Math B17 - Spring 1999 - Midterm Exam No. 2 - Lerma (answers)

ANSWERS

1. Let A and B be the following matrices:

$$A = \begin{bmatrix} -2 & 2 & -4 \\ -2 & 3 & -3 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Compute $P^{-1}\,A\,P$.

Answer:

$$P^{-1} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \qquad P^{-1}\,A\,P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. Is the row vector $\mathbf{r} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$ in the row space of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$?

Answer:

Since the rows of A are not proportional, rank$A = 2$. If $\mathbf{r}$ is in the row space of A then the following matrix should have also rank 2:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{bmatrix}.$$

In fact, after using Gaussian reduction, that matrix becomes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so rank$B = 2$, and $\mathbf{r}$ does belong to the row space of A.

Another way to prove that $\mathbf{r}$ belongs to the row space of A is to note that in can be obtained by subtracting twice the first row of A from its second row.

3. Solve the following system of equations:

$$\left\{\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2\,x_3 & = & 1 \\ & & x_2 & + & 2\,x_3 & = & 3 \\ x_1 & & & + & 4\,x_3 & = & 4 \end{array}\right.$$

Answer:

The augmented matrix is: $\left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 4 \end{array}\right]$

After using Gauss-Jordan reduction we get: $\left[\begin{array}{rrr|r} 1 & 0 & 4 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

i.e.:

$$\left\{\begin{array}{rcrcrcl} x_1 & & & & 4\,x_3 & = & 4 \\ & & x_2 & + & 2\,x_3 & = & 3 \end{array}\right.$$

The solution is $x_1 = 4 - 4\,x_3$, $x_2 = 3 - 2\,x_3$, or:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 4\,x_3 \\ 3 - 2\,x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$$

4. Find a basis and the dimension of the solution space for the system:

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \\ x_1 + 3x_2 + x_3 + 3x_4 = 0 \end{cases}$$

Answer:

The coefficient matrix is:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}.$$

After using Gauss-Jordan reduction it becomes: $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Hence, the general solution of $A\mathbf{x} = \mathbf{0}$ is:

$$\begin{cases} x_1 = 2x_3 + 3x_4 \\ x_2 = -x_3 - 2x_4 \end{cases}.$$

In matrix form: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$

Hence, the following set is a basis for the solution space:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and its dimension is 2.

5. Find the coordinates of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ respect to the basis:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Answer:

Calling $\mathbf{v}' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ the coordinates of $\mathbf{v}$ respect to the new basis, we have $P\,\mathbf{v}' = \mathbf{v}$, where $P = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$. Hence, we must solve the system:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

The augmented matrix is: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$

After Gauss-Jordan reduction it becomes: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Hence the solution is:

$$\mathbf{v}' = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

6. Diagonalize the following matrix: $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$. Find the matrix P such that $P^{-1} A P$ is diagonal.

Answer:

The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{bmatrix} = \lambda^2 - \lambda - 2$$
$$= (\lambda - 2)(\lambda + 1)$$

Its roots are $\lambda = 2$ and $\lambda = -1$.

For $\lambda = 2$ we get $A - 2I = \begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix}$.

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

The solutions of $(A - 2I)\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So we can take the following eigenvector: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, or any non-zero multiple of it.

For $\lambda = -1$ we get $A + I = \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix}$.

After using Gauss-Jordan the matrix becomes: $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$.

The solutions of $(A + I)\mathbf{v} = 0$ are:

$$\mathbf{v} = \begin{bmatrix} x_2/2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So we can take the following eigenvector: $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, or any non-zero multiple of it.

Hence:[1]

$$P = [\mathbf{v_1}, \mathbf{v_2}] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

$$P^{-1}\,A\,P = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

[1] There are other possible choices for P, for instance: $P = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$.