Sabbah 3

Purity (1): Punctual purity

Weil number / pure of weight w alg nmbr x $\quad |x| = p^{w/2}$

all conjugates of x

Purity for $\overline{\mathbb{Q}}_\ell$-sheaves

All eigenvalues of F_x on X, $x \in |X|$ are pure of weight w wrt $p^{\deg x}$

With this in mind:

Thm (Katz, Laumon '85)

$$K \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \qquad X = \mathbb{A}^1_{\mathbb{F}_p}$$

Assume it is pure of weight w.

$\forall \psi$ ~~$\mathcal{F}_\psi$~~ $\mathcal{F}_\psi K|_{\hat{U} \underset{\text{open}}{\subset} \hat{\mathbb{A}}^1}$ smooth; pure of same weight

(it is a sheaf, i.e. concentrated in one degree)

Katz-Laumon had a wish: formulate something similar for $\mathcal{D}$-modules.

Pure lisse (smooth) $\overline{\mathbb{Q}}_\ell$-sheaves

$\updownarrow$ variations of polarized Hodge structures

X cplx mfld

V holo bdle w/ integrable ∇

$\ker \nabla = \mathcal{V}$ loc const sheaf of $\mathbb{C}$-vect spaces

H : C^∞-bundle, $\bar\partial$ op $\nabla + \bar\partial$

integrable C^∞ connection

<u>Dfn</u> Hodge decomposition:

$$H = \bigoplus_p H^{p,w-p}$$

C^∞ bundles

- $F^pH = \bigoplus_{p' \geq p} H^{p',w-p'}$ is stable by $\bar\partial$

$\rightsquigarrow F^pV \subset V$ holomorphic

$$\nabla(F^pV) \subset F^{p-1}V \otimes \Omega^1_X$$

(Griffiths transversality)

Q : operator on H:

s.-s. with eigenvalues

$p - w/2$ on $H^{p,w-p}$

<u>Polarization:</u>

$$H \otimes H \to C^\infty_X$$

$(-1)^w$ Hermitian flat sesquilinear form, nondeg

<u>Positivity:</u> $h(u,v) = k(e^{\pi i Q} u, v)$ pos def Hermitian

This is the standard dfn of a variation of polarized complex Hodge structures.

<u>Thm</u> (Griffiths, Schmid '73)

$X^* \subset X$ $(V, F^\bullet V, \nabla, k)$

punctured disc disc $\subset \mathbb{C}$ variation of polarized Hodge structures

<u>Then</u> $(j_* V)^{bn} \subset j_* V$ = sections whose norm is locally bounded

<u>And</u> ∇ has log pole.

<u>Thm</u>: $(j_* V)^{lb} \subset j_* V$

is a <u>free</u> $\mathcal{O}_X$-mod

$\equiv$

$(V, F^p V, \nabla, k)$ variation of polarized Hodge structures on $\mathbb{A}^1_{\mathbb{C}} \setminus \{c_1, \ldots, c_n\}$

<u>holomorphic</u> bundles ; no algebraic structure.

$\rightsquigarrow M$ reg hol $\mathcal{D}$-module on $\mathbb{P}^1$.

$\rightarrow$ algebraic

$\rightarrow$ ${}^F M$ a $\mathbb{C}[\tau]\langle \partial_\tau \rangle$-module

Sing: 0 reg $\quad \hat{U} = \mathbb{C}^*$

∞ irreg

<u>Yesterday</u>: explained how the behavior at ∞ is expressed through singularities of M.

<u>So</u>: This is NOT a variation of a Hodge structure.

<u>So</u>: how to <u>redefine</u> purity?

(Now, due to Mochizuki, know in any dimension)

Extending the notion of a Hodge strre:

Simpson, twistor structure...

<u>Twistor variable</u> $\left\{\begin{array}{l}\lambda \\ i\hbar \\ z \\ u\end{array}\right.$ $V, F^{\bullet}V, \nabla, k$

$$R_F V = \bigoplus_k F^k V \cdot z^{-k} = \bigoplus_\ell F_\ell V \cdot z^\ell$$

(loc) free $\mathcal{O}_X[z]$-module

<u>Pairing</u> σ: $\mathbb{P}^1 \circlearrowleft$

$\mathbb{C}_z \cup \mathbb{C}_{z'}$

$z = 1/z'$

$z \longmapsto -1/\bar{z}$

invol, no fixed pts

$\mathbb{C}^* \to \bar{\mathbb{C}}^*$

$\mathcal{H} = R_F V$

holomorphic bundle

$\mathcal{H}$ hol on $X \times \mathbb{C}_z$

$\sigma^* \bar{\mathcal{H}}$ hol on $\bar{X} \times \mathbb{C}_{z'}$

$S^1 = \{|z|=1\}$

$$
\begin{array}{c}
\mathcal{H}\big|_{X\times S^1} \otimes \sigma^* \overline{\mathcal{H}}\big|_{S^1} \\
\downarrow \\
\mathcal{C}^{\infty}_{X\times S^1}
\end{array}
$$

