SRA & QHR

Losev 2

Plan

1) Affine Nakajima quiver varieties
2) Example: V/Γ_n
3) $eH_{un}e$ as QHR
4) Resolution & its deformation
5) Weakly Procesi bundle
6) Derived equivalences

1. Setting: Quiver = oriented graph

$$Q = (Q_0, Q_1),\ \underset{\text{tail}}{t},\ \underset{\text{head}}{h} : Q_1 \to Q_0$$

$$t(a) \xrightarrow{a} h(a)$$

• U_i, $i \in Q_0$, - vector space. $\dim U_i = u_i$

$\mathbb{Z}_{\geq 0} \ni w_i$, $i \in Q_0$

$$R_0 = \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(U_{t(a)}, U_{h(a)})$$

$$\oplus \bigoplus_{i \in Q_0} U_i^{\oplus w_i} \quad \text{framings}$$

$$R = T^*R_0 = R_0 \oplus R_0^*$$

Structures on R.

$$G = \prod_{i \in Q_0} GL(V_i) \circlearrowright R_0, R$$

$$\mathbb{C}^\times \circlearrowright R$$

$$\mathbb{C}^\times = \{z \in \mathbb{C} \mid z \neq 0\} \qquad (t, r) \mapsto t^{-1} r$$

Symplectic form Ω on R, G-inv

<u>Moment map</u> $\mu : R \to \mathfrak{g}^* \longleftrightarrow$ comoment map

$\mathbb{C}[R] \leftarrow \mathfrak{g}$

functions on R

$$\mathfrak{g} \to \mathfrak{sp}(R) \qquad \mathbb{C}[R]_2 \xleftarrow{\sim} \mathfrak{sp}(R)$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d] & \mathbb{C}[R] \\ \mathfrak{sp}(R) \arrow[r, "\sim"] & \mathbb{C}[R]_2 \arrow[u, hook] \end{tikzcd}$$

μ^* is G-equivariant $(G \circlearrowright \mathbb{C}[R])$

$\to \mathfrak{g} \circlearrowright \mathbb{C}[R]$

Affine quiver variety:

$$X_0 = \mu^{-1}(0)/\!/G :=$$

$$= \operatorname{Spec}\left(\mathbb{C}[R]/\mathbb{C}[R]\mu^*(\mathfrak{g})\right)^G$$

$X_0 \circlearrowleft \mathbb{C}^\times$; natural Poisson structure

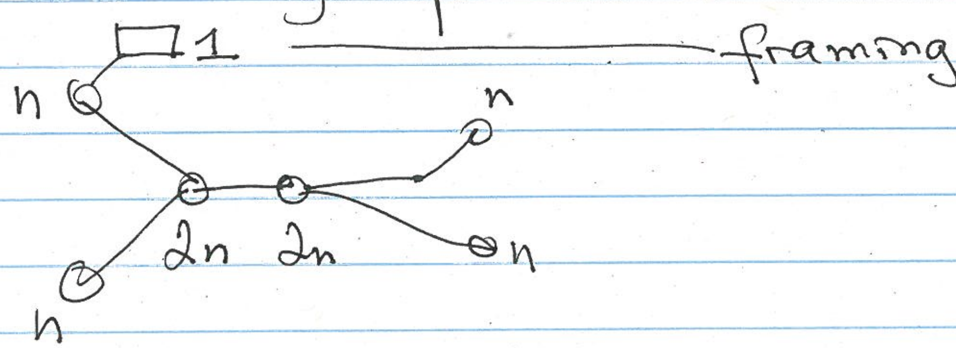
2) $\Gamma_1 \subset SL_2(\mathbb{C})$, $n > 0 \rightsquigarrow$

$$\rightsquigarrow \Gamma_n = S_n \ltimes \Gamma_1^n \circlearrowleft V = \mathbb{C}^{2n}$$

McKay quiver Q, $u = n\delta$
└ indecomposable imaginary root

$w = \epsilon_1 = (1, 0, \dots, 0)$
↑
coord vector

E.g. dihedral group w/ 12 elements



$\rightsquigarrow X_0$

Thm (Gan–Ginzburg)

$\mu^{-1}(0)$ is reduced complete intersection.

$$X_0 \xrightarrow{\sim} V/\Gamma_n$$

$\mathbb{C}^\times$

3). Quantization.

$\mathbb{C}(R) \rightsquigarrow$ homogen. Weyl algebra

$$A_h = T(R)[h]/([r_1, r_2] = \Omega(r_1, r_2)h)$$

$G \circlearrowright$

$\Big\{$ + quantum comoment map $\Phi\Big\}$

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & A_h \\ \downarrow & & \uparrow \\ sp(R) & \to & A_h = sp(R) \oplus \mathbb{C}h \end{array}$$

Properties. $[\Phi(\xi), f] = h \cdot \xi_{A_h} f$

Deformation of X_0.

$$\mathbb{D} = [A_h / A_h \Phi([\mathfrak{g}, \mathfrak{g}])]^G$$

$$\mathbb{D} = [A_{\hbar}/A_{\hbar} \cdot \Phi([\mathfrak{g},\mathfrak{g}])]^G$$

$$\uparrow$$

$$\mathfrak{p}' = \mathbb{C}\hbar \oplus \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$$

