SYMPLECTIC REFLECTION ALGEBRAS AND DEFORMATION QUANTIZATION

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1. Introduction

The goal of these lectures is to review known constructions of quantizations of the varieties of the form V/Γ. Here V is a vector space (everything is defined over the field $\mathbb{C}$ of complex numbers) equipped with a symplectic form ω, and Γ is a finite subgroup of the symplectic group $\mathrm{Sp}(V)$. We understand a *quantization* in a very algebraic way. Namely, we can consider the symmetric algebra SV of V and its subalgebra $A := (SV)^\Gamma$ of all Γ-invariant elements. The latter can be thought as the algebra of polynomial functions on the quotient V/Γ. This algebra has a natural grading and a natural Poisson bracket $\{\cdot,\cdot\}$, both are inherited from SV. The grading and the bracket are compatible in the following way: if $f, g \in A$ are homogeneous elements of degrees i, j, respectively, then $\{f, g\}$ is homogeneous of degree $i+j-2$. By a *quantization* of A we mean a unital associative algebra $\mathcal{A}$ equipped with an ascending exhaustive algebra filtration $0 = \mathrm{F}_{-1}\mathcal{A} \subset \mathbb{C} = \mathrm{F}_0\mathcal{A} \subset \mathrm{F}_1\mathcal{A} \subset \mathrm{F}_2\mathcal{A} \subset \ldots \subset \mathcal{A}$ such that $[\mathrm{F}_i\mathcal{A}, \mathrm{F}_j\mathcal{A}] \subset \mathrm{F}_{i+j-2}\mathcal{A}$ and the associated graded algebra $\mathrm{gr}\,\mathcal{A} := \sum_{i=0}^{+\infty} \mathrm{F}_i\mathcal{A}/\mathrm{F}_{i-1}\mathcal{A}$ is isomorphic to A as a graded Poisson algebra. Recall that the bracket on $\mathrm{gr}\,\mathcal{A}$ is defined via $\{a + \mathrm{F}_{i-1}\mathcal{A}, b + \mathrm{F}_{j-1}\mathcal{A}\} := [a, b] + \mathrm{F}_{i+j-3}\mathcal{A}$.

There are several reasons to be interested in quantizations of V/Γ. First, in some special cases they serve as algebras of observables for some interesting integrable systems (quantum Calogero-Moser spaces). Second (this is more important for the author), these quantizations happen to have a very interesting representation theory.

There is a general construction that produces (at least some) quantizations of V/Γ. This construction was introduced in the full generality by Etingof and Ginzburg in [EG]. The Etingof-Ginzburg quantizations are obtained from so called *Symplectic reflection algebras* (SRA, for short). These are deformations of an "orbifold resolution of singularities" of V/Γ that is a certain non-commutative algebra (a skew group ring). This construction will be considered in Section 2.

For some special groups Γ the variety V/Γ as well as its quantizations can be obtained by an alternative construction. This class of groups is called *wreath-products*, it is obtained as follows. Take a finite subgroup $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$. Then the semidirect product $\Gamma_n := S_n \ltimes \Gamma_1^n$ acts by linear symplectomorphisms on $V := \mathbb{C}^{2n}$. It turns out that V/Γ_n can be realized as a Nakajima quiver variety, i.e., a Hamiltonian reduction of the representation space of an appropriate quiver under an action of a certain reductive group. Quantizations can be constructed using the procedure called a *quantum Hamiltonian reduction.* We will explain how to do this in Section 3.

It turns out that the quantizations produced from the SRA and by quantum Hamiltonian reduction are the same. This was first proved in [EGGO] and then in [L] by an alternative method that we are going to explain. An ideological reason for the coincidence is that both quantizations can be lifted to resolutions of singularities. An SRA quantization has this property by its construction: it is obtained from a quantization of an orbifold resolution.

A quantization obtained by the quantum Hamiltonian reduction can be lifted to a usual algebro-geometric resolution of singularities that happens to be a non-affine quiver variety. The two resolutions can be related via a tilting vector bundle (a weakly Procesi bundle obtained in [BK]) on the algebro-geometric resolution. Quantizing this bundle, we can prove that the two constructions of quantizations give the same answer. This will also be explained in Section 3. Finally, we will explain an application of our construction to producing some derived equivalences.

2. Symplectic reflection algebras

2.1. A skew-group ring. Recall that V denotes a finite dimensional vector space over $\mathbb{C}$ equipped with a symplectic form ω, and Γ is a finite subgroup of $\mathrm{Sp}(V)$. We want to study quantizations of the graded Poisson algebra $A := (SV)^\Gamma$. A somewhat informal reason why this task is non-trivial is that V/Γ is not smooth. Fortunately, one can replace A with a closely related algebra that will no longer be commutative but will be smooth (in the sense that its global dimension is finite). This algebra is obtained by a skew group ring construction.

Namely, consider a vector space $SV \otimes \mathbb{C}\Gamma$, where $\mathbb{C}\Gamma$ is a group algebra of Γ. Equip this space with a multiplication by setting $(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1 g_1(f_2) \otimes g_1 g_2, f_1, f_2 \in SV, g_1, g_2 \in \Gamma$, where $g_1(f_2)$ stands for the image of f_2 under the action of g_1. Below we will write $SV\#\Gamma$ for this algebra.

The algebra $SV\#\Gamma$ has finite global dimension because the category of finitely generated $SV\#\Gamma$-modules is the same as the category of Γ-equivariant coherent sheaves (and so the category of $SV\#\Gamma$-modules is what we expect from the category of coherent sheaves of the quotient by a free action). The algebra $SV\#\Gamma$ is still graded (with $\mathbb{C}\Gamma$ being the component of degree 0) but is no longer commutative.

In the next subsection we will classify graded deformations of $SV\#\Gamma$ and then, in Subsection 2.4, relate them to quantizations of A.

