CONSTRUCTION OF A NUMBER GREATER THAN ONE WHOSE POWERS ARE UNIFORMLY DISTRIBUTED MODULO ONE

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Abstract. We study how to construct a number greater than one whose powers are uniformly distributed modulo 1. Also we prove that for every $\lambda > 0$ there is a dense set of computable numbers $\alpha > 1$ such that the discrepancy of $\{\lambda\alpha^n\}_{n=1}^N$ is $O(N^{-\frac{1}{4+\varepsilon}})$.

1. Introduction.

It is well known that, as a consequence of Koksma's Theorem [1], for almost every number $\alpha > 1$ (in the sense of Lebesgue measure), the sequence $\{\alpha^n\}_{n=1}^\infty$ is uniformly distributed modulo 1 (u.d. mod 1), but only examples from the exceptional set, such as P.V.-numbers and Salem numbers, are known [2, pag. 71]. Another well known "metric" result is Weyl's theorem: given any $\alpha > 1$, the sequence $\{\lambda\alpha^n\}_{n=1}^\infty$ is u.d. mod. 1 for almost every real number λ [3]. However, in this case explicit examples of such λ are known. When α is an integer, the problem reduces to the construction of a normal number to the base α [4, chap. 1, secc. 8]. If α is not an integer, the construction of λ is somewhat more complicated, but still possible, as proven by Kulikova [5], by using an idea of Lebesgue permitting the effective use of metrical theorems [6].

It is interesting to note that some "purely existential" results can be transformed into constructive procedures suitable to produce mathematical objects with the required property. For instance, Gray [7] has used Cantor's result to design an algorithm which generates the digits of a transcendental number η in the interval $(0, 1)$. Basically, his algorithm generates (suitable decimal approximations of) all algebraic numbers by orderly generating all polynomials of integral coefficients, and approximating its roots up to some point. Then the digits of η are defined by a diagonal method. Similarly, Kulikova's result also takes advantage of a metrical result to give a procedure which generates the digits of a number with the required property [5]. A similar idea will be used here.

2. **Previous results.**

Given $\lambda > 0$, we are going to define a procedure to construct a number $\alpha_0 > 1$ such that the discrepancy of $\{\lambda\alpha_0^n\}_{n=1}^N$ approaches zero as $N \to \infty$.

The idea is to start with some closed interval $I = [a,b]$, with $1 < a < b$, and take open subsets $J_1, J_2, \ldots$ of points α for which $\{\lambda\alpha^n\}_{n=1}^N$ has "high" discrepancy. Those subsets will have the following features:

(i) Each J_k is a union of finitely many open intervals with computable endpoints.

(ii) $G = I \setminus \cup_{k=1}^\infty J_k \neq \emptyset$, and for every $\alpha \in G$ the discrepancy of $\{\lambda\alpha^n\}_{n=1}^N$ tends to zero as $N \to \infty$.

(iii) There is a computable sequence of nested closed intervals I_i of length approaching zero such that $I_i \setminus \cup_{k=1}^\infty J_k \neq \emptyset$ for every i.

The number α_0 will be determined as the intersection of the I_i's.

In order to control the discrepancy we are going to use the following bound [8]:

$$\Delta(x_1,\ldots,x_N) \;\le\; \frac{1}{H+1} \;+\; 2\sum_{h=1}^{H}\left\{\frac{1}{\pi h} + \frac{1}{H+1}\right\}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}\right| \tag{1}$$

where $\Delta(x_1,\ldots,x_N)$ is the discrepancy of $x_1,\ldots,x_N$. The following lemma gives bounds for the size of the subsets of I with "high" discrepancy.

