A SIMPLE DERIVATION OF THE EQUATION FOR THE BRACHISTOCHRONE CURVE

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Abstract. This is a brief summary of a derivation of the equation of the curve of fastest descent between two points, also called brachistochrone.

1. Introduction

Assume you have two points A and B in a vertical plane, and a particle sliding frictionless along a curve joining A and B under the influence of gravity. The problem is to find the equation of the curve joining A and B that minimizes the time taken by the particle to travel from A to B. The problem was solved in 1696 by Leibniz, L'Hôpital, Newton, and Jakob and Johann Bernoulli. Here we summarize a solution following Johann Bernoulli's approach, based on *Fermat's principle of least time.*

2. Equation of the brachistochrone

2.1. **Fermat's principle of least time.** Fermat's principle of least time states that light rays passing through different media follow the fastest path between two points. This implies *Snell's law:* the ratio of sines of the angle of incidence α_1 and the angle of refraction α_2 is equal to the ratio of the velocities v_1, v_2 in the two media, i.e.,

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2}\,.$$

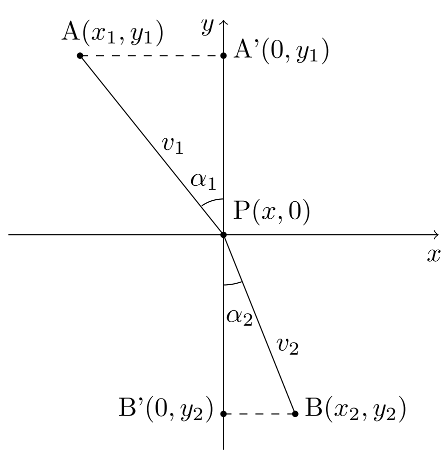

Figure 1. Fermat's principle.

To prove it, assume the given points have coordinates $A(x_1, y_1)$, $B(x_2, y_2)$ respectively, and the light enters the other media at point $P(x, 0)$. Then, the time for the light to travel from A to B is

$$T(x) = \frac{\overline{AP}}{v_1} + \frac{\overline{PB}}{v_1} = \frac{\sqrt{(x-x_1)^2+y_1^2}}{v_1} + \frac{\sqrt{(x-x_2)^2+y_2^2}}{v_2}\,,$$

where $\overline{AP}$ and $\overline{PB}$ are the lengths of segments AP and PB respectively. Next, differentiating with respect to x we get

$$\begin{aligned} T'(x) &= \frac{x - x_1}{v_1\sqrt{(x-x_1)^2 + y_1^2}} + \frac{x - x_2}{v_2\sqrt{(x-x_2)^2 + y_2^2}} \\ &= \frac{\overline{AA'}}{v_1\,\overline{AP}} - \frac{\overline{B'B'}}{v_1\,\overline{PB}} = \frac{\sin\alpha_1}{v_1} - \frac{\sin\alpha_2}{v_2}\,. \end{aligned}$$

The value of x that minimizes the time $T(x)$ verifies $T'(x) = 0$, hence $\frac{\sin\alpha_1}{v_1} - \frac{\sin\alpha_2}{v_2} = 0$, and Snell's law follows.

In order to show the converse, i.e., that Snell's law implies least time, we will show that $T(x)$ is convex. In fact, its second derivative is:

$$T''(x) = \frac{y_1^2}{v_1\sqrt{(x-x_1)^2 + y_1^2)^3}} + \frac{y_1^2}{v_1\sqrt{(x-x_1)^2 + y_1^2)^3}} > 0\,,$$

and the result follows.

2.2. **Derivation of the equation for the brachistochrone.** Here we derive the equation for the curve of fastest descent using Fermat's principle of least time and Snell's law.

We assume that the particle is dropped with initial velocity $v_0 = 0$ at the initial point A with coordinates $(0, 0)$. The position P of the particle at time t will be (x, y), where x is the horizontal distance to the right of the origin of coordinates, and y is the vertical distance dropped from the origin of coordinates—so the y axis is pointing down rather that up.

The conservation of energy implies that the kinetic energy of the particle equals the decrease in potential energy due to gravity:

$$\frac{1}{2}mv^2 = mgy\,,$$

where m = mass of the particle, v = speed or the particle, g = acceleration of gravity (we note that the potential energy depends only on y). Hence, he have $v^2 = 2gy$.

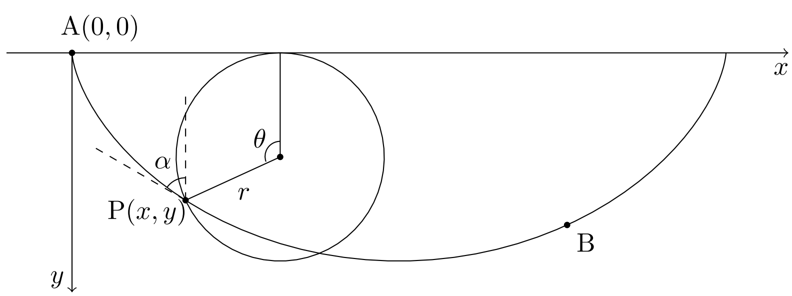

Figure 2. Cycloid

Next, Snell's law for a trajectory that verifies the principle of least time implies that the ratio of the particle's velocity and the sine of the angle α formed by the tangent to the trajectory with the vertical is constant, hence

$$\frac{v^2}{\sin^2(\alpha)} = \frac{2gy}{dx^2/ds^2} = 2gy\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 4gr\,,$$

where $ds^2 = dx^2 + dy^2$ is the line element, and r is a constant. From here we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{2r}{y} - 1\,. \tag{1}$$

This differential equation is satisfied by an inverted cycloid:

$$x = r(\theta - \sin\theta)$$
$$y = r(1 - \cos(\theta))$$

where θ and r are the angle rotated by the generating circle and r is its radius. In fact, differentiating the equation of the cycloid we have:

$$dx = r(1 - \cos\theta)\,d\theta\,,$$
$$dy = r\sin\theta\,d\theta\,,$$

hence

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{1-\cos^2\theta}{(1-\cos\theta)^2} = \frac{1+\cos\theta}{1-\cos\theta}\,,$$
$$\frac{2r}{y} - 1 = \frac{2}{1-\cos\theta} - 1 = \frac{1+\cos\theta}{1-\cos\theta}\,,$$

and (1) follows.

Finally, we can reintroduce time dependence by using $v^2 = 2gy$.

We have

$$v^2 = r^2(1-\cos\theta)^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2$$
$$= 2r^2(1-\cos(\theta))\left(\frac{d\theta}{dt}\right)^2$$

and

$$2gy = 2gr(1-\cos(\theta))\,,$$

hence

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{r}\,.$$

This implies $\theta = \omega t$, were $\omega = \sqrt{\frac{g}{r}}$, and $t =$ time elapse since the particle was dropped at $A(0,0)$.

Hence, the final parametric equation of the brachistochrone with time as parameter is

$$\boxed{\begin{aligned} x &= r(\omega t - \sin\omega t) \\ y &= r(1 - \cos(\omega t)) \end{aligned}}$$