Pantev

NC Hodge structures and Mirror Symmetry

Recall: a pure $\mathbb{Q}$-HS is the data

$$(\underbrace{V}_{\text{complex vector space}}, \underbrace{V_{\mathbb{Q}}}_{\text{rational subspace } V_{\mathbb{Q}} \otimes \mathbb{C} = V_{\mathbb{C}}}, \underbrace{F^{\bullet}V}_{\text{Hodge filtration } w\text{-opposed to } \overline{F}^{\bullet}V})$$

w-opposed:

$$gr^{p}_{F^{\bullet}} \, gr^{q}_{\overline{F}^{\bullet}} = 0$$

unless $p+q=w$

This is what we have on $H^{\bullet}(X)$, X compact Kähler.

Question: Find an extension of pure HS applicable to NC geometry.

Idea: replace the data of a filtration by some geom proxy data on V.

1). Rees construction: get a vector space w/ $\mathbb{C}^*$-action.

2) Connection on a vector bundle on the punctured disc.

Def A rational pure nc Hodge structure is a triple $(H, \nabla, \mathcal{E}_B, \underline{iso})$ where

- $H \to \mathbb{A}^1$ algebraic vector bundle/$\mathbb{C}$, $\mathbb{Z}/2$-graded
- ∇ merom connection in H with pole of ord ≤ 2 @ 0
- $(\mathcal{E}_B, \underline{iso})$ - rational

$\mathcal{E}_B \to \mathbb{A}^1 \setminus \{0\}$ local system of rational ~~aff~~ vector bdles/$\mathbb{Q}$, $\mathbb{Z}/2$-graded

$$\underline{iso} : \mathcal{E}_B \xrightarrow{\sim} (H_{|\mathbb{A}^1 \setminus \{0\}})^\nabla$$

Properties
- ∇ has a pole of order ~~of~~ ≤ 1 at ∞. (in addition to having a pole ord ≤ 2 at 0)
- $\mathbb{Q}$-structure axiom: $(\mathcal{E}_B, \underline{iso})$ is compatible w/ Stokes data.

(i.e. the Stokes filtration is defined over $\mathbb{Q}$ on $(H|_{S^1})^{\nabla}$ via iso)

(Opposedness condition):

The complex conjugation $\tau : H|_{S^1}^{\nabla} \circlearrowleft$ induced from the rational structure allows us to glue

$$H\Big|_{\{|u|\leq 1\}} \quad \text{with} \quad \gamma^* \overline{H}_{\{|u|\geq 1\}}$$

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1$$
$$u \mapsto 1/\bar{u}$$

so we get $\hat{H} \to \mathbb{P}^1$ holom $\Rightarrow$ algebraic

Then the opposedness condition:

$$\hat{H} \xrightarrow{\sim} \mathcal{O}^{\oplus r}$$

Facts (1) We get a category of pure nc Hodge structures

($\mathbb{Q}$-nc HS)

which is a $\mathbb{Q}$-linear abelian category similar to ($\mathbb{Q}$-HS)

Polarized nc HS: ...

Similar to usual Hodge theory:

{Polarizable nc HS} semisimple.

(2) If $(V, F^\bullet V, V_{\mathbb{Q}})$ pure HS of weight w

we can construct a pure nc-$\mathbb{Q}$-HS:

Consider: $\mathcal{T}_{w/2} = \left(\mathcal{O}_{\mathbb{A}^1_0}, d - \frac{w}{2}\frac{du}{u}\right)$

$H \to \mathbb{A}^1_{\mathbb{C}}$. corresponds via Rees module construction to $F^\bullet V$

H corresponds to $\mathbb{C}[u]$-module

$$\sum_p u^{-p} F^p V \subset \mathbb{C}[u, u^{-1}] \otimes V$$

$V \otimes \mathcal{T}_{w/2}$; $\nabla = \left(d - \frac{w}{2}\frac{du}{u}\right) \otimes \mathrm{id}_V$

preserves the subbundle H.

