GL_3 Elementary factorization | **Gekhtman 1**

$$a_{ij} = \sum_{i \to j} \prod_{\text{path}} \text{weights}$$

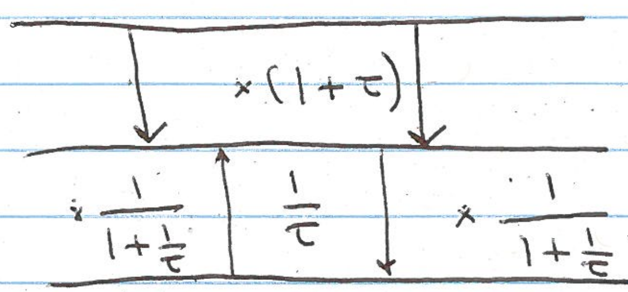
Minor of A

$$\Delta_I^J = \sum_{I \rightrightarrows J} \prod \text{(weights)}$$

"Special" minors: $\Delta_1^1 \quad \Delta_1^3 \quad \Delta_3^1$

$\Delta_{12}^{12} \quad \Delta_{12}^{13} \quad \Delta_{13}^{12} \quad \Delta_{12}^{23} \quad \Delta_{23}^{12}$

Δ_{123}^{123}



Poisson-Lie structure (standard) on GL_n

$$B_+^{(2)} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \qquad B_-^{(2)} = \begin{bmatrix} a & 0 \\ b & a^{-1} \end{bmatrix}$$

Standard Poisson-Lie:

$$B_\pm^{(2)} \to GL_n \text{ is Poisson}$$

Cluster Algebra

① Initial seed: $(x_1, \ldots, x_m; x_{m+1}, \ldots, x_n)$
and an $m \times (m+n)$ integral matrix B
$= m[\underset{m}{B_0} \mid \underset{n}{B_1}]$ s.t. B_0 is skew-symmetrizable
(input data...)

$$(\underbrace{x_1, \ldots, x_m}_{\text{cluster var.}}; \overbrace{x_{m+1}, \ldots, x_{m+n}}^{\text{stable}})$$

② $$x_i \mapsto x_i' = \frac{1}{x_i}\left(\prod_{b_{ij} \geq 0} x_j^{b_{ij}} + \prod_{b_{ij} < 0} x_j^{-b_{ij}}\right)$$
(Cluster transformations)

③ (Matrix mutation)

$$B \to B' \qquad b'_{jk} = \begin{cases} -b_{jk} & \text{if } j = i,\ k = i \\ b_{jk} + \dfrac{b_{ji}|b_{ik}| + |b_{ji}| b_{ik}}{2} \end{cases}$$

Ex B skew symmetric:

nicer way to describe in terms of quivers

① Can repeat for $\bar{x}' = (\bar{x} - \{x_i\}) \cup x_i'$

Do this in all possible ways.
Obtain collection of <u>all</u> possible cluster variables.

<u>Laurent Phenomenon</u> (F.-Z.)

Every cluster variable is a Laurent polynomial in terms of the initial seed.

<u>τ-coordinates</u> (or X- or Y- or ℓ-...)

$$\forall i \qquad \tau_i = \prod_j x_j^{b_{ij}}$$

The transformation:

fix i. ~~τ~~- Transf. in the direction:

$$\tau_i \mapsto 1/\tau_i \qquad \tau_j \mapsto \tau_j (1+\tau_i)^{b_{ij}} \quad b_{ij} > 0$$

$$\tau_j \mapsto \tau_j \left(\frac{1}{1+\tau_i}\right)^{-b_{ij}} \quad b_{ij} < 0$$

Rings of rational functions on:
$\bigcap$(Bruhat cells); Teichmüller spaces;
...

How do you recognize this from the ring of rational functions?

Compatible Poisson structures

$(M, \{,\})$ is a Poisson manifold:
we call a coord system $\{x_1, \dots, x_n\}$
Compatible:

$$\{x_i, x_j\} = c_{ij} \, x_i x_j$$

If we define on an initial seed a Poisson bracket

$$\{x_i, x_j\} = \omega_{ij} \, x_i x_j$$

is it possible to pick $\Omega = (\omega_{ij})$
so that every cluster transformation preserves $\{,\}$?

Yes: If B is of full rank

then the condition is

$$B\Omega = [D \mid 0]$$

diagonal;
the (a?) skew-symmetrizer
of B_0.

a) A cluster algebra becomes a Poisson algebra

b) Philosophically: $\{ , \}$ helps find cluster algebra structures on your manifold.