Lemma. *Let λ be any fix positive real number. Let $I = [a,b]$ any closed interval of real numbers such that $1 < a < b$. For positive integers H and N, let $\phi_{H,N}(\alpha)$ be the function:*

$$\phi_{H,N}(\alpha) \;=\; \frac{1}{H+1} \;+\; 2\sum_{h=1}^{H}\left\{\frac{1}{\pi h} + \frac{1}{H+1}\right\}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h \lambda\alpha^n}\right|$$

Also define the set:

$$E(H,N,\tau) \;=\; \left\{\alpha \in I \,:\, \frac{1+\log H}{N^\tau} < \phi_{H,N}(\alpha),\right\}$$

and let N_0 be a fix integer such that $N_0 \geq \dfrac{a}{\lambda(a-1)^2(b-a)}$. Then:

(i) For $N \geq N_0$ the following inequality holds:

$$\mu(E(H,N,\tau)) \;<\; (b-a)\,\frac{N^\tau}{1+\log H}\left\{\frac{1}{H+1} \;+\; \frac{4}{\sqrt{N}}\left(\frac{1+\log H}{\pi} + \frac{H}{H+1}\right)\right\}$$

where μ represents the Lebesgue measure.

(ii) Let ν be any real number greater that 1. Let E_M be the set:

$$E_M \;=\; E([M^\nu],[M^{2\nu}],\frac{1}{2\nu})$$

where $[x] =$ *integer part of* x. *Then, for* $M \geq N_0^{\frac{1}{2\nu}}$:

$$\mu(E_M) \;<\; \frac{12(b-a)}{M^{\nu-1}}$$

(iii) *Given any pair of real numbers* $\varepsilon > \varepsilon' > 0$, *assume* $\nu = 2 + \varepsilon'/2$. *Let* $J(M_1, M_2)$ *be the (possibly empty) set*

$$J(M_1, M_2) \;=\; \bigcup_{M=M_1+1}^{M_2} E_M$$

where $M_2 \geq M_1 \geq N_0^{\frac{1}{2\nu}}$ *(M_2 may be ∞). Then*

$$\mu(J(M_1, M_2)) \;<\; \frac{24\,(b-a)}{\varepsilon'\, M_1^{\varepsilon'/2}}$$

For any integer $M_0 > \max\{(24/\varepsilon')^{2/\varepsilon'}, N_0^{\frac{1}{2\nu}}\}$, *the set*

$$G_{M_0} \;=\; I \setminus J(M_0, \infty)$$

has positive measure. If $\alpha \in G_{M_0}$ *then the discrepancy* Δ_N *of* $\{\lambda\alpha^n\}_{n=1}^N$ *verifies*

$$\Delta_N \;=\; O\Big(\frac{\log N}{N^{\frac{1}{4+\varepsilon'}}}\Big) \;=\; O(N^{-\frac{1}{4+\varepsilon}})$$

Proof.

(i) In the inequality defining $E(H, N, \tau)$ integrate the left hand side over $E(H, N, \tau)$ and the right hand side over the whole interval $[a, b]$:

$$\frac{1+\log H}{N^\tau}\,\mu(E(H,N,\tau))$$
$$<\; \frac{(b-a)}{H+1} \;+\; 2\sum_{h=1}^{H}\Big\{\frac{1}{\pi h} + \frac{1}{H+1}\Big\}\,\frac{1}{N}\int_a^b \Big|\sum_{n=1}^N e^{2\pi i h\lambda\alpha^n}\Big|\,d\alpha$$

By Jensen's inequality:

$$\left(\frac{1}{(b-a)}\int_a^b \Big|\sum_{n=1}^N e^{2\pi i h\lambda\alpha^n}\Big|\,d\alpha\right)^2 \;\leq\; \frac{1}{(b-a)}\int_a^b \Big|\sum_{n=1}^N e^{2\pi i h\lambda\alpha^n}\Big|^2\,d\alpha$$

The integral on the right hand side can be bounded in the following way:

$$\int_a^b \Big|\sum_{n=1}^N e^{2\pi i h\lambda\alpha^n}\Big|^2\,d\alpha \;=\; \int_a^b \Big(\sum_{1\leq n,m\leq N} e^{2\pi i h\lambda(\alpha^m-\alpha^n)}\Big)\,d\alpha$$
$$=\; N\,(b-a) \;+\; 2\sum_{1\leq n<m\leq N}\Big|\int_a^b e^{2\pi i h\lambda(\alpha^m-\alpha^n)}\,d\alpha\Big|$$