2.2. Symplectic reflection algebras. Let $\mathfrak{p}$ be a finite dimensional vector space, a "space of parameters". We are going to study graded algebras H over $S\mathfrak{p}$ such that $\mathfrak{p}$ has degree 2, H is a free graded module over $S\mathfrak{p}$, and $H/(\mathfrak{p}) = SV\#\Gamma$ (the choice of degree has to do with the degree of the Poisson bracket on A). It turns out that there is a universal such deformation, H_{un} over $S(\mathfrak{p}_{un})$ that can be given by explicit generators and relations. When we say "universal" we mean that for any other H over $S\mathfrak{p}$ there is a unique linear map $\mathfrak{p}_{un} \to \mathfrak{p}$ such that $H = S\mathfrak{p} \otimes_{S(\mathfrak{p}_{un})} H_{un}$. The algebra H_{un} will be called the *universal* SRA.

Let us first describe the space $\mathfrak{p}_{un}$. We say that an element $s \in \Gamma$ is a *symplectic reflection* if $\mathrm{rk}(s - \mathrm{id}) = 2$. We remark that the restriction of ω to $V_s := \mathrm{im}(s - \mathrm{id})$ is non-degenerate for any $s \in H$ and so 2 is the minimal possible positive dimension of V_s. Let $S_1, \ldots, S_r$ be all classes of symplectic reflections in H. Then let $\mathfrak{p}_{un}$ be an $r+1$-dimensional vector space.

To write the relations in (our candidate for) H_{un} we need some more notation. Namely, let us notice that $V = V_s \oplus V^s$, where V^s is the fixed point subspace of s. For $s \in \Gamma$ let ω_s be the skew-symmetric form equal to the sum of $\omega|_{V_s}$ and the zero form on V^s.

Theorem 2.1. *Suppose that the group* Γ *is* symplectically irreducible*, i.e., there is no* Γ*-stable proper symplectic subspace in* V*. Consider the algebra* H_{un} *defined as the quotient of*

$S(\mathfrak{p}_{un})\otimes TV\#\Gamma$ by the relations

$$[u,v]=h\omega(u,v)+\sum_{i=1}^{r}c_i\sum_{s\in S_i}\omega_s(u,v)s, u,v\in V, \tag{1}$$

where $h,c_1,\ldots,c_r$ is a basis in $\mathfrak{p}_{un}$. Then H_{un} is a free graded $S(\mathfrak{p}_{un})$-module and satisfies the universality condition above.

For $\dim V=2$ (where every nontrivial element of Γ is a symplectic reflection) the construction of H_{un} (together with the proof of the freeness) appeared in [CBH]. The construction in the general case is due to Etingof and Ginzburg, [EG]. Their proof that the algebra given by relations (1) is flat over $S(\mathfrak{p}_{un})$ was based on the so called *Koszul deformation principle*. Strangely, it seems that the universality property was not emphasized in the literature before [L] although it was definitely known to the experts.

Proof. An important ingredient in the proof of the theorem (and actually the first step in studying deformations of an associative algebra) is the computation of the Hochschild cohomology. Recall that deformations of an arbitrary associative algebra B are parameterized by the second cohomology $\mathrm{HH}^2(B)$ (of the B-bimodule B) with obstructions lying in $\mathrm{HH}^3(B)$. Our algebra B is graded and we are only interested in graded deformations, where the parameter space has degree 2. Such deformations are parameterized by $\mathrm{HH}^2_{-2}(B)$ and the obstructions lie in $\mathrm{HH}^3_i(B)$ for $i\leqslant -4$. Here the subscript indicates the degree (recall, that the Hochschild cohomolgy of a graded algebra is naturally graded). The following lemma implies that all graded deformations of $SV\#H$ are unobstructed implying the existence of a universal deformation H_{un} over the parameter space $\mathfrak{p}_{un}$.

Lemma 2.2. *Under the assumptions of Theorem 2.1, we have $\mathrm{HH}^2_{-2}(SV\#\Gamma)\cong\mathfrak{p}_{un}$ and $\mathrm{HH}^3_i(SV\#\Gamma)=0$ for $i\leqslant -4$.*

Proof. Using the definition of $\mathrm{HH}(B)$ as $\mathrm{Ext}(B,B)$ in the category of B-bimodules, we easily see that

$$\mathrm{HH}(SV\#\Gamma)=\left(\bigoplus_{g\in\Gamma}\mathrm{HH}(SV,SV\cdot g)\right)^{\Gamma}, \tag{2}$$

where $SV\cdot g$ stands for the SV bimodule, where the B-action on the left is as on SV, while the action on the right is twisted by g. The group $\mathrm{HH}(SV,SV\cdot g)$ can be computed as follows. The element g is diagonalizable in some basis $x_1,\ldots,x_n$ of V, say $g=\mathrm{diag}(g_1,\ldots,g_n)$. Then we have the following isomorphism that respects both the cohomological grading and that induced by the grading on SV:

$$\mathrm{HH}(SV,SV\cdot g)=\bigotimes_{i=1}^{n}\mathrm{HH}(\mathbb{C}[x_i],\mathbb{C}[x_i]\cdot g_i), \tag{3}$$

where we view g_i as an element of a cyclic group acting on $\mathbb{C}$. Take the Koszul resolution of the $\mathbb{C}[x_i]$-bimodule $\mathbb{C}[x_i]$: $\mathbb{C}[x_i]\otimes\mathbb{C}[x_i]\to\mathbb{C}[x_i]\otimes\mathbb{C}[x_i]$, where the map given by $a\otimes b\to ax\otimes b-a\otimes xb$. So $\mathrm{HH}(\mathbb{C}[x_i],\mathbb{C}[x_i]\cdot g_i)$ is the cohomology of the complex $\mathbb{C}[x_i]\cdot g_i\to\mathbb{C}[x_i]\cdot g_i$, where the map is a left $\mathbb{C}[x_i]$-module homomorphism given by $1\cdot g\mapsto(x-g(x))\cdot g$. So when $g=1$, then $\mathrm{HH}(\mathbb{C}[x_i],\mathbb{C}[x_i])$ is naturally identified with the algebra of polyvector fields on $\mathbb{C}[x_i]$. For $g\neq 1$, the complex $\mathrm{HH}(\mathbb{C}[x_i],\mathbb{C}[x_i])$ is concentrated in cohomolgical degree 1 and in the usual degree -1. This computation together with (2),(3) easily imply the required result. □

It remains to check that the deformation H_{un} is given by (1). This is done in several steps.