∇ has monodromy $(-1)^w \cdot \mathrm{id}_V$.

$\mathcal{E}_B$ = local system of flat sections whose value @ 1 is in $\mathbb{Q}$.

No Stokes data (regular singul.)

Opposedness axiom $\Leftarrow$ opposedness of the Hodge filtrn

This gives a functor

$$N : (\mathbb{Q}\text{-HS}) \to (\mathbb{Q}\text{-ns HS})$$

which factors through $\otimes \mathbb{Q}(1)$

im : reg sing, mon $= \pm 1$

$$\bar{N} : (\mathbb{Q}\text{-HS}) \big/ (\cdot \otimes \mathbb{Q}(1)) \hookrightarrow (\mathbb{Q}\text{-nc HS})$$

Main conjectures

Conjecture: The periodic cyclic homology of a smooth and proper $\mathbb{Z}/2$ graded dg category carries a natural pure $\mathbb{Q}$-nc HS, of regular type. $\leftarrow$ reg sing

There are some candidates for the Hodge filtration etc., but it is hard to check axioms.

Ex: Fukaya (Smooth compact CY);
DCoh (Smooth compact scheme); ...

A-model HS:

(X, ω) compact symplectic manifolds of dim $2d$

This gives a family

$$\begin{array}{ccc} (\mathbb{X}, \omega) & \longrightarrow & \mathbb{A}^1 \\ \| & & \ni \\ X \times \mathbb{A}^1 & & q \end{array}$$

$$\omega = (\log q) \cdot \omega$$

or: a B-field, i.e. a circle-valued closed two-form...

The 3-point GW invariants give

$$*_q : H^\bullet(X, \mathbb{C})^{\otimes 2} \to H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}_q$$

Novikov ring...

Assumption: $*_q$ is convergent in a nbhd of zero.

Consider $\mathcal{H}^0 = \bigoplus_{k \equiv d \,(\mathrm{mod}\, 2)} H^k(X, \mathbb{C}) \otimes \mathbb{C}\{q, u\}$

$\mathcal{H}^1 = \bigoplus_{k \equiv d+1 \,(\mathrm{mod}\, 2)}$

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

There is a connection ∇ on $\mathcal{H}$:

$$\nabla_{\partial/\partial u} = \frac{\partial}{\partial u} + u^{-2}\left(K_X *_q \cdot\right) + u^{-1}\,Gr$$

$$\nabla_{\partial/\partial q} = \frac{\partial}{\partial q} \cancel{\pm}\; q^{-1}u^{-1} \cdot \left([\omega] *_q \cdot\right)$$

$$Gr\Big|_{H^k} = \frac{k-d}{2}\,\mathrm{id}$$

Remark: nc HS came earlier from Fourier-Laplace transforms of variations of Hodge structures.

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$$\hat{\Gamma}(X) = \prod_{i=1}^{d} \Gamma(1+\delta_i)$$

$\delta_1, \ldots, \delta_d$ Chern roots of T_X

$$e^{\log u \cdot Gr} \quad e^{\frac{\log u}{u}(K_X *_q)} \quad \lim_{q \to 0} e^{\frac{\log u}{u}\omega \wedge} \quad \mathcal{E}_B \subset$$

$$\subset H^{\bullet}(X, \mathbb{C})$$

A-model:

$$H^\cdot(X,\mathbb{Q}) \to H^\cdot(X,\mathbb{C}) \to H^\cdot(X,\mathbb{C})$$

$$\& \qquad (2\pi i)^{k/2} \qquad \text{prod w/ } \hat{\Gamma}(X)$$

Rmks (1) If X is a Toric Fano

($\mathcal{E}_B$ iso) is the mirror of the usual rational structure of the LG model. (Iritani)

(2) This rational structure is compatible with stokes data.