We have [4, lemma 1.2.1]:

$$\left|\int_a^b e^{2\pi i h\lambda(\alpha^m-\alpha^n)}\,d\alpha\right| \;<\; \frac{1}{h\lambda(ma^{m-1}-na^{n-1})}$$

From here we get:

$$\begin{aligned}\int_a^b \left|\sum_{n=1}^{N} e^{2\pi i h\lambda\alpha^n}\right|^2 d\alpha \;&<\; N\,(b-a) \;+\; 2\sum_{1\le n<m\le N} \frac{1}{h\lambda(ma^{m-1}-na^{n-1})}\\ &\le\; N\,(b-a) \;+\; \frac{2a}{h\lambda(a-1)^2}\end{aligned}$$

Going back we get:

$$\begin{aligned}\frac{1}{(b-a)}\int_a^b \left|\sum_{n=1}^{N} e^{2\pi i h\lambda\alpha^n}\right| d\alpha \;&<\; \left(N \;+\; \frac{2a}{h\lambda(a-1)^2(b-a)}\right)^{1/2}\\ &\le\; \sqrt{N}\left(1 \;+\; \frac{a}{h\lambda(a-1)^2(b-a)N}\right)\end{aligned}$$

For $N \ge \dfrac{a}{\lambda(a-1)^2(b-a)}$ we get:

$$\frac{1}{(b-a)}\int_a^b \left|\sum_{n=1}^{N} e^{2\pi i h\lambda\alpha^n}\right| d\alpha \;<\; 2\sqrt{N}$$

Hence:

$$\begin{aligned}\frac{1+\log H}{N^\tau}\,\mu(E(H,N,\tau)) \;&<\; \frac{(b-a)}{H+1} \;+\; 2\sum_{h=1}^{H}\left\{\frac{1}{\pi h}+\frac{1}{H+1}\right\}\frac{1}{N}\,2\sqrt{N}\,(b-a)\\ &<\; (b-a)\left\{\frac{1}{H+1} \;+\; \frac{4}{\sqrt{N}}\left(\frac{1+\log H}{\pi}+\frac{H}{H+1}\right)\right\}\end{aligned}$$

From here the announced result follows.

(ii) From (i) we get:

$$\begin{aligned}\mu(E_M) \;&<\; (b-a)\,M\left\{\frac{1}{M^\nu}+\frac{4}{\sqrt{[M^{2\nu}]}}\left(\frac{1}{\pi}+1\right)\right\}\\ &\le\; (b-a)\,\frac{1}{M^{\nu-1}}\left\{1+\frac{4M^\nu}{[M^\nu]}\left(\frac{1}{\pi}+1\right)\right\}\\ &\le (b-a)\,\frac{1}{M^{\nu-1}}\left\{1+8\left(\frac{1}{\pi}+1\right)\right\}\\ &<\; \frac{12(b-a)}{M^{\nu-1}}\end{aligned}$$

(iii) We have:

$$\begin{aligned}\mu(J(M_1,M_2)) &\le \sum_{M=M_1+1}^{M_2} \mu(E_M)\\ &< 12\,(b-a) \sum_{M=M_1+1}^{\infty} \frac{1}{M^{\nu-1}}\\ &< 12\,(b-a) \int_{M_1}^{\infty} \frac{dx}{x^{\nu-1}}\\ &= \frac{24\,(b-a)}{\varepsilon'\, M_1^{\varepsilon'/2}}\end{aligned}$$

Using the condition on M_0, we get that $J(M_1, M_2)$ has measure less than $(b-a)$, hence G_{M_0}, which is its complement respect to $I = [a, b]$, has positive measure.