Step 1. Let π denote the natural projection $H_{un} \twoheadrightarrow SV\#\Gamma$. Clearly, π identifies the degree 0 component of H_{un} with $\mathbb{C}\Gamma$ and the degree 1 component with $V\otimes\mathbb{C}\Gamma$. In particular, there is a natural inclusion of V into H_{un}. For $u,v\in V\subset H_{un}$, the element $[u,v]$ has degree 2 and lies in $\ker\pi$. But the degree 2 component of $\ker\pi$ is $\mathfrak{p}\otimes\mathbb{C}\Gamma$. So there is a map $\kappa:\bigwedge^2 V\to\mathfrak{p}\otimes\mathbb{C}\Gamma$ such that $[u,v]=\kappa(u,v)$. We remark that κ is Γ-equivariant. This is because of the inclusion $\Gamma\subset H_{un}$.

Step 2. Let I denote the ideal in $S(\mathfrak{p}_{un})\otimes TV\#\Gamma$ generated by the elements $u\otimes v-v\otimes u-\kappa(u,v)$ and $\tilde{H}_{un}:=S(\mathfrak{p}_{un})\otimes TV\#H/I$. Then, thanks to step 1, we have an epimorphism $\tilde{H}_{un}\to H_{un}$. We claim that this is an isomorphism. Indeed, the natural grading on TV induces a filtration on $\tilde{H}_{un}$ (with $\mathfrak{p}$ in degree 0). The algebra $\operatorname{gr}\tilde{H}_{un}$ is bigraded and we have epimorphisms $S(\mathfrak{p}_{un})\otimes SV\#\Gamma\twoheadrightarrow\operatorname{gr}\tilde{H}_{un}\twoheadrightarrow\operatorname{gr}H_{un}$. But the Poincare series of $S(\mathfrak{p}_{un})\otimes SV\#H$ and H_{un} with respect to the initial grading coincide. Hence $\operatorname{gr}\tilde{H}_{un}=\operatorname{gr}H_{un}$ and therefore $\tilde{H}_{un}=H_{un}$.

Step 3. We claim that there are $h,c_1,\ldots,c_r\in\mathfrak{p}_{un}$ such that $\kappa(u,v)$ coincides with the right hand side of (1). We will use an original argument from [EG] for this. Namely, for every $u,v,w\in V$ we have the Jacobi identity

$$[[u,v],w]+[[v,w],u]+[[w,u],v]=0 \tag{4}$$

in H_{un}. Let $\kappa=\sum_{g\in\Gamma}\kappa_g g$ with $\kappa_g(u,v)\in\bigwedge^2 V^*\otimes\mathfrak{p}$. Then $[[u,v],w]=\sum_{g\in\Gamma}\kappa_g(u,v)[g,w]=\sum_{g\in\Gamma}\kappa_g(u,v)(g(w)-w)g$. Since H_{un} is free over $S(\mathfrak{p}_{un})$, we see that $\kappa_g(u,v)(g(w)-w)+\kappa_g(v,w)(g(u)-u)+\kappa_g(w,u)(g(w)-w)=0$. From here it is easy to deduce that $\kappa_g=0$ when $\operatorname{rk}(g-\mathrm{id})\geqslant 4$. Our claim easily follows from the Γ-equivariance of κ.

Step 4. The elements $h,c_1,\ldots,c_r$ from Step 3 form a basis in $\mathfrak{p}_{un}$ because H_{un} is a universal deformation. □

For $p\in\mathfrak{p}_{un}^*$ let H_p denote the specialization of H_{un} at p, i.e., the quotient of H_{un} by the maximal ideal of p in $S(\mathfrak{p}_{un})$. This is a filtered algebra whose associated graded is $SV\#H$.

Let us finish the subsection by explaining the name "symplectic reflection algebra". Let Γ^0 denote the subgroup of Γ generated by the symplectic reflections and let H^0_{un} be the corresponding SRA. Then clearly $H_{un}=H^0_{un}\#_{\Gamma^0}\Gamma$, where as a vector space the right hand side is $H^0_{un}\otimes_{\mathbb{C}\Gamma^0}\mathbb{C}\Gamma$ and the product is introduced by analogy with $SV\#\Gamma$. So, basically, only a symplectic reflection group (=a group generated by symplectic reflections) matters.

2.3. **A class of symplectic reflection groups.** We are going to introduce a certain class of symplectic reflection groups (usually called *wreath-products*) that will be used in the sequel. Namely, we take a finite subgroup $\Gamma_1\subset\mathrm{SL}_2(\mathbb{C})$ and consider the semidirect product $\Gamma_n:=\mathfrak{S}_n\leftthreetimes\Gamma^n$. This semidirect product acts naturally on $V:=(\mathbb{C}^2)^{\oplus n}$: an element γ_i in the ith copy of $\Gamma_1\subset\Gamma_1^n\subset\Gamma_n$ corresponding to γ acts as γ on the ith summand $\mathbb{C}^2$ and trivially on the other summands, while $\mathfrak{S}_n\subset\Gamma_n$ permutes the summands $\mathbb{C}^2$.

