

Preygel 1.

Categorifying D-module operations

Notation X finite type scheme / $k (= \mathbb{C})$
char $k = 0$

Familiar categorification:

Constructible fn / $X^{\text{an}}(\mathbb{C})$ (values in \mathbb{Z})
 \uparrow
 X

Constructible sheaves / $X^{\text{an}}(\mathbb{C})$ (val k)
} topological

Less familiar:

D-mods on X (val $k((u))$)
 \uparrow
 X (HP.)

Quasi-coh dg cat / X

Motivation: Thm (FT) HP. $(\text{Perf } X) \xrightarrow{\text{DR}} C^*(u)$
"true" def \uparrow

2) topological $\underline{\mathbb{C}}_X$ on X^{an}

$\mathcal{D}(X) := !\text{-crystals}$

$$RH^{-1}(\underline{\mathbb{C}}_X) \in \mathcal{D}(X)$$

$R\text{Hom}_{\mathcal{D}(X)}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X)$

Derived De Rham complex:

iii) $\text{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[-1], d_{int} + d_{dR})$

Thus,

$$HP'_{\bullet, K}(\mathcal{D}\text{Coh}(X)) \simeq C_{\bullet}^{dR, BM}(X)$$

$$RH^{-1}(\omega_X^{top}) = \omega_X$$

$$\pi_* (\omega_X)$$

Ring R	R -lin dg cat	Constr Shf \mathcal{D} -mod
\mathcal{O}_X	$\mathcal{D}\text{Coh}(X)$ $\text{Perf}(X)$	ω_X^{top} $\underline{\mathbb{C}}_X$ $RH^{-1}(\underline{\mathbb{C}}_X)$
k	R -linear periodic cyclic...?	
	$\pi_* \mathcal{D}\text{Coh} X$ $\pi_* \text{Perf} X$	$R\Gamma(\omega_X^{top})(k)$ etc. $R\Gamma(\underline{\mathbb{C}}_X)$ $\pi_* \dots$

$$2) M \xrightarrow{f} A^1$$

$$\begin{array}{l} D\text{Coh}(M) \text{ linear} / \mathcal{O}_M \\ f_* D\text{Coh}(M) \text{ linear} / k[x] \\ \pi_* f_* D\text{Coh}(M) \text{ linear} / k \end{array} \left. \begin{array}{l} \omega_M(u) \\ f_* \omega_M(u) \\ \uparrow \end{array} \right\}$$

$$\pi_* f_* (\omega_M(u)) =$$

$$= C_*^{\text{BM}}(M)(u)$$

$$MF(M, f) \text{ linear over } k[\beta]$$

$$\downarrow$$

$$\text{deg } -2$$

$$\rightarrow \text{Fourier}(f_* \omega_M)(u)$$

Primer on QC!

$QC(X) := \text{dg-cat of quasi-coh cpxs}$

$QC(X)^{<N} = \text{dg cat of quasicoh cpxs}$; bdded above + homol in deg $< N$
 termwise injective

$QC(X) = \text{dg cat of q-coh cxs}$

" $\text{Ind [Perf}(X)]$ homotopy-injective

[easy "homotopy-projective" replacement functor] X affine

$QC^!(X) = \text{dg cat of q coh cplex inj}$

[$\omega_X \otimes \mathcal{O}_X$ termwise proj]

injective model for the dualizing functor

Rmks

Getzler-Gauss-Mann connexion

R comm ring $\mathcal{C} \in \text{dgcat}_R$

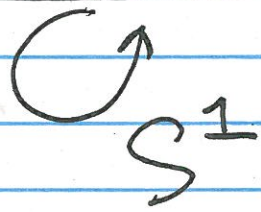
Rough idea:

$$\text{id} \in \text{Fun}_R(\text{Ind } \mathcal{C}, \text{Ind } \mathcal{C})$$

$$\text{Ind}[\mathcal{C}^{\otimes p} \otimes_R \mathcal{C}] \xrightarrow{\text{tr}} R\text{-mod}$$

$$\mathcal{F} \otimes_R \mathcal{G} \mapsto R\text{Hom}(\mathcal{F}, \mathcal{G})$$

$$\text{HH}^{\bullet, R}(\mathcal{C}) = \text{tr}(\text{id})$$



"Connes B "

$$\text{HC}^{\bullet, R}(\mathcal{C}) = [\text{HH}^{\bullet, R}(\mathcal{C})] S^1$$

linear / $C^*(BS^1) \simeq k[u]$

2) $\text{HH}^{\bullet, k}(\mathcal{C})$ module / $\text{HH}^{\bullet, k}(R)$

$$\text{HH}^{\bullet, R}(\mathcal{C}) \stackrel{S^1\text{-equivariantly}}{=} \text{HH}^{\bullet, k}(\mathcal{C}) \otimes_{\text{HH}^{\bullet, k}(R)} \text{HH}^{\bullet, k}(k)$$

$$HH^{\cdot R}(\mathcal{C}) = HH^{\cdot k}(\mathcal{C}) \otimes_{HH(R)} \text{End}_{HH(R)}^{(R)}$$

Prop $\text{End}_{HH(R)}^{(R)} \xrightarrow[\cong]{\text{Tate}} \mathcal{D}_R((u))$

Some derived version of differential ops

Thm

$$QC^!(LX) \xrightarrow[\cong]{\text{Tate}} QC^!(X_{\downarrow R})^{(u)}$$

Tate construction
t-reg
dg cat

$$\text{Ind} [DCoh(LX)]^{\text{Tate}}$$

$$\omega_{\hat{X}^S}((u)) \simeq \omega_{\hat{X}^{S^2}} \in QC^!(X^2)$$