Sabbah 2

Stokes structures

Aim to find a category of

hol $\mathcal{D}_X$ - mod $\underset{RH}{\longleftrightarrow}$ Stokes-perverse sheaves

- X _ curve
- partial results in higher dim

Interests: notion of a $\mathbb{Q}$-structure on a holom $\mathcal{D}$ mod (Mochizuki...)

Y top space ; $\mathcal{J}$ sheaf of ordered ab grps on Y

$\mathcal{J}^{\text{ét}} \to Y$ loc homeo

$U \subset Y$ open $\quad \Gamma(U, \mathcal{J}) = \{ \underset{\text{CONT}}{s : U \to \mathcal{J}^{\text{ét}}}, \mu \circ s = id$

$y \in Y \quad \mathcal{J}_y = \mu^{-1}(y)$ discrete top

$\forall y \in Y, \forall \varphi, \psi \in \mathcal{J}_y, \quad \varphi \underset{y}{\leq} \psi$ order defined

$\Rightarrow \exists V \ni y$ s.t. $\varphi \leq \psi$

(~~filtrati~~ order is OPEN)

Pre-$\mathcal{I}$-filtration of a sheaf $\mathcal{F}$ on Y

subsheaf $\mathcal{F}_{\leq} \subset \mu^{-1}\mathcal{F}$

$$(\mu^{-1}\mathcal{F})_{\varphi} = (\mu^{-1}\mathcal{F})_{\psi} = \mathcal{F}_y$$

s.t. $\forall y \in Y$, $\forall \varphi, \psi \in \mathcal{I}_y$,

$$\varphi \leq_y \psi \Rightarrow \mathcal{F}_{\leq\varphi} \subset \mathcal{F}_{\leq\psi} \subset \mathcal{F}_y$$

<u>Example</u> $\mathcal{I}$ = constant sheaf $\mathbb{Z}_Y$

$\mathcal{F}_{\leq p} \subset \mathcal{F}_y$ - <u>usual</u> filtration indexed by $\mathbb{Z}$

$$\mathcal{I}^{\text{ét}} = Y \times \mathbb{Z}$$

$\mathcal{M}$ a <u>germ</u> of $\mathcal{D}_X$-modules in one variable holonomic:

$\mathcal{M} = \mathbb{C}\{t\}\langle\partial_t\rangle$ $\forall m$ $\exists P$: $Pm = 0$

$\hat{\mathcal{M}} = \mathbb{C}[[t]]\langle\partial_t\rangle$

<u>Theorem</u> (Levelt-Turrittin)

① $\hat{\mathcal{M}} = \hat{\mathcal{M}}^{reg} \oplus \hat{\mathcal{M}}^{irreg}$

② $\hat{\mathcal{M}}^{irreg}$ is a $\mathbb{C}((t))$-vector space (t. bijective)

③ $\hat{\mathcal{M}}^{irreg} = \bigoplus El(\rho, \varphi, R)$

$\rho \in u\,\mathbb{C}[\![u]\!] \quad v_u(\rho) = p \geq 1$

$\varphi \in \mathbb{C}(\!(u)\!)/\mathbb{C}[\![u]\!]$ ord of pole $= q$

$El(\rho, \varphi, R_\varphi) := \rho_+ (R_\varphi, \nabla + d\varphi \cdot Id)$

is a $\mathbb{C}(\!(t)\!)$-vector space

(An irregular connection, pushed forward by a ramification of order p).

Basic inv.

Slope: q/p

rk: pr irregularity number: qr

This is the formal case.

What about $\mathcal{M}$? (on $\mathbb{C}\{t\}\langle\partial_t\rangle$)

$\mathbb{C}\{t\}$ = ring of convergent series

Answer: add the Stokes data.

Stokes filtration.

First: the sheaf $\mathcal{I}$. Also space Y? sheaf $\mathcal{F}$?

Y: assume that $\mathcal{M}$ is a $\mathbb{C}\{t\}$ - vector space
i.e. $t : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$.