The symplectic reflections in Γ_n are as follows. First, each γ_i with $\gamma\neq 1$ is a symplectic reflection. Also each transposition $s_{ij}\in\mathfrak{S}_n\subset\Gamma_n$ is a symplectic reflection as well. These elements generate Γ_n hence Γ_n is a symplectic reflection group.

The conjugacy classes of symplectic reflections are as follows. First, for $n>1$, there is a single class, say S_1, containing s_{ij}, it consists of the elements of the form $s_{ij}\gamma_i\gamma_j^{-1}$. Second, let $\Gamma_1\setminus\{1\}=\bigsqcup_{i=2}^r S_i^0$ be the decomposition into conjugacy classes. Then $\{\gamma_j|j=1,\ldots,n,\gamma\in S_i^0\}$ form a conjugacy class, say S_i.

Let us finish this subsection by recalling a classification of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. These subgroups are in one-to-one correspondence with affine Dynkin quivers (McKay correspondence). Namely, let $N_1, \ldots, N_r$ be the irreducible Γ_1-modules, where N_1 is the trivial module. The vertices in our quiver are $1, \ldots, r$ and the number of arrows between i and j (regardless the orientation) is $\dim \mathrm{Hom}(\mathbb{C}^2 \otimes N_i, N_j)$, where $\mathbb{C}^2$ is the tautological Γ_1-module coming from the inclusion $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$. Then it is known that this quiver is an affine Dynkin quiver, and 1 is the extending vertex. Moreover, the vector $(\dim N_i)_{i=1}^r$ is the minimal imaginary root δ. The type $\tilde{A}_r$ quiver corresponds to the cyclic group of order $r+1$, while the type $\tilde{D}_r, r \geqslant 4$, quiver represents the dihedral group of order $4(r-2)$.

We remark that the universal parameter space $\mathfrak{p}_{un}$ in this case has the dimension one bigger than the number of vertices of Q.

2.4. **Spherical subalgebras.** Now let us return to our original problem: to construct quantizations of $A = (SV)^\Gamma$. First of all, let us recover A from $SV\#\Gamma$. Consider the trivial idempotent $e := \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g$. We can view e as an element of $SV\#\Gamma$. The map $a \mapsto ea$ defines an embedding of A into $SV\#\Gamma$. Its image is identified with the *spherical subalgebra* $e(SV\#\Gamma)e$ that is an associative subalgebra of $SV\#\Gamma$ with unit e. Now, we can take the spherical subalgebras $eH_{un}e \subset H_{un}$ and $eH_pe \subset H_p$ for $p \in \mathfrak{p}_{un}^*$. Then eH_pe is a filtered algebra with $\mathrm{gr}\, eH_pe = e\,\mathrm{gr}\, H_pe = (SV)^\Gamma$. It was shown by Etingof and Ginzburg, the end of Section 2 in [EG], that the induced Poisson bracket on $(SV)^\Gamma$ is $p(x)\{\cdot,\cdot\}$ for some $x \in \mathfrak{p}$ proportional to h. In fact, $x = h$. Indeed, if we choose p with $p(h) = 1, p(c_i) = 0$, then $H_p = \mathbb{A}\#\Gamma$, where $\mathbb{A}$ is the Weyl algebra of V. Then $eH_pe = \mathbb{A}^\Gamma$ is a quantization of $(SV)^\Gamma$ in the sense explained above.

3. Quantum Hamiltonian reduction

In this section we are going to relate certain quotient singularities to Nakajima quiver varieties.

In Subsection 3.1 we are going to explain how to relate V/Γ_n to a quiver variety starting with some special cases. Then in Subsection 3.2 we will explain how to use this relationship to produce quantizations of V/Γ_n using quantum Hamiltonian reduction.

3.1. **Quotient singularities vs quivers.** First, we will explain a general construction and then try to motivate it by examples. Take the group $\Gamma = \Gamma_n (= \mathfrak{S}_n \leftthreetimes \Gamma_1^n)$ as in Subsection 2.3. Let Q denote the quiver corresponding to Γ with some orientation and $\delta = (\delta_1, \ldots, \delta_r)$ be the indecomposable imaginary root.

Let us recall a general definition of a Nakajima quiver variety. Take a quiver Q with set of vertices Q_0 and set of arrows Q_1. For an arrow $a \in Q_1$ let $t(a), h(a) \in Q_0$ denote its tail and its head. Fix two vectors $u = (u_i)_{i\in Q_0}, w = (w_i)_{i\in Q_0}$.

We consider the double quiver DQ obtained from Q by adding an opposite arrow for any arrow in Q. Pick vector spaces U_i, W_i of dimensions $u_i, w_i, i \in Q_0$. Consider the space $R := R(DQ, n\delta, \epsilon_1)$ of representations of DQ with dimension vector u and framing w. By definition, this space equals

$$\bigoplus_{a\in Q_1} \left(\mathrm{Hom}(U_{t(a)}, U_{h(a)}) \oplus \mathrm{Hom}(U_{h(a)}, U_{t(a)})\right) \oplus \bigoplus_{i\in Q_0} \left(\mathrm{Hom}(W_i, U_i) \oplus \mathrm{Hom}(U_i, W_i)\right).$$

We will write an element of this space as $(A_a, B_a, C_i, D_i)_{a\in Q_1, i\in Q_0}$, where $A_a \in \mathrm{Hom}(U_{t(a)}, U_{h(a)}), B_a \in \mathrm{Hom}(U_{h(a)}, U_{t(a)}), C_i \in \mathrm{Hom}(W_i, U_i), D_i \in \mathrm{Hom}(U_i, W_i)$.

