Math B17 - Winter 1999 - Final Exam (solutions)

SOLUTIONS

1. Determine if the following infinite series converge or diverge:

(a) $\displaystyle\sum_{n=1000}^{\infty} \frac{(-1)^n}{\ln \ln \ln n}$

Solution:

It *converges*, because it is an alternating series and the n-th term is decreasing and tends to zero.

(b) $\displaystyle\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Solution:

We have

$$\lim_{n\to\infty} n \sin \frac{1}{n} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

so the series *diverges* because the n-th term does not tend to zero.

2. Determine if the following infinite series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

Solution:

By comparison:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

The series on the right hand side converges by the integral test, hence the given series *converges.*

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+n+2}{n^3+3n+7}$

Solution:

We have

$$\lim_{n\to\infty} \frac{\frac{n^2+n+2}{n^3+3n+7}}{1/n} = \lim_{n\to\infty} \frac{n^3+n^2+2n}{n^3+3n+7} = 1$$

so by the Limit Comparison Test the given series behaves the same as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, i.e.: *diverges.*

3. Use power series to compute the following limit:

$$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)=$$

Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\ln(1+x)}{x\,\ln(1+x)}\\
&= \lim_{x\to 0}\frac{x-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}{x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{2}-\frac{x^3}{3}+\cdots}{x^2-\frac{x^3}{2}+\frac{x^4}{3}-\cdots}\\
&= \lim_{x\to 0}\frac{\frac{1}{2}-\frac{x}{3}+\cdots}{1-\frac{x}{2}+\frac{x^2}{3}-\cdots}\\
&= \frac{1}{2}
\end{aligned}$$

4. Solve the following system of equations or show that it has no solutions:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = 14 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 = 18 \\ 4x_1 + 5x_2 + 6x_3 + 7x_4 = 22 \end{cases}$$

Solution:

The augmented matrix is $A' = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 2 & 3 & 4 & 5 & 14 \\ 3 & 4 & 5 & 6 & 18 \\ 4 & 5 & 6 & 7 & 22 \end{array}\right]$.

After using Gauss-Jordan reduction we get: $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

i.e.:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases}$$

The solution is $x_1 = -2 + x_3 + 2x_4$, $x_2 = 6 - 2x_3 - 3x_4$, or:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

5. Solve the following system of equations or show that it has no solutions:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 3x_2 + 4x_3 = 5 \\ x_2 + 2x_3 = 0 \end{cases}$$

Solution:

The augmented matrix is: $A' = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 0 \end{array}\right]$.

After applying Gauss-Jordan we get: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

We see that the rank of the coefficient matrix is 2, and the rank of the augmented matrix is 3, hence the system has no solutions.

6. Which of the following matrices are orthogonal? Why?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$$

Solution:

A way to see if a matrix is orthogonal is to check if its inverse equals its transpose:

$$A^t\,A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Since $A^t\,A \neq I$, A is not orthogonal.

$$B^t\,B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

So B is not orthogonal either.

$$C^t\,C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence C is orthogonal.

7. In $\mathbb{R}^2$ find the change of basis matrix for a 60^0 clockwise rotation. What are the new coordinates of the point $(1,1)$? (Note: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$.)

Solution:

The rotation transforms the standard basis $\left\{ \mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ into $\left\{ \mathbf{v_1} = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \right\}$, hence the change of basis matrix is

$$P = [\,\mathbf{v_1}\ \mathbf{v_2}\,] = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Since P is orthogonal, its inverse is its transpose: $P^{-1} = P^t = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

The new coordinates of the point $(1,1)$ are:

$$P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1-\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{bmatrix},$$

i.e.: $\left(\dfrac{1-\sqrt{3}}{2}, \dfrac{1+\sqrt{3}}{2}\right)$.

8. Find the dimension and an *orthonormal* basis for the column space of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

Solution:

After using Gauss reduction on A the matrix becomes: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which shows that the first three columns of A form a basis for its column space:

$$\left\{ \mathbf{v_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Since the basis has three vectors, the dimension of the column space is 3.