$\mathcal{M}$ is defined on some small disc Δ_t

$$\Delta_t^* \leftarrow \text{local system } \mathcal{L} = \operatorname{Ker} D_{\partial_t}$$

$$\tilde{\Delta}_t = S^1 \times [0, \varepsilon) \to \Delta_t$$

$$(e^{i\theta}, r) \mapsto r \cdot e^{i\theta}$$

Extend $\mathcal{L}$ to $\tilde{\Delta}$; restrict to S^1.

$$S^1 \times \{0\} = Y.$$

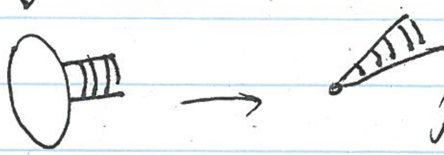
Sheaf $\mathcal{I} = \varinjlim_{d \in \mathbb{N}} \mathcal{I}_d$

$\mathcal{I}_1$ = constant sheaf on S^1 with fiber

$$\mathcal{P}_t = \mathbb{C}((t)) / \mathbb{C}[[t]]$$

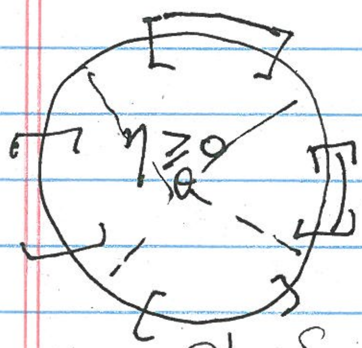
order : $\eta = \eta_q(t)\, t^{-q}$ $\quad \eta_q(0) \neq 0$

$\eta \underset{\theta}{\leq} 0 \iff \eta \equiv 0$ or $\mathrm{Re}(\eta) < 0$
in the neighborhood of θ



$$\mathrm{Arg}(\eta_q(0) - q \cdot \theta) \in \left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)$$

$q=2$



$S^1 \times \{0\}$

$2q$ intervals of same length.

On boundary points: η and 0 not comparable.

$\mathcal{J}_d$ locally constant sheaf on S^1 with fiber $\mathcal{P}_u = \mathbb{C}((u))/\mathbb{C}[[u]]$

$t = u^d$

Monodromy $T(\varphi(u)) = \varphi(\zeta_d u)$

$$\zeta_d = \exp\left(\frac{2\pi i}{d}\right)$$

Similarly we defined ~~monodromy~~ order on $\mathcal{J}$.

$$\mathcal{J}_1 \subseteq \mathcal{J}_d \subseteq \mathcal{J}_{d'}$$

$$d | d'$$

$\mathcal{J} = \varinjlim_{d} \mathcal{J}_d$ contains all possible ramified polar parts.

$Y = S^1$
$\mathcal{J} = \varinjlim \mathcal{J}_d$

$\mathcal{F}$ = loc syst defined by $\mathcal{M}|_{\Delta^*}$

<u>Def</u> A Stokes filtration is a $\mathcal{J}$-filtration on $\mathcal{L}$ (Deligne '78).

$$\mu : \mathcal{J}^{\text{ét}} \to S^1$$

$$\mathcal{J}_1^{\text{ét}} \underset{\text{top}}{=} \coprod_{\varphi \in P_t} S^1 \times \{\varphi\} \xrightarrow{\mu} S^1$$

$$\mathcal{J}_1 = \coprod_{d \geq 1} \coprod_{\varphi \in P_u^d} S_u^1 \times \{\varphi\} \xrightarrow{\text{deg } d \text{ covering}} S_t$$

$$\mathcal{L}_{\leq} \subset \mu^{-1}\mathcal{L} \qquad \text{pre-}\mathcal{J}\text{-filt}^{\text{on}}$$

($\mathcal{F}_{\leq} \subset \mu^{-1}\mathcal{F}$; may define subsheaf $\mathcal{F}_{<} \subset \mathcal{F}_{\leq} \subset \mu^{-1}\mathcal{F}$
if $\mathcal{J}^{\text{ét}}$ is Hausdorff then $\mathcal{F}_{\leq}$ is a sheaf...)

pre $\mathcal{J}$-filtration $\mathcal{L}_{\leq} \subset \mu^{-1}\mathcal{L}$ pre $\mathcal{J}$-filt$^{\text{on}}$

$$\mathcal{L}_{\leq} \supset \mathcal{L}_{<}$$

$\mathcal{L}_{<}$ defined; $\text{gr}\,\mathcal{L} = \mathcal{L}_{\leq}/\mathcal{L}_{<}$

$\mu_!\, \text{gr}\,\mathcal{L}$ is a local system of same rank as $\mathcal{L}$.

