G. Dito

Star-exponentials (w, in deformation quantization (w/ Pierre Schapira)

$$(M, \omega) \rightsquigarrow \{\ ,\ \}$$

$$\begin{cases} f * g = fg + \sum_{r \geq 1} \hbar^r C_r(f, g) \\ C_1(f, g) - C_1(g, f) = \{f, g\} \\ \text{associative} \end{cases}$$

An algebra structure on $\left(C^\infty(M)[[\hbar]], *\right)$

$$\mathrm{Exp}_{*\hbar}\left(\frac{tH}{i\hbar}\right) = \sum_{n \geq 0} \frac{1}{n!}\left(\frac{t}{i\hbar}\right)^n H^{*n}$$

<u>Example</u> $T^*\mathbb{R} = \mathbb{R}^2$ w/ Moyal product

$$f \underset{M}{*} g = \exp\left(\frac{\hbar}{2}\{\ ,\ \}\right)(f, g)$$

$$\mathrm{Exp}_{*}\left(\frac{tH}{i\hbar}\right) = \sum \frac{1}{\cos\frac{t}{2}} \exp\left(\frac{x^2 + \xi^2}{i\hbar} \tan\frac{t}{2}\right)$$

~~where $\pi_n(x, \xi)$~~

$$= \sum_{n \geq 0} \exp\left(-i\left(n+\tfrac{1}{2}\right)t\right)\pi_n(x, \xi)$$

where $\pi_n(x, \xi) = 2(-1)^n \exp\left(-\frac{x^2 \xi^2}{\hbar}\right) \cdot L_n\left(\frac{\cdots}{\hbar}\right)$

Laguerre polynomials

Normal $\star$ product:

$$(f \star_N g)(\bar{z}, z) = fg + \sum_{n \geq 1} \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial z^n} \frac{\partial^n g}{\partial \bar{z}^n}$$

Can be represented by path integral

... (Faddeev, Les Houches, 1975)

Heuristically:

$$\exp(\ \cdots\ ) = \text{F. P. I.}$$

Feynmann path integral

Sheaf of microdiff operators $\mathcal{E}_{T^*X}$

Sato-Kashiwara-Kawai

Boutet de Monvel

$$\sigma_{tot}(P)(x,\xi) = \sum_{-\infty < j \leq m} p_j(x,\xi)$$

$$m \in \mathbb{Z} \qquad p_j \in \Gamma(U, \mathcal{O}_{T^*X}(U))$$

Canonical estimates:

$\forall K \Subset U$ $\exists$ positive $C, \varepsilon$:

$$\sup_{(x;\xi)} |p_j(x;\xi)| \leq C\varepsilon^{-j}(-j)!$$

$$\forall j < 0$$

$$\sigma_{tot}(P \circ Q) = \sum_{\alpha \in \mathbb{N}} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{tot}(P)_x$$

$$\times \partial_x^\alpha \sigma_{tot}(Q)$$

Filtered by the order.

$$\mathrm{gr}\,\mathcal{E}_{T^*X} \simeq \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{T^*X}(j)$$

- Consider a homogeneous symplectic transformation

$$\varphi : T^*X \supset U \xrightarrow{\sim} V \supset T^*X$$

induces an isom of $\mathcal{E}_U \simeq \mathcal{E}_V$ (<u>not</u> uniquely)

$\hat{k} = \mathbb{C}[[\hbar, \hbar^{-1}]$

Same growth condition

Forget the homogeneity: get a (no more conic) filtered sheaf of algs $\mathcal{W}_{T^*X}$

Kontsevich '2001
Polesello - Schapira 2004
Kashiwara '96
] Algebroid stack

$$\left(\sigma_{tot}\, P\right)(x, \xi) = \sum p_n(x, \xi)\cdot \hbar^{-n}$$

— same growth estimates

$\mathcal{W}_{T^*X}$ directly from $\mathcal{E}_{T^*X}$:

$$\mathcal{E}_{T^*(X\times\mathbb{C}),\hat{t}} = \{P \in \mathcal{E}_{T^*(X\times\mathbb{C})} : [P, \partial_t] = 0\}$$

Set $T^*_{\tau\neq 0}(X\times\mathbb{C}) = \{(x,t,\xi,\tau) : \tau \neq 0\}$

...