The vector space R is identified with $T^*R(Q,u,w) = R(Q,u,w) \oplus R(Q,u,w)^*$, where $R(Q,u,w) = \bigoplus_{a\in Q_1} \mathrm{Hom}(U_{t(a)}, U_{h(a)}) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(U_i, W_i)$ and hence carries a natural symplectic form. Explicitly (we will not need this formula in the sequel), we set

$$\Omega((A'_a, B'_a, C'_i, D'_i), (A''_a, B''_a, C''_i, D''_i)) = \sum_{a\in Q_1} \mathrm{tr}(A'_a B''_a - A''_a B'_a) + \sum_{i\in Q_0} (C'_i D''_i - C''_i D'_i).$$

Here we identify $\mathrm{Hom}(U_{h(a)}, U_{t(a)})$ with $\mathrm{Hom}(U_{t(a)}, U_{h(a)})^*$ and $\mathrm{Hom}(U_i, W_i)$ with $\mathrm{Hom}(W_i, U_i)^*$ by means of the trace pairing.

Also the group $G := \prod_{i\in Q_0} \mathrm{GL}(u_i)$ acts naturally on R: for $g = (g_i) \in G$ we have $g.(A_a, B_a, C_i, D_i) = (g_{h(a)} A_a g_{t(a)}^{-1}, g_{t(a)} B_a g_{h(a)}^{-1}, gC_i, D_i g^{-1})$. This action preserves the form Ω and therefore admits a *moment map*, a G-equivariant morphism $R \to \mathfrak{g}^*$ with the following property: $\{\mu^*(\xi), f\} = \xi_* f$ for all $\xi \in \mathfrak{g}, f \in \mathbb{C}[R]$. Here $\mu^* : \mathfrak{g} \to \mathbb{C}[R]$ is the dual map and ξ_* in the right hand side stands for the operator produced by the representation of $\mathfrak{g}$ in $\mathbb{C}[R]$ coming from the G-action. There are several choices of a moment map but we need a homogeneous one obtained as follows. Recall that one can realize the algebra $\mathfrak{sp}(R)$ as the space $\mathbb{C}[R]_2$ of homogeneous quadratic elements in $\mathbb{C}[R]$. The Lie bracket on $\mathfrak{sp}(R)$ corresponds to the Poisson bracket on $\mathbb{C}[R]_2$. So for μ^* we take the composition $\mathfrak{g} \to \mathfrak{sp}(R) \xrightarrow{\sim} \mathbb{C}[R]_2$. Again, one can write an explicit formula:

$$\mu(A_a, B_a, C_i, D_i) = (\sum_{a, t(a)=i} B_a A_a - \sum_{a, h(a)=i} A_a B_a + C_i D_i)_{i\in Q_0}, \tag{5}$$

where we again identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the trace pairing.

By definition, the affine Nakajima quiver variety $\mathcal{M}_0(DQ, v, w)$ is the Hamiltonian reduction $\mu^{-1}(0)//G := \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$. Here we view $\mu^{-1}(0)$ as a possibly non-reduced scheme with ideal generated by $\{\mu^*(\xi), \xi \in \mathfrak{g}\}$. It is not difficult to see that the algebra $\mathbb{C}[\mu^{-1}(0)]^G$ has a natural grading and also a natural Poisson bracket of degree -2.

We have the following theorem relating quotient singularities to special Nakajima quiver varieities.

Theorem 3.1. *Let Q be the McKay quiver, $v = n\delta, w = (1, 0, \ldots, 0)$. Then every irreducible component of $\mu^{-1}(0)$ contains a free G-orbit and therefore $\mu^{-1}(0)$ is a reducible complete intersection. Further, there is an isomorphism $\mathbb{C}[\mu^{-1}(0)]^G \cong (SV)^{\Gamma_n}$ of graded Poisson algebras.*

In the remainder of this subsection we will try to persuade the reader that this statement is natural by explaining two special cases.

The first case is $n = 1$. Both varieties V/Γ_1 and $\mu^{-1}(0)//G$ appear when we try to parameterize certain semisimple representations of $S(V)\#\Gamma_1$. The representations we need are those that are isomorphic to $\mathbb{C}\Gamma_1$ over $\Gamma_1 \subset S(V)\#\Gamma_1$.

First of all, let us describe the irreducible $S(V)\#\Gamma_1$-modules. The subalgebra $S(V)^{\Gamma_1} \subset S(V)\#\Gamma_1$ embedded via $S(V)^{\Gamma_1} \hookrightarrow S(V) \hookrightarrow S(V)\#\Gamma_1$ is central and so acts by a scalar on every finite dimensional irreducible $S(V)\#\Gamma_1$-module. So to any finite dimensional irreducible $S(V)\#\Gamma_1$-module (=Γ_1-equivariant coherent sheaf with finite support on V) M we assign a point in V/Γ_1, say x. As an $S(V)$-module, M is supported on $\pi^{-1}(x)$, where π denotes the quotient morphism $V \to V/\Gamma_1$. But the action of Γ_1 on $V/\{0\}$ is free, so $\pi^{-1}(x)$ consists of $|\Gamma_1|$ points and $\mathbb{C}[\pi^{-1}(x)]$ is already an irreducible $S(V)\#\Gamma_1$-module. Our analysis implies that this is the only irreducible module with central character x.

Now suppose that $x = 0$. This means that V acts by nilpotent endomorphisms on M. Since M is irreducible, V must act by 0. It follows that M is irreducible already as a Γ_1-module.

The conclusion is that each $S(V)\#\Gamma_1$-module, semisimple and isomorphic to $\mathbb{C}\Gamma_1$ as a Γ_1-module, has a central character, a point in V/Γ_1, and that a central character determines the module uniquely up to an isomorphism. So, at least set theoretically, the moduli space for the modules under consideration is V/Γ_1.