Now we orthonormalize it by using the Gram-Schmidt process. Since $\mathbf{v_1}$ is already orthogonal to $\mathbf{v_2}$ and $\mathbf{v_3}$, we need to apply Gram-Schmidt to $\{\mathbf{v_2}, \mathbf{v_3}\}$ only:

$$\mathbf{v_3'} = \mathbf{v_3} - \frac{\mathbf{v_2} \cdot \mathbf{v_3}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1 \\ 1/2 \\ 1 \end{bmatrix}.$$

After normalizing we get the following orthonormal basis:

$$\left\{ \mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_3} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ -2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

9. Find the principal axes and classify the central conic:

$$5\,x^2 + 5\,y^2 - 6\,xy = 8$$

Solution:

The conic can be represented as $\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 8$, where $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$.

We must diagonalize A as $D = P^t A P$ for some *orthogonal* matrix $P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}$, where $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A.

The eigenvalues of A are the roots of the characteristic polynomial:

$$\det\,(A - \lambda\, I) = \det \begin{bmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{bmatrix} = \lambda^2 - 10\,\lambda + 16 = (\lambda - 2)\,(\lambda - 8)$$

The eigenvalues are $\lambda = 2$ and $\lambda = 8$.

For $\lambda = 2$ we must solve $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as eigenvector.

For $\lambda = 8$ we must solve $\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $x_1 = -x_2$, or: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so we take $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Note that $\mathbf{v_1}$ and $\mathbf{v_2}$ are already orthogonal, so all we need is to normalize them: $\mathbf{u_1} = \frac{1}{\sqrt{2}}\mathbf{v_1}$, $\mathbf{u_2} = \frac{1}{\sqrt{2}}\mathbf{v_2}$. The matrix for the change of basis is:

$$P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In the new basis the conic is $\begin{bmatrix} x' & y' \end{bmatrix} D \begin{bmatrix} x' \\ y' \end{bmatrix} = 8$, where

$$D = P^t\,A\,P = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix},$$

and

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = P^t \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.:

$$\begin{cases} x' = \frac{1}{\sqrt{2}} (x + y) \\ y' = \frac{1}{\sqrt{2}} (-x + y) \end{cases}$$

Hence the conic is $2\,x'^2 + 8\,y'^2 = 8$, or equivalently: $\dfrac{x'^2}{4} + y'^2 = 1$, which is an *ellipse*. Its principal axes are given by the basic vectors

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Note: An alternative solution is $x'^2 + \dfrac{y'^2}{4} = 1$, and

$$\mathbf{u_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

10. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$. Find a matrix P such that $D = P^{-1} A\, P$ is diagonal.

Solution:

The solution is of the form $P = [\, \mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\,]$, were $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors for A.

We have

$$\det(A - \lambda\, I) = \det \begin{bmatrix} 1-\lambda & 2 & -2 \\ 2 & 1-\lambda & -2 \\ 0 & 0 & -1-\lambda \end{bmatrix} = 3 + 5\lambda + \lambda^2 - \lambda^3 = -(\lambda - 3)(\lambda + 1)^2$$

So the roots of the characteristic polynomial are $\lambda = 3$ and $\lambda = -1$ (double).

For $\lambda = 3$ we get: $A - 3\, I = \begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & -2 \\ 0 & 0 & -4 \end{bmatrix}$

After using Gauss-Jordan that matrix becomes: $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The solution of $(A - 3\, I)\, \mathbf{x} = \mathbf{0}$ is $x_1 = x_2$, $x_3 = 0$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So we take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as the first eigenvector.

Next, for $\lambda = -1$ we get $A + I = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

After using Gauss-Jordan that matrix becomes $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, the solution of $(A + I)\,\mathbf{x} = \mathbf{0}$ is $x_1 = -x_2 + x_3$, i.e.:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So, we take the eigenvectors $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The matrix P is:[1]

$$P = [\,\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\,] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$D = P^{-1}\,A\,P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[1]Other solutions, obtained by permuting the columns of P, are also possible. Also the vectors $\mathbf{v_2}$ and $\mathbf{v_2}$ could be chosen differently, provided they span the same subspace.