NOT a regular Hodge structure! Second order singularity is precisely $u^{-2} * _q K_X$.

B-model: $\mathbb{Q}$-nc HS associated with LG models

Y quasi-projective, $w : Y \to \mathbb{C}$ alg fn

$\underset{\sim}{K_Y} = 0$, w is projective

Want: NC HS on $H^\cdot(Y,w)$ e.g. on

$$H_B^\cdot(Y, w^{-1}(t); \mathbb{C}) \quad \left(\begin{array}{c} t \text{ regular} \\ \text{value of } w \end{array}\right)$$

Thm (dual description of nc HS)

There is an equivalence of categories

$$\boxed{\begin{array}{c} (H,\nabla), (\mathcal{E}_B, \underline{iso}) \\ \text{satisfying} \\ (\mathbb{Q}\text{-structure})^{exp} \\ (\text{filtration})^{exp} \end{array}}$$

$((H,\nabla), \mathcal{F}_B, f)$

(H,∇) (nc-filtration)^{ex}

$\mathcal{F}_B \in \mathrm{Constr\,Shf}(\mathbb{A}^1, \mathbb{Q})$

$R\Gamma(\mathbb{A}^1, \mathcal{F}_B) = 0$

$\mathcal{F}_B \otimes \mathbb{C} \xrightarrow{\sim} DR(i_* H_{\mathbb{A}^1 - \{0\}}, \nabla)$

exponential type: no ramification (as in Claude Sabbah)

Fourier transform

Thm $((H,\nabla), (\mathcal{E}_B, iso))$ — nc HS of exp type

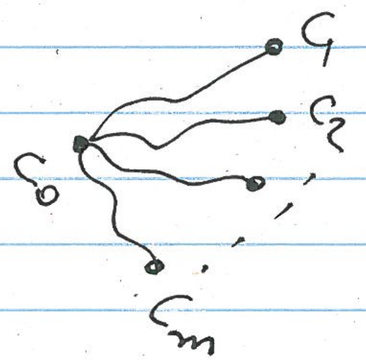
then the data $(H,\nabla), (\mathcal{E}_B, \underline{iso})$ is equivalent:

(regular type) $\{(R_i, \mathcal{E}_{B,i}, \underline{iso}_i)\}_{i=1}^{m}$

(gluing data) $c_0 \in \mathbb{A}^1_{\mathbb{C}}$ – gluing point

$c_1, \dots, c_m \in \mathbb{C}$;

paths

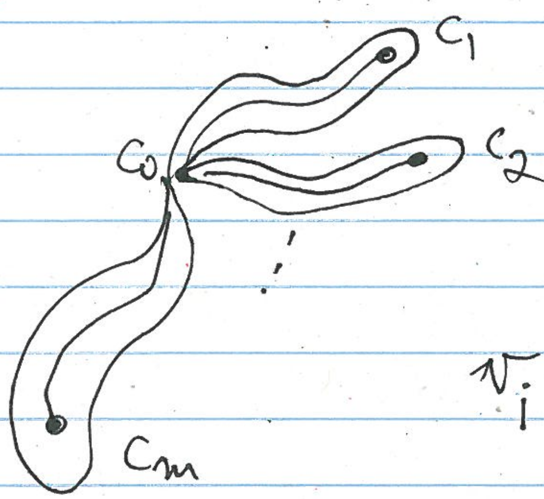


$$\boxed{T_{ij} : (\mathcal{E}_{B_i})_\infty \xrightarrow{\sim} (\mathcal{E}_{B_j})_\infty}$$

(they come from using the Fourier transform...) gluing data for the perverse sheaf $\mathcal{F}_B$...

