Categorifying D-module operations

Notation X finite type scheme$/k$ $(=\mathbb{C})$
char $k = 0$

Familiar categorification:

$$\begin{gathered}\text{Constructible fn}/X^{an}(\mathbb{C}) \text{ (values in } \mathbb{Z})\\ \Big\uparrow{\scriptstyle \chi} \\ \underbrace{\text{Constructible sheaves}/X^{an}(\mathbb{C}) \text{ (val } k)}_{\text{topological}}\end{gathered}$$

Less familiar:

$$\begin{gathered}\mathcal{D}\text{-mods on } X \text{ (val } k((u)))\\ \Big\uparrow{\scriptstyle \chi} \quad (HP_\bullet) \\ \text{Quasi-coh dg cat}/X\end{gathered}$$

Motivation: 1) Thm (FT) $HP_\bullet(\mathrm{Perf}\, X) \xrightarrow{\sim} C^\bullet_{dR}((u))$

"true" dfn

2) topological $\underline{\mathbb{C}}_X$ on X^{an}

$\mathcal{D}(X) :=$!-crystals

$$RH^{-1}(\underline{\mathbb{C}}_X) \in \mathcal{D}(X)$$

$$R\mathrm{Hom}_{\mathcal{D}(X)}(\ -\ ,\ -\ )$$

Derived De Rham cplex:

iii) $$\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathbb{L}_X[-1], d_{int} + d_{dR})$$

Thm' $$HP'^{\,k}_{\bullet}(\mathrm{DCoh}(X)) \simeq C^{BM}_{\bullet}(X)$$

$$RH^{-1}(\omega_X^{top}) = \omega_X$$

$$\pi_*(\omega_X)$$

=

Ring R	R-lin dg cat	Constr Shf	D-mod
$\mathcal{O}_X$	$\mathrm{DCoh}(X)$	ω_X^{top}	ω_X
	$\mathrm{Perf}(X)$	$\underline{\mathbb{C}}_X$	$RH^{-1}(\mathbb{C}_X)$
	R-linear periodic cyclic...?		
k	$\pi_* \mathrm{DCoh}\, X$	$R\Gamma(\omega^{top})((u))$	etc.
	$\pi_* \mathrm{Perf}\, X$	$R\Gamma(\mathbb{C}_X)$	|
			$\pi_* \ldots$

2) $M \xrightarrow{f} \mathbb{A}^1$

$D\mathrm{Coh}(M)$ linear / $\mathcal{O}_M$ — $\omega_M((u))$

$f_* D\mathrm{Coh}(M)$ linear / $k[x]$ — $f_*(\omega_M((u)))$

$\pi_* f_* D\mathrm{Coh}(M)$ linear / k

$$\pi_* f_* (\omega_M((u))) =$$

$$= C_*^{BM}(M)((u))$$

$$\boxed{MF(M, f) \text{ linear over } k[\beta], \quad \beta \text{ deg } -2}$$

$\longrightarrow \widetilde{\mathrm{Fourier}}(f_* \omega_M)((u))$

Primer on $QC^!$

$QC(X) :=$ dg-cat of quasi-coh cpxs / quasi-iso

$QC(X)^{<N}$ = dg cat of quasicoh cpxs ; bdded above + homol in deg $<N$
termwise injective

$QC(X)$ = dg cat of q-coh cxs
$\parallel$
Ind [Perf(X)] homotopy-injective

[easy
$\approx$
replacement functor "homotopy-projective"]
X affine

$QC^!(X)$ = dg cat of q coh cplex inj

[$\xleftarrow{\omega_X \otimes_{O_X} -}$ termwise proj ...]

injective model for the dualizing functor

Rmks

Getzler-Gauss-Manin connexion

R comm ring $\quad \mathcal{C} \in \mathrm{dgcat}_R$

<u>Rough idea</u>:

$$\mathrm{id} \in \mathrm{Fun}_R(\mathrm{Ind}\,\mathcal{C}, \mathrm{Ind}\,\mathcal{C})$$
$$\wr\!\wr$$
$$\mathrm{Ind}[\mathcal{C}^{op} \otimes_R \mathcal{C}] \xrightarrow{tr} R\text{-mod}$$

$$\mathcal{F} \otimes \mathcal{G} \mapsto R\mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

$$\boxed{HH_\bullet^{/R}(\mathcal{C}) = tr(\mathrm{id})}$$

$\circlearrowright S^1 \quad$ "Connes B"

$$HC^{-/R}(\mathcal{C}) = [HH_\bullet^{/R}(\mathcal{C})]^{S^1}$$

linear / $\quad C^*(BS^1) \simeq k[u]$

2) $HH_\bullet^{/k}(\mathcal{C})$ module / $HH_\bullet^{k}(R)$

S¹-equivariantly

$$HH_\bullet^{/R}(\mathcal{C}) \simeq HH_\bullet^{k}(\mathcal{C}) \otimes_{HH_\bullet(R)} HH_\bullet(k)$$

$$HH_\bullet^{R}(\mathcal{C}) = HH_\bullet^{k}(\mathcal{C}) \underset{HH_\bullet(R)}{\otimes} \mathrm{End}_{HH_\bullet(R)}(R)$$

Prop $\mathrm{End}_{HH_\bullet(R)}(R)^{\mathrm{Tate}} \simeq \mathcal{D}_R((u))$

Some derived version of differential ops

Thm $QC^!(LX)^{t\,\mathrm{Tate}}_{4} \xrightarrow{\sim} QC^!(X_{dR})((u))$

Tate construction
t-reg
dg cat

$\mathrm{Ind}[\mathrm{DCoh}(LX)]^{\mathrm{Tate}}$

$\omega_{\hat{X}S}((u)) \simeq \omega_{\hat{X}S^2} \in QC^!(X^2)$