__Outline__

$$\begin{tikzcd}
\mathcal{O}_X^{s,\hbar} \arrow[r, rightsquigarrow] \arrow[d] & \mathcal{W}^s_{T^*X} \text{ (resolvent)} \arrow[d] \\
\mathcal{O}^{t,\hbar} \arrow[r, rightsquigarrow] & \mathcal{W}^t_{T^*X} \text{ (expon.)}
\end{tikzcd}$$

- $\dfrac{1}{s-H} \in \mathcal{W}^s_{T^*X}$
- $\exp\left(\dfrac{tH}{i\hbar}\right) \in \mathcal{W}^t_{T^*X}$

Df $\mathcal{O}_X^{\hbar}$

$$f(x,\hbar) = \sum_{-\infty < j \leq m} f_j(x)\hbar^{-j}$$

$$f_j \in \mathcal{O}_X(U)$$

$$\sup_K |f_j| \leq C\varepsilon^{-j}(-j)!$$

$\mathbb{C}_s = \mathbb{C}$ w/ coordinate $s$. $\quad a: \mathbb{C}_s \times X \to X$

$$\mathcal{O}_X^{s,\hbar} = R^1 a_! \mathcal{O}_X^{\hbar}{}^{\times}$$

$\mathcal{O}_X^{s,\hbar}$ = Convolution algebra $H^1_c(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$

$$H^1_K(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) =$$

$$= \Gamma(\mathbb{C} \setminus K, \mathcal{O}_{\mathbb{C}}) / \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$$

## The sheaf $\mathcal{O}_X^{s,\hbar}$

$$f(s,x,\hbar) = \sum_{j \le m} f_j(s,x)\hbar^{-j}$$

$f_j(s,x)$ holom on $(\mathbb{C}_s \setminus K_0) \times X$

$K_0$ compact indep of $j$

Estimates on $f_j \ldots$

$U \subset X \qquad f(t,x,\hbar) = \sum_{j\in\mathbb{Z}} f_j(t,x)\hbar^{-j}$

$$f_j \in \Gamma(U, \mathcal{O}_{\mathbb{C}_t \times X}|_{t=0})$$

$\times$ $f_j(t,x)$ holo around $t=0$

$$\sup_{x\in K} |f_j(t,x)| \le \frac{R^{j-m}}{(j-m)!} \cdots \quad \forall j \ge m$$

$\times$ $\hbar^{-1} : \mathcal{O}_X^{t,\hbar}(m) \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}(m+1)$

$\times$ $f \in \mathcal{O}_X^{t,\hbar}(m) \quad g \in \mathcal{O}_X^{t,\hbar}(m')$

$$fg \in \mathcal{O}^{t,\hbar}(m+m')$$

Thus * A sheaf of algebras
* No formal counterpart, if forget about growth conditions.

Laplace transform

The sheaves $\mathcal{O}_X^{t,\hbar}$ and $\mathcal{O}_X^{s,\hbar}$ related by a kind of Laplace transform

~~$\mathcal{L}(s^{-1-a})$~~ =

$$\mathcal{L} : \mathcal{O}_X^{s,\hbar} \xrightarrow{\sim} \mathcal{O}_X^{t,\hbar}.$$

Note: $\hat{\mathcal{O}}_X^{s,\hbar}$, a formal $\mathcal{O}_X^{s,\hbar}$, does exist. But the Laplace transform cannot be applied.

Example ...

## The sheaf $\mathcal{W}^{s}_{T^*X}$

$$\mathcal{W}_{\mathbb{C}_s \times T^*X} = \{ P \in \mathcal{W}_{T^*(\mathbb{C}_s \times X)} \mid [P, s] = 0 \}$$

$$\mathcal{W}^{s}_{T^*X} = R^1 q_! \, \mathcal{W}_{\mathbb{C}_s \times T^*X}$$

## The sheaf $\mathcal{W}^{t}_{T^*X}$

a filtered sheaf of algebras

$p_j(t; x, \xi)$ holom around $\{|t| < \eta\} \times K$

$\mathcal{W}^{t}_{T^*X}$ contains usual def. qua.

$\times$ $\frac{1}{s-P}$ defines an element

of $\Gamma(U, \mathcal{W}^{s}_{T^*X})$

Expand in $s^{-k}$ and apply

Laplace transform. get exp.

Theorem For $P \in \mathcal{W}_{T^*X}(0)$ (order 0),
there is a section
$$\exp\left(\frac{t}{\hbar}P\right) \in \mathcal{W}^t_{T^*X} \quad \text{s.t.}$$
($X$ affine):
$$\sigma_{tot}\left(\exp\frac{t}{\hbar}P\right) = \sum_{n\geq 0} \frac{t\hbar^{-1}\sigma_{tot}^{\#n}(P)}{n!}$$

Boutet de Monvel:
pseudo-norms (needed even to show that $\mathcal{E}$ is an algebra)