{x-1} &= \lim_{x\to 1} \frac{(x^2-1) - (x^2-1)^2/2 + (x^2-1)^3/3 - \cdots}{x-1} \\ &= \lim_{x\to 1} \{(x+1) - (x+1)(x^2-1)/2 + (x+1)(x^2-1)^2/3 - \cdots\} \\ &= 2\end{aligned}$$

An alternative (and simpler) way consists of using $\ln(x^2) = 2\ln x$:

$$\begin{aligned}\lim_{x\to 1} \frac{\ln(x^2)}{x-1} &= 2\lim_{x\to 1} \frac{\ln x}{x-1} \\ &= 2\lim_{x\to 1} \frac{(x-1) - (x-1)^2/2 + (x-1)^3/3 - \cdots}{x-1} \\ &= 2\lim_{x\to 1} \{(1 - (x-1)/2 + (x-1)^2/3 - \cdots\} \\ &= 2\end{aligned}$$