Concerning the order of Δ_N, we start noting that $\alpha \in G_{M_0}$ implies

$$\Delta_N \le \phi_{H,N}(\alpha) \le \frac{1+\log H}{N^{1/2\nu}} = O\Big(\frac{\log N}{N^{\frac{1}{4+\varepsilon'}}}\Big)$$

for $H = [M^\nu]$ and $N = [M^{2\nu}]$. Hence, it remains only to prove the result for other values of N.

Assume that $[M^{2\nu}] < N < [(M+1)^{2\nu}]$, and $H = [M^\nu]$. For simplicity, put $N' = [M^{2\nu}]$ and $N'' = [(M+1)^{2\nu}]$. Then:

$$\begin{aligned}\Delta_N &\le \phi_{H,N}(\alpha)\\ &\le \phi_{H,N'}(\alpha) + \frac{1}{H+1} + 2\sum_{h=1}^{H}\Big\{\frac{1}{\pi h} + \frac{1}{H+1}\Big\}\Big|\frac{1}{N}\sum_{n=N'+1}^{N} e^{2\pi i h\lambda\alpha^n}\Big|\\ &< \phi_{H,N'}(\alpha) + \frac{1}{H} + 2\Big(\frac{1+\log H}{\pi} + 1\Big)\frac{N''-N'}{N}\\ &= O\Big(\frac{\log N}{N^{\frac{1}{4+\varepsilon'}}}\Big) + O\Big(\frac{1}{N^{\frac{1}{4+\varepsilon'}}}\Big) + O(\log N)\,\frac{N''-N'}{N}\end{aligned}$$

Finally we have:

$$\frac{N''-N'}{N} < \frac{[(M+1)^{2\nu}] - [M^{2\nu}]}{[M^{2\nu}]} = O(M^{-1}) = O(N^{-\frac{1}{4+\varepsilon'}})$$

and from here the announced result follows.

□

3. **The procedure.**

Given any closed interval of real numbers $I = [a, b]$ with $1 < a < b$, any $\lambda > 0$, and any $\varepsilon > 0$, we are going to get a number α_0 such that $\alpha_0 \in I$, and the discrepancy Δ_N of $\{\lambda\alpha_0^n\}_{n=1}^N$ is $O(N^{-\frac{1}{4+\varepsilon}})$, hence $\{\lambda\alpha_0^n\}_{n=1}^\infty$ is u.d. mod 1.

Take $\nu = 2 + \varepsilon'$, where $\varepsilon > \varepsilon' > 0$. Let E_M subsets of $I = [a,b]$ as in the lemma. We define a sequence of open sets $\{J_k\}_{k=1}^{\infty}$, where J_k is of the form $J_k = J(M_{k-1}, M_k)$, and a sequence of nested closed intervals $\{I_k\}_{k=0}^{\infty}$, where $I_k = [a_k, b_k]$, in a recursive way.

In step 0, we take any $M_0 > \max\{(24/\varepsilon')^{2/\varepsilon'}, N_0^{\frac{1}{2\nu}}\}$, as in part (iii) of the lemma, and $I_0 = [a_0, b_0]$ equal to $I = [a,b]$. Hence we have $\mu(I_0 \setminus J(M_0,\infty)) > d_0 = (b_0 - a_0) - 24(b-a)/(\varepsilon' M_0^{\varepsilon'/2}) > 0$

In step $k \geq 1$, assume that $I_{k-1} = [a_{k-1}, b_{k-1}]$ and J_i for $i = 1, \dots, k-1$ have already been found, and that $\mu(I_{k-1} \setminus J(M_0,\infty)) > d_{k-1} > 0$. Next, we take $M_k \geq M_{k-1}$ such that $d_{k-1}/2 > 24(b-a)/(\varepsilon' M_k^{\varepsilon'/2})$. Now we form the set $J_k = J(M_{k-1}, M_k)$, add it to the previously found to form the set $J(M_0, M_k) = \cup_{i=1}^{k} J_i$, and determine the measure of $I_k \setminus J(M_0, M_k)$ for I_k equal to each one of $I_k' = [a_{k-1}, \frac{a_{k-1}+b_{k-1}}{2}]$ and $I_k'' = [\frac{a_{k-1}+b_{k-1}}{2}, b_{k-1}]$. Since