On the other hand, to define an action of $S(V)\#\Gamma_1$ on $\mathbb{C}\Gamma_1$ we just need to specify the action of $V = \mathbb{C}^2$ and make sure the actions of the two basis elements commute. The action is given by a Γ_1-equivariant map $\mathbb{C}^2 \otimes \mathbb{C}\Gamma_1 \to \mathbb{C}\Gamma_1$ or, since, $\mathbb{C}\Gamma_1 = \bigoplus_{i=1}^r \mathbb{C}^{\delta_i} \otimes N_i$, by an element $\bigoplus_{i,j} \mathrm{Hom}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j})^{n_{ij}}$, where $N_{ij} = \dim \mathrm{Hom}(\mathbb{C}^2 \otimes N_i, N_j)$. So $\mathrm{Hom}_{\Gamma_1}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma_1, \mathbb{C}\Gamma_1)$ is basically the space R considered above but with the modification that we take $w = 0$ instead $w = \epsilon_1$.

The commutativity condition just says that the moment map is 0. So the representations of $S(V)\#\Gamma_1$ are in one-to-one correspondence with points of $\mu^{-1}(0)$. But different points can correspond to isomorphic representations. It is easy to see that two points of $\mu^{-1}(0)$ correspond to isomorphic representations if they lie in the same G-orbit (the group G just acts by changing bases in the spaces $\mathbb{C}^{\delta_i}$). Furthermore, one can show that a representation is semisimple if and only if its G-orbit is closed. Furthermore, the quotient morphism $\mu^{-1}(0) \to \mu^{-1}(0)//G$ has a general property that every fiber contains a single closed orbit. So the variety $\mu^{-1}(0)//G$ also parameterizes semisimple representations of $S(V)\#\Gamma_1$ in $\mathbb{C}\Gamma_1$.

Now let us consider the opposite case: $\Gamma_1 = \{1\}$. In this case $\Gamma_n = S_n$ acts on $V = \mathbb{C}^n \oplus \mathbb{C}^n$ diagonally. An easy special case of the Chevalley restriction theorem says that, for $\mathfrak{g} = \mathfrak{gl}_n$, $\mathbb{C}[\mathfrak{g}]^G$ is isomorphic to $\mathbb{C}[\mathbb{C}^n]^{S_n}$, the isomorphism is provided by the restriction to the subalgebra $\mathfrak{h}$ of diagonal matrices. This theorem has a double analog. Namely, consider the subscheme C of $\mathfrak{g} \oplus \mathfrak{g}$ given by $\{(X,Y) | [X,Y] = 0\}$. It is a well-known open problem on whether C is reduced, in any case, let C_{red} denote the corresponding reduced scheme. It is not difficult to show that the restriction to $\mathfrak{h}^2$ gives rise to the isomorphism $\mathbb{C}[C_{red}]^G \cong \mathbb{C}[\mathbb{C}^{2n}]^{S_n}$. In fact, as was proved by Etingof and Ginzburg in [EG], the algebra $\mathbb{C}[C]^G$ has no nilpotents and so $\mathbb{C}[C]^G \cong \mathbb{C}[\mathbb{C}^{2n}]^{S_n}$.

So we more or less see an isomorphism of the theorem in the two special cases. However, we have not used any framing. It turns out that the framing does not change the reduction $\mu^{-1}(0)//G$ but insures good properties of the moment map: that $\mu^{-1}(0)$ is a reduced complete intersection. This is true even without framing (modulo a minor modification) for $n = 1$ but is false for $\Gamma_1 = \{1\}$: the dimension of C is $n^2 + n$ instead of n^2.

3.2. **Quantum Hamiltonian reduction and an isomorphism theorem.** We can construct a graded deformation of $\mathbb{C}[\mu^{-1}(0)]^G$ using the quantum Hamiltonian reduction. Namely, consider the homogeneous Weyl algebra $\mathbb{A}_h(R) := T(R)[h]/(r_1 \otimes r_2 - r_2 \otimes r_1 - h\Omega(r_1, r_2))$. This is a graded algebra (with h being of degree 2) acted on by G. The G-action admits a *a quantum comomoment map* Φ. Namely, we can naturally embed $\mathfrak{sp}(R)$ into $\mathbb{A}_h(R)_2$ so that $\mathbb{A}_h(R)_2 = \mathfrak{sp}(R) \oplus \mathbb{C}h$. This is a $\mathrm{Sp}(R)$-equivariant embedding of Lie algebras, where the bracket on $\mathbb{A}_h(R)_2$ is given by $[a,b] = \frac{1}{h}(ab - ba)$. Then for Φ we take the composition of $\mathfrak{g} \to \mathfrak{sp}(R)$ and this embedding. This is a G-equivariant linear map satisfying $[\Phi(\xi), f] = h\xi f$ for any $\xi \in \mathfrak{g}, f \in \mathbb{A}_h(R)$.

Consider the space $\mathrm{D} := (\mathbb{A}_h(R)/\mathbb{A}_h(R)\Phi([\mathfrak{g},\mathfrak{g}]))^G$. This space has a natural algebra structure induced from $\mathbb{A}_h(R)$. Moreover, Φ induces a linear map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \mathrm{D}$ with central image (of degree 2). So D is a graded algebra over $S(\mathfrak{p}')$, where $\mathfrak{p}' := \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \oplus \mathbb{C}h$.

The quotient $\mathrm{D}/(\mathfrak{p}')$ is naturally identified with $\mathbb{C}[\mu^{-1}(0)]^G$. The fact that $\mu^{-1}(0)$ is a complete intersection implies that D is flat (=free) over $S(\mathfrak{p}')$. For $p \in \mathfrak{p}'^*$ we can define a specialization D_p. This is a filtered algebra whose associated graded is $\mathbb{C}[\mu^{-1}(0)]^G$ and it is a quantization precisely when $p(h) = 1$.

