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Hochschild-Witt complex

X/k smooth$/k$

$H^\bullet_{DR}(X) = H(X, \Omega^\bullet_X)$ $\qquad$ $CH_\bullet(A)$

$HH_\bullet(A) \qquad \to A^{\otimes n} \to \cdots \to A^{\otimes 2} \to A$

Thm (HKR, 1962) $\;$ A commutative, $X = \operatorname{Spec} A$, smooth alg variety$/k$

$$HH_i(A) \simeq H^0(X, \Omega^i_X)$$

$k = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

Cartier: $\quad H^i_{DR}(X) = \Omega^i_X \qquad X$ affine

$$df = 0 \Rightarrow \begin{cases} f \text{ loc const, char } k = 0 \\ f = g^p, \text{ char } k = p \end{cases}$$

In characteristic p: crystalline cohomology.

$H^\bullet_{cris}(X) \qquad \mathbb{Z}_p$ - modules

If $X = \tilde{X} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ then $H^\bullet_{cris}(X) \simeq H^\bullet_{DR}(\tilde{X})$

Original dfn of Grothendieck: very hi-tech
Later, '77: Deligne-Illusie (paper by Illusie)

De Rham-Witt complex

(Bloch: constructed in small degrees using $K_\bullet$)

$$\mathbb{H}^\bullet(X, W_\bullet \Omega^\bullet_X) \simeq H^\bullet_{cris}(X)$$

<u>Goal</u>: to find NC generalization of $W\Omega_X$

<u>Result</u>: $\exists\, W_\bullet CH_\bullet(A)$ functorial in A, such that

$$W_\bullet HH_\bullet(A) = W_\bullet \Omega_X \quad \text{in the HKR case.}$$

1936 Witt vectors.

A commutative ring

$$\mathbb{W}(A) = 1 + tA[[t]] \hookrightarrow A[[t]]^* \to A^* \to 0$$

Nontrivial observation: this has a ring structure.

<u>Ex</u> $V = A^n$ $\quad a : V \circlearrowleft$ $\qquad \det(1 - ta) \in \mathbb{W}(A)$

<u>Prop</u> $\exists!$ functorial ring structure on $\mathbb{W}(A)$ such that

$$\det(1 - ta) * (1 - ta') = \det(1 - t(a \otimes a'))$$

For example, take $A = \mathbb{Z}$. infinite linear

Then $W(\mathbb{Z}) = \mathbb{Z}\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots \rangle$ combs

$\{i,j\} = $ l.c.m. $\qquad \varepsilon_i \cdot \varepsilon_j = \dfrac{ij}{\{i,j\}} \cdot \varepsilon_{\{i,j\}}$

$$\varepsilon_n = 1 - t^n \in 1 + t\mathbb{Z}[t]$$

S - finite set ; $\sigma : S \to S$

$\sigma : \mathbb{Z}[S] \circlearrowleft$; $\quad S = \mathbb{Z}/n$, $\sigma(x) = x+1$:

$$\det(1 - t\sigma) = 1 - t^n$$

$S \otimes S' = \oplus S'' \quad \ldots$

Take $A = k = \mathbb{F}_p$

$W(A) = \mathbb{Z}_p \langle \varepsilon_i \rangle, \quad p \nmid i$

((Same product as for $\mathbb{Z}$, subject to: $\varepsilon_{p^n} = p^n$)

$$= \prod_{p \nmid i} \mathbb{Z}_p$$

$\varepsilon_i^2 = i\,\varepsilon_i \qquad \frac{1}{i}\varepsilon_i$ is an idempotent

A/k $\quad W\!\!\!J(A)$ is a $W\!\!\!J(k)$-algebra

all the idempotents act on $W\!\!\!J(A)$

Def $\quad W(A) = \bigcap_{p \nmid i} \ker \frac{1}{i}\varepsilon_i \subset W\!\!\!J(A)$

p-typical Witt vectors

This gives a functorial lifting of a comm alg $/k$ to char 0. In particular, we have $W.\mathcal{O}_X$

The de Rham-Witt complex is an extension of this to diff. forms.