Thm The category of Stokes-filtered local systems is abelian. Any morphism is strict.

Morphism: $\lambda = (\mathcal{L}, \mathcal{L}_\bullet) \to (\mathcal{L}', \mathcal{L}'_\bullet)$

by definition: λ is a morphism of loc systs of sheaves; $\mu^{-1}(\mathcal{L}_\leq) \subseteq \mathcal{L}'_\leq$

Strict: $\mu^{-1}(\lambda)(\mathcal{L}'_\leq) = \mathcal{L}'_\leq \cap \cancel{\mu^{-1}(\lambda)(\mu^{-1}\mathcal{L})}$

$\mu^{-1}(\lambda)(\mu^{-1}(\mathcal{L}))$
$\parallel$
$\mu^{-1}(\lambda)(\mathcal{L})$

Thm (Deligne) <u>Equivalence</u>

$\mathcal{M}$ Merom Connect $\mathcal{M} \xrightarrow{SDR}$ $\mathbb{C}$-Stokes-filtered loc systs $(\mathcal{L}, \mathcal{L}_\bullet)$

$\parallel$

$\mathbb{C}\{t\}$ - vect space fin dim w/connection

$\downarrow$ graded

formal $\widehat{\ }$

$\mathrm{gr}\,\mathcal{L}$

$\widehat{\mathcal{M}}$ Formal merom connections $\xrightarrow{\quad SDR \quad}$ <u>Graded</u> Stokes filtered local systems

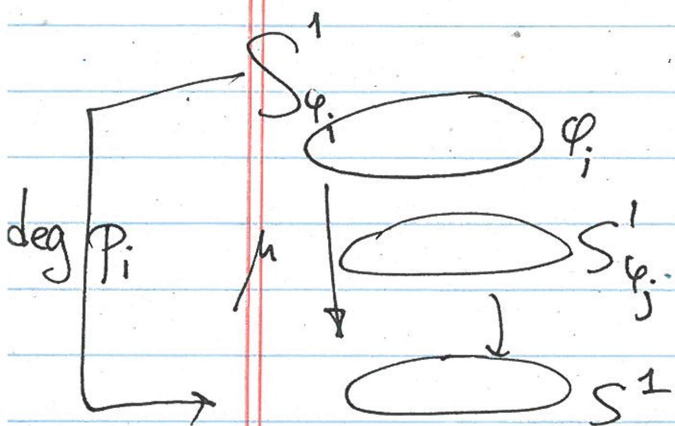
$\mu_!\,\mathrm{gr}\,\mathcal{L} \underset{loc}{\sim} \mathcal{L}$

<u>glob</u>

More explicit description of $(\mathcal{L}, \mathcal{L}_\cdot)$ on S^1

Finite # of $\varphi_i \in \mathbb{C}((t_i))/\mathbb{C}[[t_i]]$

ex. $\varphi_i = \frac{c_i}{t}$



On each circle:

$\mathrm{gr}_{\varphi_i} \mathcal{L}$

local system on $S^1_{\varphi_i}$

loc on S^1

$$\mathcal{L}_\theta \underset{\text{loc}}{\simeq} \bigoplus_i \mu_! \, \mathrm{gr}_{\varphi_i} \mathcal{L}_\theta$$

dep on θ

Given φ, what is $\mathcal{L}_{\leq\varphi}$?

Each circle is decomposed into finite # of segments ("green"

$$\mathcal{L}_{\leq\varphi,\theta} \subset \mathcal{L}_\theta$$

$$\| $$

$$\bigoplus_{\substack{\text{segments} \\ \text{"green at } \theta\text{"}}} \mu_! \, \mathrm{gr}_{\varphi_i} \mathcal{L}$$

② Define the RH functor.