So we again get a graded deformation of $A := (SV)^{\Gamma_n}$. In fact, this deformation is the same as $eH_{un}e$. More precisely, the following holds.

Theorem 3.2. *There is an isomorphism* $\mathfrak{p}_{un} \xrightarrow{\sim} \mathfrak{p}'$ *such that* $\mathrm{D} = S(\mathfrak{p}') \otimes_{S(\mathfrak{p}_{un})} eH_{un}e$ *(an isomorphism of graded algebras).*

One can explicitly write an isomorphism $\mathfrak{p}_{un} \xrightarrow{\sim} \mathfrak{p}'$ (that happens to be determined uniquely up to a certain finite group action) but this is somewhat technical so we are not going to do this here.

We are going to sketch a proof below, for the details the reader is referred to [L]. It involves basically two ideas. First, as we have already seen in the case of SRA, it is beneficial to work with a resolution rather than with the original singular variety. For our choice of Γ, the variety $\mu^{-1}(0)//G$ admits a usual algebro-geometric resolution, say X, that is again produced by Nakajima's quiver variety construction. This resolution is symplectic so it makes sense to speak about its quantization. One can produce a quantization via a suitable version of quantum Hamiltonian reduction.

So we have quantizations (or, more precisely, deformations) of two resolutions: an orbifold resolution $S(V)\#\Gamma_n$ and the algebro-geometric resolution X. A problem is to relate the two. There is a classical recipe how to approach this problem that is used, for example, to establish McKay correspondence as an equivalence of derived categories. Namely, one needs a tilting (=without higher self-extensions) bundle $\mathcal{P}$ on X with $\mathrm{End}(\mathcal{P}) = S(V)\#\Gamma$. The existence of such a bundle was proved by Bezrukavnikov and Kaledin. Being tilting, the bundle is deformable and this together with a universality property of the SRA, basically, leads to the coincidence of quantizations.

3.3. **Algebro-geometric resolutions and their quantizations.** Let us explain how to construct X and also its (commutative and non-commutative) deformations.

We first need to fix a *stability condition*. This should be a "general enough" character, say θ, of G. For example, one can take $\theta(g_1, \ldots, g_r) = \prod_{i=1}^r \det(g_i)$, this is a stability condition used in Nakajima's papers. Then one can define the subset $R^{ss} \subset R$ of semi-stable points. In our example, this is going to be the set of all (A_a, B_a, C_i, D_i) satisfying the following condition: there is no nonzero collection $U_i' \subset U_i$ such that this collection is stable under all A_a, B_a and is annihilated by the D_i's. Now the condition that θ is generic means that the action of G on R^{ss} is free (this is not difficult to check directly in our example). There are conditions on the u_i's and v_i's that guarantee that R^{ss} is non-empty, they are satisfied for our choice of parameters. So the quotient $X := (\mu^{-1}(0) \cap R^{ss})/G$ is smooth and symplectic. Also we have a natural morphism $\pi : X \to \mu^{-1}(0)//G$. A general GIT (Geometric Invariant Theory) result implies that this morphism is projective. So we do get a symplectic resolution of singularities. We remark that there is an action of the one-dimensional torus $\mathbb{C}^\times$ on X (coming from the dilations on R). The action contracts X to $\pi^{-1}(0)$ (where 0 means the image of $0 \in R$ in $\mu^{-1}(0)//G$).

Now let us explain how to construct deformations of X using Hamiltonian reduction. Again, consider the homogenized Weyl algebra $\mathbb{A}_h(R)$ but then complete with respect to the powers of h. We get an algebra over $\mathbb{C}[[h]]$ and it can be localized to a sheaf on R. In particular, we can restrict this sheaf to R^{ss}, the resulting sheaf is denoted by $\mathbb{A}_h(R)|_{R^{ss}}$. Then we can form a reduction

$$\mathcal{D} := \left(\mathbb{A}_h(R)|_{R^{ss}}/\mathbb{A}_h(R)|_{R^{ss}}\Phi([\mathfrak{g},\mathfrak{g}])\right)^G.$$

What we get is a sheaf over $\tilde{X} := \mu^{-1}((\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*)/G$ that is a deformation of X. It is not difficult to see that, due to the freeness of the action, $\mathcal{D}(\mathbb{A}_h(R), G)$ is a flat formal deformation (and, in fact, a quantization) of the structure sheaf $\mathcal{O}_{\tilde{X}}$.

Now let us compare the local reduction $\mathcal{D}$ and the global one, D. We can take the global sections, $\Gamma(\tilde{X}, \mathcal{D})$ and this is an algebra over $\mathbb{C}[[h]]$, complete with respect to the h-adic filtration. But it is still acted by $\mathbb{C}^\times$ and we can take all elements that are $\mathbb{C}^\times$-finite, meaning, lying in a finite dimensional $\mathbb{C}^\times$-submodule. The subalgebra $\Gamma(\tilde{X}, \mathcal{D})_{\mathbb{C}^\times-fin}$ of $\mathbb{C}^\times$-finite elements can be shown to coincide with D.

3.4. **Weakly Procesi bundles.** By a *weakly Procesi* bundle on X we mean a $\mathbb{C}^\times$-equivariant vector bundle $\mathcal{P}$ with the following two properties:

(P1) $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P})$ is isomorphic to $S(V)\#\Gamma_n$ as a $\mathbb{C}[X] = S(V)^{\Gamma_n}$-module. The latter equality holds because $X \to V/\Gamma_n$ is a resolution of singularities.

(P2) $\mathrm{Ext}^i(\mathcal{P}, \mathcal{P}) = 0$ for $i > 0$.