$\cancel{R_F k\left(\sum v_p z^{-p}, \sigma\right.}$

<u>Proposition</u> V Pol HS $\Rightarrow$ pairing nondegenerate on $X \times \mathbb{P}^1$ obtained by $\mathcal{H}^\vee$ with $\sigma^* \mathcal{H}$, is trivial in fibers $\{x\} \times \mathbb{P}^1$

$$R_F k\left(\sum v_p z^{-p}, \sigma^*\left(\sum w_q z^{-q}\right)\right) =$$

$$= \sum_{p,q} k(v_p, w_q) \cdot (-1)^q z^{p-q}$$

${}^F M \qquad (V, \nabla) \rightsquigarrow \mathcal{M}$

$(V, F^\bullet V, \nabla) \longrightarrow (\mathcal{M}, F_\bullet \mathcal{M}) \qquad R_F \mathcal{M}$

good filtration by Griffiths–Schmid

module / $\mathbb{C}[t, z]\langle z\partial_t \rangle$

${}^F(R_F M) = R_F M$ as $\mathbb{C}[z]$-mod;

∂_τ acts as t and ~~∂~~ τ as $-\partial_t$.

Integral formula:

$$\hat{\pi}_+ \left(\pi^+ R_F M \otimes E^{t\tau/z} \right)$$

<u>Punctual purity</u> on $U = \{\tau \neq 0\}$

Fix $\tau = 1$ for instance.
Get the standard Fourier $t \leftrightarrow 1/z$

Fibre at $\tau = 1$ of ${}^F(R_F M)$ is
a free $\mathbb{C}[z]$-module of finite rank

$(\mathcal{H})$

2) the pairing $\mathcal{C}$ defined from the
Fourier transform of k.

<u>Thm</u> This $(\mathcal{H}, \mathcal{C})$ produced by gluing
is a TRIVIAL vector bundle on $\mathbb{P}^1$.

× <u>So</u>: a "correct" definition:

<u>tame</u> twistor structure
≈ Fourier of such: tame twistor str^re on $\hat{U}$
by T. Mochizuki: can be extended to a [WILD TW. $\mathcal{D}$-mod

Sabbah 4

(Lect. 4 of the notes on the WEB: skipped (for now)).

Twisted de Rham complex.

Witten: $f: M \longrightarrow \mathbb{R}$ Morse

smooth compact

$\mathcal{A}^{\bullet}(M)$, $d + \tau\, df$ $\qquad \tau$ large

$\rightsquigarrow \Delta_\tau$ elliptic

acts on L^2 forms

eigenvectors concentrate their L^2 norm $\tau \to \infty$, in the neighborhood of the critical pts of f

$\forall K \subset M \quad K \cap \mathrm{Crit}(f) = \emptyset$

$\Rightarrow \exists A, C \quad \forall$ eigenvector η, $|\lambda| \leq A$,

$$\Rightarrow \qquad \|\eta\|_{L^2(K)} \leq \frac{C}{\tau} \|\eta\|_{L^2}$$

Alg geom$/\mathbb{C}$ $\qquad f: U \to \mathbb{A}^1_{\mathbb{C}}$ $\qquad U$ quasiproj

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, hook, "j"'] & \mathbb{A}^1_{\mathbb{C}} \\ X \arrow[ur, "F"'] & \end{tikzcd}$$

$X \setminus U$ divisor

F proj map

X quasiproj

(Compactify the fibers, not the values)

Complex of sheaves of vanishing cycles:

Vanishing cycles

$\phi_f \mathbb{C}_U$, also $\phi_{f-c} \mathbb{C}_U$

value $f=0$ $\quad c \in \mathbb{C}$

supported on $f^{-1}(\{c\})$

$\forall c \quad Rj_* \phi_{f-c} \mathbb{C} \qquad \phi_{F-c}(Rj_* \mathbb{C}_U)$

supp on $F^{-1}(c)$

$$f : U \to \mathbb{A}^1_{\mathbb{C}} \qquad \begin{array}{ccc} G_f & \hookrightarrow & U \times \mathbb{A}^1_{\mathbb{C}} \\ & & \downarrow p_2 \\ & t_c = t - c & \mathbb{A}^1_{\mathbb{C}} \end{array}$$

$\mu_{U \times \{0\}}(\mathbb{C}_{G_f})$ on $T^*_{U \times \{0\}} U \times \mathbb{A}^1$

section dt_c

Pull by this section.
Get a complex on U
supported on $t_c = 0 \cap G_f$

This sheaf comes w/ monodromy. (coming from $\mathbb{C}^*$ action).