$W(A)/p \to A$ $\quad$ not isom, unless A perfect, i.e. $x \mapsto x^p$ isom

R/k assoc alg

M $\quad$ R-bimodule, projective $/ R \overset{o}{\underset{k}{\otimes}} R$.

$T^{\bullet}(M/R) = \bigoplus M \underset{R}{\otimes} \cdots \underset{R}{\otimes} M$

$T^{\leq n}$ - truncated.

$$0 \to W_n(M/R) \to K_1(T^{\leq}(M/R)) \xrightarrow{\text{augm}} K_1(R) \to 0$$

They form a projective system.

$$0 \to \mathrm{Cycl}^n(M/R) \to \mathbb{W}_n(M/R) \to \mathbb{W}_{n-1}(M/R) \to 0$$

If $R = k$, $\mathrm{Cycl}^n(M/k) = (M^{\otimes n})_\sigma$

$$HH_0(R, M^{\otimes_R N}) = M^{\otimes_k n}$$

$mr_1 \otimes m_2 \otimes \dots =$
$= m_1 \otimes r_1 m_2 \otimes \dots,$
$\dots$
$rm_1 \otimes \dots \otimes m_n =$
$= m_1 \otimes \dots \otimes m_n r$

$K_1^+(T^\cdot(M/R)) =$

$= HH_0(T^\cdot(M/R)) = \bigoplus \mathrm{Cycl}^n(M/R)$

V – a free finite-rank R-module

$$a : V \to M \otimes_R V \qquad (V, a)$$

form an exact category

(R, M)-mod

<u>Def</u> $K_\bullet(R, M) = K_\bullet((R, M)\text{-mod})$

(Dundas - McCarthy)

$$K_\bullet(R, M) = K(R) \oplus \bar{K}_\bullet(R, M)$$

There is a natural map $\bar{K}_0(R, M) \to \mathbb{W}(M/R)$

$$(V, a) \quad P = V \underset{R}{\otimes} T^{\bullet}(M/R)$$

$$\hat{a} : P \circlearrowleft \quad \text{over } T^{\bullet}(M/R)$$

extends canonically.

$$ch : \bar{K}_0(R, M) \to \mathbb{W}.(M/R)$$

$$\det(1 - \hat{a})$$

<u>Product</u>:

$$\bar{K}_0(R_1, M_1) \otimes \bar{K}(R_2, M_2)$$
$$\downarrow$$
$$\bar{K}_0(R_1 \otimes R_2, \ M_1 \otimes M_2)$$

($\otimes$ of V's).

<u>Prop</u> $\exists !$ product on $\mathbb{W}(M/R)$ compatible with ch.

<u>Corollary</u> If R/k, $\mathbb{W}(M/R)$ is a module over

$$\mathbb{W}(k/k) = \mathbb{W}(k).$$

$$W(M/R) = \bigcap_{p \nmid i} \ker \tfrac{1}{i} \varepsilon_i \subset \mathbb{W}(M/R)$$

$$0 \to \mathrm{Cycl}^{p^i}(M/R) \to W_{i+1}(M/R) \to W_i(M/R) \to 0$$

M <u>not</u> projective: $B_\bullet(R, M)$

$R^\bullet \otimes M \otimes R^\bullet$ is a <u>simplicial</u> R-bimodule

Def $\overline{W_\bullet(H(R,M))} = W_\bullet(B_\bullet(R,M)/R)$

<u>Comparison thm</u> In the HKR situation,

$$W_\bullet HH_{\underset{L}{\bullet}}(A, A) \simeq W_\bullet \Omega^i_X$$

$M = R$: <u>this</u> simplicial object ~~was~~ <u>extends</u> to a cyclic object.

$F, V, d \ldots$ <u>exists</u> on the left.

The Chern character extends to

$$K_\bullet(A)_p \to W_\bullet HH_\bullet(A)^{hF}$$

Conjecture: <u>this</u> is an isomorphism for a reasonable class of A.