$$\begin{aligned} \mu(I_k' \setminus J(M_0,M_k)) \;+\; \mu(I_k'' \setminus J(M_0,M_k)) \;&=\; \mu(I_{k-1} \setminus J(M_0,M_k)) \\ &>\; \mu(I_{k-1} \setminus J(M_0,\infty)) \\ &>\; d_{k-1} \end{aligned}$$

at least one of $I_k' \setminus J(M_0,M_k)$ or $I_k'' \setminus J(M_0,M_k)$ should have measure greater than $d_{k-1}/2$. So, we take I_k with the condition $\mu(I_k \setminus J(M_0,M_k)) > d_{k-1}/2$. If we now subtract $24(b-a)/(\varepsilon' M_k^{\varepsilon'/2})$, which by part (iii) of the lemma is an upper bound for $\mu(J(M_k,\infty))$, we get a positive lower bound $d_k = d_{k-1}/2 - 24(b-a)/(\varepsilon' M_k^{\varepsilon'/2}) > 0$ for $\mu(I_k \setminus J(M_0,\infty))$. At this point we have found I_k and J_k, and we have that $\mu(I_k \setminus J(M_0,\infty)) > d_k > 0$, so everything is ready to proceed with step $k+1$.

Let α_0 be the unique point in $\cup_{k=0}^{\infty} I_k$. If G_{M_0} is the closed set $I \setminus J(M_0,\infty)$, we have $I_k \cap G_{M_0} = I_k \setminus J(M_0,\infty) \neq \emptyset$ for every k, hence $\alpha_0 \in G_{M_0}$, and by part (iii) of the lemma, α_0 verifies the requirements.

4. **Computational considerations and summary.**

The claim that α_0 is a computable number rests on the the fact that the sets E_M are finite unions of open intervals with computable endpoints, and so are the sets J_k. To be more precise, the endpoints are solutions in α of equations of the form:

$$\frac{1}{H+1} \;+\; 2\sum_{h=1}^{H} \left\{ \frac{1}{\pi h} + \frac{1}{H+1} \right\} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \lambda \alpha^n} \right| - \frac{1+\log H}{N^\tau} = 0 \tag{2}$$

In practice it is impossible to compute exactly such solutions, so it is necessary to deal with approximations. Than can be done in such way that the approximations verify

$$\frac{1+\log H}{N^\tau} \;\leq\; \phi_{H,N}(\alpha) \leq\; \frac{1+\log H}{N^{\tau'}}$$

for some fix $\tau' \leq \tau$ close to τ, say $\tau' = \tau - \varepsilon_0$. The sets E_M' computed this way will be slightly smaller than the sets E_M used above, so the process can still be carried out successfully and an $\alpha_0 \in I$ be found. However the speed of convergence of the discrepancy Δ_N of $\{\lambda\alpha_0^n\}_{n=1}^N$ will be relaxed to:

$$\Delta_N = O\Big(\frac{\log N}{N^{\frac{1}{4+\varepsilon'}-\varepsilon_0}}\Big)$$

But the result is still of the form

$$\Delta_N = O(N^{-\frac{1}{4+\varepsilon}})$$

if $0 < \varepsilon_0 < \frac{1}{4+\varepsilon'} - \frac{1}{4+\varepsilon}$.

The procedure described here could be unpractical because it would require a very high computational load. However it does provide a result of at least theoretical interest: there is a dense set of computable numbers $\alpha > 1$ such that $\{\lambda\alpha^n\}_{n=1}^\infty$ is u.d. mod. 1 and the discrepancy of $\{\lambda\alpha^n\}_{n=1}^N$ is $O(N^{-\frac{1}{4+\varepsilon}})$.

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