<u>Now</u>: for $w : Y \to \mathbb{C}$

Fix $c_0 \in Y - \mathrm{crit}(w)$



$$v_i = H^k(w^{-1}(\gamma_i), w^{-1}(c_0); \mathbb{Q})$$

$$v = \bigoplus_{i=1}^{m} v_i$$

$T_{ii} : v_i \to v_i$ — monodromy on ∂D_i

<u>Fact</u>: <u>this</u> is a perverse sheaf (i.e. the constr. sheaf constructed by using these data)

$$\mathcal{F}_B = H^K(Y, w^{-1}(c_0)); \mathbb{Q})$$

$$R\Gamma(\mathbb{A}^1, \mathcal{F}_B) = 0$$

$$\equiv$$

There is a Betti-de Rham iso

$$\mathbb{C}\{u\} \otimes H^{\cdot}_B(Y, w^{-1}(c_0); \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^{\cdot}_{Zar}(\Omega^{\cdot}\{u\}, ud + dw)$$

Conjecture: This defines a $\mathbb{Q}$ - nc HS

A theorem from Sabbah's lectures:

R.H.S. $\simeq$ (same w/out ud); decomposes into a product ~~ae~~ of nearby cycles of critical pts.

This is our decomposition (into V_i?)

MS Conjecture If (X, ω) mirrors (Y, w)

then $\begin{pmatrix} \text{A-model} \\ \text{nc HS} \end{pmatrix} \simeq \begin{pmatrix} \text{B-model} \\ \text{nc HS} \end{pmatrix}$

$\mathbb{Q}$-structures: easy...
Opposedness: difficult.

(conjecturally...)

This can be extended beyond the smooth compact case.

Recall: A $\mathbb{Q}$-MHS is

$$(V, V_{\mathbb{Q}}, F^{\bullet}, W_{\bullet})$$

opposedness: $gr_k^{W_\bullet}(V, V_{\mathbb{Q}}, F)$

is a pure HS wrt weight k.

or: $gr_k^W \, gr_F^p \, gr_{\bar{F}}^q = 0$.

A $\mathbb{Q}$-nc MHS is the data

$$\left((H, \nabla); \; (\mathcal{E}_B, \underset{W.\mathcal{E}_B;}{\underline{iso}}), \; W.H\right)$$

$W.$ ∇-horizontal; $W.$-graded quotients are pure nc HS.

<u>Asymptotic nc MH theory</u>:

<u>Recall</u>: If $\mathcal{X} \to \Delta$ degenerating family of projective mflds i.e.

$\cdot\ \mathcal{X} \to \Delta$; $\mathcal{X}$ smooth; $\mathcal{X}_t$ smooth proj, $t \neq 0$;

• $\mathcal{X}_0$ - strict normal crossings divisor.

Δ - 1 dim disc. centered at 0.

In this case: on $H^{\cdot}(\mathcal{X}_0, \mathbb{C})$ canonical functorial Deligne MHS

$H^{\cdot}(\mathcal{X}_t, \mathbb{C})$ canonical functorial pure HS

We have a deformation retract $c: \mathcal{X} \to \mathcal{X}_0$ restriction to $\mathcal{X}_t$ gives $c_t: \mathcal{X}_t \to \mathcal{X}_0$

continuous, not at all analytic.

This induces a map

$$c_t^*: H^*(\mathcal{X}_0, \mathbb{C}) \longrightarrow H^*(\mathcal{X}_t, \mathbb{C})$$

which is asymptotic to a map of Hodge structures.

For this we need to perturb the HS on the nearby fiber so that it depends on the family $\mathcal{X} \to \Delta$ and not just on the fiber $\mathcal{X}_t$.

This is the context of the nilpotent orbit thm!

