Math B17 - Winter 1999 - Midterm Exam No. 1 (solutions)

SOLUTIONS

1. Determine if the following infinite series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{\ln n}{1+\ln(n+7)}$$

Solution:

Using l'Hôpital's rule we check that the n-th term does not converge to zero:

$$\lim_{n\to\infty} \frac{\ln n}{1+\ln(n+7)} = \lim_{x\to\infty} \frac{\ln x}{1+\ln(x+7)} = \lim_{x\to\infty} \frac{1/x}{1/(x+7)} = \lim_{x\to\infty} \frac{x+7}{x} = 1$$

Hence, by the n-th Term Test for Divergence, the series diverges.

2. Use the integral test to determine if the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$$

Solution:

First, note that $f(x) = \frac{1}{x\,(\ln x)^2}$ is continuous, positive and decreasing for $x \geq 2$. Next, we compute the following integral:

$$\int_2^n \frac{1}{x\,(\ln x)^2}\,dx = \left[-\frac{1}{\ln x}\right]_2^n = \frac{1}{\ln 2} - \frac{1}{\ln n}\;.$$

So:

$$\lim_{n\to\infty} \int_2^n \frac{1}{x\,(\ln x)^2} = \frac{1}{\ln 2}\;.$$

Since the integral converges, the series converges.

3. Let S be the sum of the following series:

$$S = \sum_{n=0}^{\infty} \frac{\cos^2 n}{5^n}$$

Determine which one of the following statements is true and show why:

1. The series diverges.
2. The series converges and $5/4 \leq S$.
3. The series converges and $0 < S < 5/4$.

Solution:

First note that $0 \leq \cos^2 n \leq 1$, hence:

$$0 \leq \frac{\cos^2 n}{5^n} \leq \frac{1}{5^n}$$

Since the following geometric series converges:

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} \,,$$

by Comparison Test the given series also converges, and its sum is $S \leq 5/4$. Note that the inequality is actually strict ($S < 5/4$), since, for instance, $\frac{\cos^2 1}{5} < 1/5$. Hence statement 3 is true.

4. Find the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-2)^n$$

Solution:

By the method at the beginning of section 11.8 of the textbook:

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1 \ ,$$

hence the radius of convergence is $R = 1/\rho = 1$, so the power series converges absolutely for $|x-2| < 1$, i.e., $1 < x < 3$.

Alternatively, using directly the Ratio Test, the series converges absolutely wherever the following limit is less than 1:

$$\lim_{n\to\infty} \frac{|x-2|^{n+1}/(n+1)}{|x-2|^n/n} = \lim_{n\to\infty} \frac{n}{n+1} |x-2| = |x-2| \ ,$$

hence the power series converges absolutely for $|x-2| < 1$, i.e., $1 < x < 3$.

Next we test the endpoints.

For $x = 1$ the series is

$$\sum_{n=1}^{\infty} \frac{1}{n} \ ,$$

which diverges.

For $x = 3$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \ ,$$

which converges.

Hence its interval of convergence is $(1, 3]$.

5. Find the power series in x of the function defined by the following integral:

$$f(x) = \int_0^x \frac{\sin t}{t}\, dt =$$

Solution:

The power series of $\sin t$ is:

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

Dividing by t we get:

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

Integrating termwise we get:

$$\int_0^x \frac{\sin t}{t}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!\,(2n+1)} = x - \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} - \frac{x^7}{7!\,7} + \cdots$$

6. Use power series to compute the following limit:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=$$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) &= \lim_{x\to 0}\frac{\sin x - x}{x\,\sin x}\\
&= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)-x}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}\\
&= \lim_{x\to 0}\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots}{x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\cdots}\\
&= \lim_{x\to 0}\frac{-\frac{x}{3!}+\frac{x^3}{5!}-\cdots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots}\\
&= \frac{0}{1}=0
\end{aligned}$$