<u>Twisted de Rham complex</u> $(\Omega^\cdot_U[\tau], d - \tau df)$

$\tau = 1$

Thm 1 $\forall i$, $\dim \mathbb{H}^i(U, (\Omega^\cdot, d - df)) =$

$$= \sum \dim \mathbb{H}^{i-1}(F^{-1}(c), \phi_{F-c}({}^R j_* \mathbb{C}_U))$$

(so: critical points at ∞ (in X) may influence the formula).

Pf $\mathbb{H}^i(\Omega^\cdot[\tau], d - \tau df) = GM^i(f)$

$\mathbb{C}[t]\langle \partial_t \rangle$ - holonomic module

Consider the Laplace transform.

${}^F(GM^i(f))$ is a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ module

$\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} {}^F GM^i(f)$ a free $\mathbb{C}[\tau, \tau^{-1}]$ - module of some rk

the rk = dim of any fibre on $\{\tau \neq 0\}$

$\dim \mathbb{H}^i(\Omega^\cdot_U, d - df)$

So, obs. 1: $\dim \mathbb{H}^i$ = rk of the fibre of the (Laplace-transformed) GM system.

For any regular holonomic module:

$$\operatorname{rk} \text{ of } \mathbb{C}[\tau,\tau^{-1}]\otimes F_M$$

$$\|$$

$$\sum_{c\in\mathbb{C}} \dim \phi_{t-c}\left(\mathcal{DR}(M)\right)$$

(mainly an application of the Kashiwara microlocal index theorem)

This is <u>not</u> vanishing cycles wrt f <u>but</u> the vanishing cycles wrt $\underline{t}$

<u>Now</u>: use compatibility of ϕ wrt <u>proper</u> pushforward.

<u>Twisted de Rham complex</u>

$$\left(\Omega^{\bullet}_U[\tau],\ d-\tau\, df\right)$$

<u>Notation</u>

$$\Big(\underbrace{H}_{\substack{\text{complex v.s.}\\ \dim<\infty}},\underbrace{T}_{\text{autom}}\Big) \xrightarrow{\widehat{RH}^{-1}} \mathbb{C}((z))\text{-vect. space}$$

$$\mathbb{C}((z))\otimes H$$

+ reg connection

$$d + M\frac{dz}{z} \qquad \exp 2\pi i M = T$$

$$(\Omega^\bullet[\tau, \tau^{-1}],\ d - \tau\, df)$$

$$ {}^F\widehat{GM}^i(f) = \mathbb{H}^i(\ \ \ ) $$

$$d - \frac{df}{z}$$

<u>Thm</u> $\quad {}^F\widehat{GM^i(f)} \cong \bigoplus_{c \in \mathbb{C}} \widehat{RH}^{-1}\Big(\mathbb{H}^{i-1}\big(F^{-1}(c),\ \phi_{F-c}(Rj_*\mathbb{C}_U)\big)\Big),\ \nabla - d(c/z)$

variant of stationary phase formula

<u>Kontsevich question</u>:

$$\mathbb{H}^i(U, (\Omega^\bullet_U((z)),\ d - df/z)\ ?$$

<u>Thm</u> (part: Kapranov '89, M. Saito, C. Sabbah)

same formula but with $\phi_f\, \mathbb{C}_U$

Pf Computation at ∞:
to see that $\Omega^\bullet((z))$ $d - df/z$ does not take into account the vanishing cycles on $X \setminus U$.

<u>Note</u>: $(\Omega^\bullet_U[z,z^{-1}],\ zd-df)$
$$\cup$$
$(\Omega^\bullet_U[z],\ zd-df)$ a complex of $\underline{\underline{\mathbb{C}[z]}}$-modules

+ a connection with pole of order 2.

<u>Q2</u> Is $\mathbb{H}^i(U, \Omega^\bullet_U[z], zd-df)$ free over $\mathbb{C}[z]$? (of the same rk, i.e. $\dim H^i(\Phi_{cpt\ fibres})$?)

<u>Meaning</u>: $\dim \mathbb{H}^\bullet(U, (\Omega^\bullet_U, df)) \overset{?}{=} \sum \dim H^{i-1}(F^{-1}(c), \phi)$?

<u>Thm</u> (Barannikov-Kontsevich): YES
Pf. C.S., Ogus-Vologodsky

<u>Use Hodge thy</u>. Analogous to degeneration at E_1 in Hodge thy.

Change settings. Use u instead of z and periodic cyclic homology.