Asymptotic nc MH thy Pantev 3

Asymptotic MHS:

$f: \mathcal{X} \to \Delta$ family of proj varieties

$\mathcal{X}$ - smooth ; $\mathcal{X}_t$ smooth $t \neq 0$;
$\mathcal{X}_0$ strict normal crossings divisor

(a) $\mathcal{X}_0$ is a strict deformation retract of $\mathcal{X}$ (Clemens)

(b) $\forall k$ the monodromy is unipotent of deg $\leq k$:

$$T : H^k(X_t, \mathbb{C}) \circlearrowleft \quad \text{(Landman)}$$

c: $\mathcal{X} \to \mathcal{X}_0$ deformation retraction lifting $\Delta \to 0$

Restrict it to a nearby smooth fiber

$(\mathcal{X}_t \doteq)$ X_t ; $c_t : X_t \to X_0$

induces $H^k(X_0, \mathbb{C}) \to H^k(X_t, \mathbb{C})$

not a map of Hodge structures
(MHS on L.H.S. ; pure HS on R.H.S. ...)

- we can modify the Hodge structure on the nearby fiber in a way that takes into account the formation of the family.

$$c_t^*: H^k(X_0, \mathbb{C}) \to H^k(X_t, \mathbb{C})$$

becomes a map of HS after the modification.

Consider $N = \log T = \sum (-1)^{i+1} \frac{(T-id)^{i+1}}{i+1}$
nilpotent on $H^k(X_t, \mathbb{C})$
associated monodromy weight filtration

$$0 \subset W_0 \subset W_1 \subset \dots \subset W_{2k} = H^k(X_t, \mathbb{C})$$

The unique filtration : $N(W_a) \subset W_{a-2}$

$$N(W_a) = W_{a-2} \cap \operatorname{Im} N$$

$$N^a: gr^W_{k+a} \xrightarrow{\sim} gr^W_{k-a}$$

is an isom or 0.

N also modifies the Hodge filtration in the following way.

Let $\mathcal{H}^k \to \Delta^\times$ be the holo bdle

$$R^k f_* (\mathbb{C}_{|\mathcal{X} \setminus X_0}) \otimes \mathcal{O}_{\Delta^\times}$$

$$\nabla: \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_{\Delta^\times}$$

Let $\mathcal{H}^k_{can}$ be Deligne's canonical extension of $\mathcal{H}^k$

to Δ. This is the unique extension so that ∇ has a ~~nilpotent po~~ log pole with nilpotent residue on $\mathcal{H}^k_{can}$.

(To construct: $\nabla^{can} = \nabla - \frac{1}{2\pi} N \frac{dt}{t}$;

no monodromy; the flat sections give you an extension $\mathcal{H}^k_{can}$.

This is a bundle that is holomorphically trivialized. The trivialization ~~lives~~ depends on coordinate but the extension does not. It really lives on the tangent ~~bundle~~ space to the disc.

<u>Thus</u> (Schmid) (1) The Hodge filtration $F^\bullet$ of $\mathcal{H}^k$ extends to a filtration by holom subbundles: F_{can}

(2) If F_{mod} is the constant filtration of $\mathcal{H}_{can}$ equal to F_{can} at 0, then $(H^k(X_{t_1}, \mathbb{C}), W_n, F^\bullet_{mod})$ is a MHS.

<u>Notation</u>: $F^\bullet_{mod} = \lim_{t\to 0} e^{\log t/2\pi i\, N} F^\bullet$

Notation:
$$F_{mod} = \lim_{t \to 0} e^{\frac{\log t}{2\pi i} N} F^{\bullet}$$

Mirrors of varieties of general type

Setup: $f : \mathcal{X} \to \Delta$ same as before.

But: require $\mathcal{X}$ to have a trivial canonical class.

Want to understand the mirror of this datum.

Question: what is the symplectic degeneration corresponding to $f : \mathcal{X} \to \Delta$?

Question: what is the limit MHS for this symplectic degeneration?

Expected answer: $\mathcal{X} \xrightarrow{f} \Delta$ is mirrored by (Y, D, ω) where (Y, ω) - symplectic of dimension$_{\mathbb{R}}$ $2n-2$ ($n = \dim_{\mathbb{C}} \mathcal{X}$) underlying $(n-1)$ dim CY

$D \subset Y$ - symplectic submanifold of codim$_{\mathbb{R}} = 2$ which underlines, typically, a variety of general type.