Thanks to (P1), each fiber of P carries a Γ_n-action and becomes a regular Γ_n-module with respect to this action. The last claim is easy to check for a general fiber, for an arbitrary one it is obtained by continuity. Because of this, the invariant subsheaf $\mathcal{P}^{\Gamma_n}$ is a line bundle. We can twist $\mathcal{P}$ with a line bundle so that $\mathcal{P}^{\Gamma_n} = \mathcal{O}_X$.

The construction of a bundle $\mathcal{P}$ satisfying (P1),(P2) is easy for $n = 1$. Namely, consider a G-equivariant bundle $\mathcal{U}_i$ on R that is trivial as a vector bundle and with each fiber equal to U_i as a G-module. We can restrict the bundle to $\mu^{-1}(0) \cap R^{ss}$ and then push it to the quotient, we get a bundle (with a natural $\mathbb{C}^\times$-equivariant structure) to be denoted by $\mathcal{N}_i$. We then can take $\mathcal{P} := \bigoplus_{i=1}^r \mathcal{N}_i^{\dim N_i}$. This bundle satisfies (P1) and (P2), this was checked in [KV].

The general case is much more involved. We still can define some tautological bundle on X, but it will have a wrong rank: $n|\Gamma_1|$ instead of $n!|\Gamma_1|$. In the (somewhat degenerate) case when $\Gamma_1 = \{1\}$, the variety X is the Hilbert scheme $\mathrm{Hilb}^2(\mathbb{C}^n)$ of pairs of points on $\mathbb{C}^n$, and there is a *Procesi bundle* on X constructed by Haiman, [H] (an alternative construction was recently given by Ginzburg, [Gi]). This bundle is very complicated but is also very important, it is one of the main ingredients of Haiman's proof of the $n!$-conjecture of Macdonald. In general, it is only known that a weakly Procesi bundle exists, [BK][1].

3.5. **The proof of the isomorphism theorem.** Now we are ready to sketch the proof of Theorem 3.2. The property (P2) implies that that there is a unique (automatically, locally free) $\mathbb{C}^\times$-equivariant right $\mathcal{D}$-module $\tilde{\mathcal{P}}_h$ that is flat over $S(\mathfrak{p}')$ and deforms $\mathcal{P}$, i.e., $\tilde{\mathcal{P}}_h/(\mathfrak{p}') = \mathcal{P}$. In more detail, it is known that a projective right module over an algebra uniquely deforms to a right module over any formal deformation and that the deformed module is automatically projective. So we can deform $\mathcal{P}$ locally. But the higher Ext vanishing

[1]There is a conjecture by Haiman, saying that there is a unique *Procesi* bundle on X that satisfies an additional condition: Γ_{n-1}-invariants in $\mathcal{P}$ is the tautological bundle of rank $n|\Gamma_1|$.

ensures that the local deformations agree on intersections. Finally, the presence of the $\mathbb{C}^\times$-action allows us to consider non-formal deformations as well. The group algebra $\mathbb{C}\Gamma_n$ still acts on $\tilde{\mathcal{P}}_h$ and the bundle of invariants $e\tilde{\mathcal{P}}_h$ is identified with $\mathcal{D}$ thanks to (P0) and the uniqueness of the deformation.

We remark that higher Ext's of $\tilde{\mathcal{P}}_h$ vanish, while the endomorphisms is a deformation of $\operatorname{End}(\mathcal{P}) = S(V)\#\Gamma_n$. Let H' denote the subalgebra of all $\mathbb{C}^\times$-finite elements in $\operatorname{End}_{\mathcal{D}^{opp}}(\tilde{\mathcal{P}}_h)$. This is now a graded algebra over $S(\mathfrak{p}')$, it is flat, and $H'/(\mathfrak{p}') = S(V)\#\Gamma_n$. The universality property for the Cherednik algebra ensures that there is a linear map $\mathfrak{p}_{un} \to \mathfrak{p}'$ such that $H' = S(\mathfrak{p}') \otimes_{S(\mathfrak{p}_{un})} H_{un}$.

Now let us recover D from H'. The bundle $\mathcal{P}$ splits into the direct sum $\mathcal{P} = e\mathcal{P} \oplus (1-e)\mathcal{P}$. The analogous decomposition for $\mathcal{P}_h$ follows. Therefore $\mathcal{D} = e\mathcal{E}nd_{\mathcal{D}^{opp}}(\tilde{\mathcal{P}}_h)e$. It follows that $eH'e = \Gamma(\tilde{X}, \mathcal{D}) = \mathrm{D}$. So we get $\mathrm{D} = S(\mathfrak{p}') \otimes_{S(\mathfrak{p}_{un})} eH_{un}e$. To prove that the map $\mathfrak{p}_{un} \to \mathfrak{p}'$ is an isomorphism and to describe it explicitly requires more work.

3.6. **Application: derived equivalence.** The original motivation of Bezrukavnikov and Kaledin to introduce a weakly Procesi bundle was to produce a derived equivalence between $\operatorname{Coh}(X)$ and $S(V)\#\Gamma_n$-mod. An equivalence is provided by the functor $\operatorname{RHom}(\mathcal{P}, \bullet)$. This equivalence can be deformed to a derived equivalence between $\mathcal{D}\text{-mod}^{\mathbb{C}^\times}$ and $H_{un}\text{-mod}^{\mathbb{C}^\times}$, see [GL] for details. Here the superscript $\mathbb{C}^\times$ indicates that we work with modules that are equivariant with respect to the $\mathbb{C}^\times$-actions introduced above (the action on H_{un} just comes from the grading). The deformed equivalence is given by $\operatorname{RHom}(\tilde{\mathcal{P}}_h, \bullet)$. This equivalences can be used to prove derived equivalences between the categories of modules over H_p and $H_{p'}$ when the difference between p and p' is integral. In the case when there are Morita equivalences between eH_pe and H_p, and between $H_{p'}$ and $eH_{p'}e$, this result becomes a special case of a more general result by McGerty and Nevins, [MN].

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