<u>Thm</u> (EGGO '05, I.L. '10)

$$eH_{un}e \xrightarrow{\sim} \mathbb{D} \qquad \mathfrak{p}_{un} \xrightarrow[\text{explicit}]{\sim} \mathfrak{p}$$

4) GIT quotient: $\theta : G \to \mathbb{C}^\times$ character

$$R^{ss} = \left\{ r \in R \,\middle|\, \exists f \in \mathbb{C}[R] \text{ s.t. } f(0) = 0;\ f(r) \neq 0;\ g \cdot f = \theta(g)^n \cdot f,\ \forall g \in G \text{ where } n \geq 0 \right\}$$

If θ is generic then $G \curvearrowright R^{ss}$ is free.

$$\text{e.g. } \theta(g) = \prod_{i \in Q_0} \det(g_i)$$

$$X := (\mu^{-1}(0) \cap R^{ss})/G$$

smooth & symplectic

$$\mathbb{C}^\times \circlearrowright X \xrightarrow[\mathbb{C}^\times\text{-equivar}]{} X_0 \quad \text{projective morphism}$$

in our case, birational.
Get a symplectic resolution.

<u>Deformations:</u> $\tilde{X} = \left(\mu^{-1}([\mathfrak{g},\mathfrak{g}]^\perp) \cap R^{ss}\right)/G$

$$\begin{array}{c} \left(\mu^{-1}([\mathfrak{g},\mathfrak{g}]^\perp) \cap R^{ss}\right)/G \\ \downarrow \\ [\mathfrak{g},\mathfrak{g}]^\perp \end{array}$$

A family of sympl alg varieties

Sheafify $\mathbb{A}_h \rightsquigarrow$ [h-adic completion]

$\hat{\mathbb{A}}^h_h$

[localize elements] sheaf on R

w/ global sections $\hat{\mathbb{A}}^h_h$

$$\mathcal{D} = p_*\left[\hat{\mathbb{A}}^h_h|_{R^{ss}} \Big/ \hat{\mathbb{A}}^h_h|_{R_{ss}}\phi([\mathfrak{g},\mathfrak{g}])\right]^G$$

sheaf of algebras $/ S(\mathfrak{p}')$, flat, h-adically complete;

(= quantization of $\tilde{X}$)

$$\mathbb{C}^\times \curvearrowright \Gamma(X, \mathcal{D}) = \mathbb{D}^{\wedge_h}$$

$$\Gamma(X, \mathcal{D})_{\mathbb{C}^\times\text{-fin}} = \mathbb{D}$$

Two deformations of V/Γ_n

$eH_{4n}e$ – "lifts" to resolution

$$\mathbb{D} \qquad\qquad \begin{matrix} S(V) \# \Gamma \\ X \end{matrix}$$

? How to relate two resolutions ?

A: Certain bundle on X.

5) Want $\mathbb{C}^\times$-equiv vect bundle $\mathcal{P}$ on X

i) $\mathrm{Ext}^i(\mathcal{P}, \mathcal{P}) = 0, \quad \forall i > 0$

ii) $\mathrm{End}(\mathcal{P}) \xrightarrow{\sim} S(V) \# \Gamma$

Alg $/ \mathbb{C}[X]$ $\qquad$ $\mathbb{C}^\times$-equiv $\qquad$ alg $/ S(V)^\Gamma$

$\rightsquigarrow$ fiberwise action $\Gamma \circlearrowright \mathcal{P}$
s.t. $\mathcal{P}_x \simeq \mathbb{C}\Gamma$

$\Rightarrow$ $\mathcal{P}^\Gamma$ – <u>line bundle</u>

(iii) $\mathcal{P}^\Gamma = \mathcal{O}_X$

Bezrukavnikov-Kaledin: $\exists \mathcal{P}$

(i) $\Rightarrow$ $\mathcal{P}$ is "deformable", i.e. loc free

$\mathbb{C}^\times$-equivariant right $\mathcal{D}$-module $\widetilde{\mathcal{P}}_h$, unique.

($\mathbb{C}^\times$-action allows to make everything <u>in</u>formal)

$$\operatorname{End}(\widetilde{\mathcal{P}}_h)/(\mathfrak{p}') = \operatorname{End}(\mathcal{P}) = S(V)\#\Gamma$$

$$H' = \operatorname{End}(\widetilde{\mathcal{P}}_h)_{\mathbb{C}^\times\text{-fin}} \text{ — } \underline{\text{SRA}}$$

<u>And</u> we have the <u>universal</u> SRA.

$$H' = S(\mathfrak{p}') \underset{S(\mathfrak{p}_{un})}{\otimes} H_{un}$$

$$\mathcal{D}' = e\,\mathcal{E}nd\,(\widetilde{\mathcal{P}}_h)\,e \Rightarrow \mathbb{D} = eH'e$$

6) $\mathcal{P}$ satisfying (i), (ii) $\rightsquigarrow$

$R\mathcal{H}om(\mathcal{P}, \bullet)$: $\quad Coh(X) \to S(V) \# \Gamma'$ — mod

$$R\,Hom(\widetilde{\mathcal{P}}_h, \bullet) : \mathcal{D}\text{-mod} \xrightarrow{\mathbb{C}^\times} H_{uh}\text{-mod}^{\mathbb{C}^*}$$

I.L. - I. Gordon: derived localization for SRA.

More in the talk of T. Nevins.