So: D of general type.

- embed into CY.
- take mirror of this CY.
- LG on that.

Interpretation: (Y, D, ω) should be viewed as a symplectic degeneration of Y.

In fact we have a 1-parameter family

$$\Big(Y, \omega + \log t \underbrace{PD(D)}_{\text{harmonic representative...}}\Big)$$

Homologically we expect:

B-side	A-side
$D^b(X_t)$	$\mathrm{Fuk}(Y, \omega + \log t \cdot PD(D))$
$D^b(X_0)$	$\mathrm{Fuk}^{\text{wrapped}}(Y - D, \omega)$
$(\mathrm{Perf}(X_0))$	$\mathrm{Fuk}(Y - D, \omega)$
$D^b(\mathcal{X}, f)$ (matrix factorizations)	$\mathrm{Fuk}(D, \omega_{\mid D})$

(switch A- and B- sides)

$(X, f, \omega) \longleftrightarrow (Y, D)$ holom

<u>Aside</u> : <u>A-side</u> <u>B-side</u>

$Fuk(X_t, \omega)$ $\qquad$ $D^b(Y)$

$Fuk(X_0, \omega)$ $\qquad$ $D^b(Y \setminus D)$

$Fuk(\mathcal{X}, f, \omega)$ $\qquad$ $D^b(D)$

$Fuk(X_0, \omega)$ is defined as an orbit category

We have a pair $(T, \varphi : id \to T)$

T is the monodromy around 0 viewed as a <u>symplectomorphism</u> of X_t ;

$$T \in Aut(Fuk(X_t, \omega))$$

$\varphi : id \to T$ because T is connected to identity by Hamiltonian isometry

$$Hom_{Fuk(X_0,\omega)}(L, L') := \varinjlim_k Hom_{Fuk(X_t,\omega)}(L, T^k L')$$

where $\lim$ is taken by using φ.

<u>Going</u> back to the original situation :

$\mathcal{X} \xrightarrow{f} \Delta$ $\qquad$ (Y, D, ω)

B-side $\qquad$ A-side

Then we can look at $(Y, \omega + \log t\, PD(D))$
and $H^k(Y, \mathbb{C})$ has a pure (nc) HS:

$$H^k(Y, \mathbb{C}),\ \nabla,\ \mathcal{E}_B,\ \text{iso}$$

rational structure given by the Γ-class; ∇- Dubrovin connection...

The log of the monodromy for this symplectic degeneration:

$$N = PD(D) \underset{q}{\star}\,.$$

Then we get a monodromy weight filtration and a nilpotent orbit modification of ∇:

Claim $\left(H^k(Y, \mathbb{C}),\ e^{\frac{\log t}{2\pi i} PD(D) \star_q} \nabla\right), W_\bullet,$

$$(\mathcal{E}_B, \text{iso})$$

is a nc MHS.

Comment about Hodge numbers

MS for CY: Hodge diamonds rotate.

Question Suppose X is Fano of dim n.

$Y \xrightarrow{W} \mathbb{C}$ LG mirror of X

How can we read the Hodge numbers of X from $Y \xrightarrow{w} \mathbb{C}$?

$Y \xrightarrow{w} \mathbb{C}$. Complex LG model   Pantev 4

s.t.
- Y quasi projective
- $\mathrm{vol}_Y \in H^0_Y(Y, K_Y)$
- w - proper

Want: define Hodge #s from this data.

$$h^{d+i}_{DR}(Y,w) = \sum_{p-q=i} h^{p,q}(X) \quad \text{if} \quad (Y,w) \text{ mirrors } (X,\omega)$$

(from Hochshild cohomology)

symplectic underlying Fano of $\dim_{\mathbb{C}} = d$.

There are 3 ways (at least) of extracting Hodge numbers:

(1) $h^{p,q}(Y,w)$ = dim of Jordan blocks of large monodromy:

$$H^K(Y, w^{-1}(t))$$

(2) $i^{p,q}(Y,w)$ = dim of $I^{p,q}$ pieces corresp. to MHS on $H^K_{dR}(Y,w)$ associated to

the irregular Hodge filtration of Deligne.
(Deligne wrote Hodge filtrations of de Rham complexes twisted by functions, at least by curves. Generalized by Yu, Sabbah).

(3) Suppose there exists compactification $\bar{Y} \xrightarrow{\bar{w}} \mathbb{P}^1$ such that: $\bar{Y}$ smooth;

$\bar{Y}_\infty$ normal crossings; vol_Y extends to poles at most of order 1 along $\bar{Y}_\infty$; same for w.
i.e. $\bar{Y}_\infty$ strict normal crossings.

$$f^{p,q}(Y,w) = \dim H^q(\bar{Y}, \Omega^{d-p}(\bar{Y},\bar{w}))$$

Conjecture They are all equal to $h^{q,d-p}(X)$

This follows if one could prove that

$$\mathbb{H}^\cdot(\bar{Y}, (\Omega^\cdot(\bar{Y},\bar{w}), d+dw)) = \mathbb{H}^\cdot(Y, \Omega^\cdot(\bar{Y},\bar{w}), d\bar{w})$$

Constructions of mirrors

$$(\mathcal{X} \xrightarrow{f} \Delta) \longleftrightarrow (Y, D, w)$$

From degenerations to symplectic varieties of general type

Start $f : \mathcal{X} \to \Delta$ (n-dim Y)

Assume : the dual intersection cplex of X_0 is a graph.

Ex $\mathcal{X} \to \Delta$ family of degenerating elliptic curves.

graph

If now D_a, D_b are two vertices joined by an edge

$$\begin{array}{l} \qquad D_a \\ D_\sigma \subset \quad \\ D_\sigma \subset \quad \\ \qquad D_b \end{array}$$

the reflection functors

$$R^a_\sigma := \mathrm{cone}\left(i^*_{\sigma a}\, i_{\sigma *} \to \mathrm{id}_{D_\sigma}\right)$$

$$R^b_\sigma := \mathrm{cone}\left(i^*_{\sigma, b}\, i_{\sigma, b *} \to \mathrm{id}_{D_\sigma}\right)$$

Explicitly: $R^a_\sigma = \otimes\, i^*_{\sigma, a} \mathcal{O}_{D_a}(-D_\sigma)$

$= \otimes N_{D_\sigma / D_a}$

Since $D_a + D_b$ is smoothable $\Rightarrow$
$\Rightarrow$ the normal bundles are inverses of each other.

$\Rightarrow R^a_\sigma R^b_\sigma$ or $R^b_\sigma R^a_\sigma$ are isomorphic to id

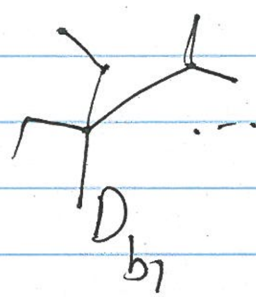
- Fix $R^a_\sigma R^b_\sigma \xrightarrow[\varphi_\sigma]{\sim} id$

We will construct (Y, D) as a gluing of mirrors of Fanos:

$$\left(D_a, \left(\overline{X_0 \setminus D_a}\right) \cap D_a\right)$$

If $a \in \text{vertex}(\Phi)$

Star(a):



The mirror of $(\mathbb{P}^1, 2 \text{ pts}) \leftrightarrow \mathbb{C}^\times \xrightarrow{w} \mathbb{C}, f$

$z \mapsto z + z^{-1}$

$Y_a \xrightarrow{w_a} \Delta_a$, P_a, w_a glue:

Y - smooth ell curve...