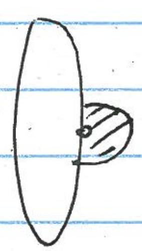
$\mathcal{M}$ meromorphic connection on Δ_t
($\mathcal{D}_{\Delta_t}$ - mod)

$\mathcal{L}$: $DR(\mathcal{M}|_{\Delta^*}$ $\quad \tilde{\Delta}$ oriented real blow-up space

$\mathcal{L}$ on S^1 : $DR(\tilde{j}_* \mathcal{O}_{X^*} \otimes \mathcal{M})|_{S^1 \times \{0\}}$

all holomorphic.
local system on S^1

Conditions on the coefficients:
Moderate growth condition



$\forall K$ nbhd of θ
cpct
$\exists$ cst C, N:

$$|f|_{K^*} \leq C r^{-N}$$

$$\mathcal{L}_{\leq 0} \subset \mathcal{L}$$
$\shortparallel$
$\mathcal{H}^0(DR(\mathcal{A}^{mod} \otimes \mathcal{M}))$; $\mathcal{L}_\varphi = \mathcal{L}_{\leq 0}$ tw. by e^φ

(replace $(\mathcal{M}, \nabla)$ with $(\mathcal{M}, \nabla + d\varphi)$)

<u>Rmk</u> $\mathcal{H}^j\left(\gamma R\left(\mathcal{A}^{\mathrm{mod}} \otimes \mathcal{M}\right)\right) = 0 \quad \forall j \geq 1$

(Hukuhara-Turritin).

i.e. can solve a D.E. with moderate coefficients.

$\equiv$

X <u>Riemann surface</u>. D reduced divisor.

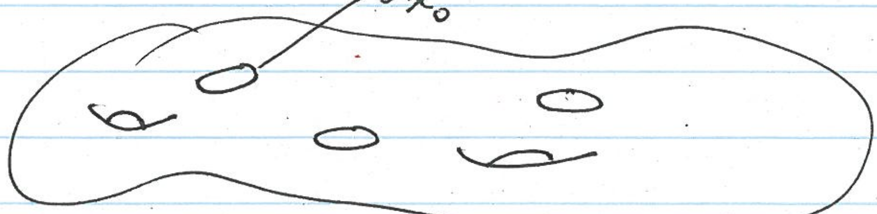
Start with meromorphic connections on X with poles on $D \Leftrightarrow \mathcal{O}_X(*D)$-free mod finite rank with connections.

($\Leftrightarrow$ holonomic $\mathcal{D}_X$-modules that are locally free $\mathcal{O}_X$-mod on X^* and that are equal to their localization).

<u>Stokes filtered local system</u>

<u>Data:</u> $\widetilde{X}(D)$ real bl.up space at D

$S^1_{x_0}$



R. S. with ∂

$$\mathcal{I} = \begin{cases} 0 & \text{on } X^* \\ \mathcal{I}_{x_0} & \forall x_0 \in D \end{cases}$$ as above

$\mathcal{F}$ loc syst on X^*; extend as a loc syst on $\tilde{X}(D)$.

$\mathcal{F}_{\leq} \subseteq \mu^{-1}\mathcal{F}$ such that over each $S^1_{x_0}$, this is a Stokes filtration of $\mathcal{F}|_{S^1_{x_0}}$.

$\mu : \mathcal{J}^{\text{ét}} \to \tilde{X}(D)$

$\mu|_{X^*}$ homeo

Poincaré duality

$$0 \to \mathcal{F}_{<} \to \mathcal{F}_{\leq} \to \mathrm{gr}\,\mathcal{F} \to 0$$

$$\mathcal{F}_{<} : \begin{cases} \mathcal{F} & \text{on } X^* \\ \mathcal{F}_{<} & \text{on } S^1_{x_0} \quad \forall x_0 \in D \end{cases}$$

$$0 \to \mathbb{D}(\mathcal{F}_{\leq}) \to \mathbb{D}(\mathcal{F}_{<}) \to \mathbb{D}(\mathrm{gr}\,\mathcal{F})[1] \to 0$$

a stokes filtration on $\mathbb{D}\mathcal{F}$

Stokes perverse sheaves

Perverse sheaf on X with sing $\subset D$.

$\longleftrightarrow$ $\mathcal{F}$ loc syst on X^* $\quad$ $\mathcal{G}_{x_0}$ perverse sheaf on Δ_{x_0} $\quad$ gluing iso $(\mathcal{G}_{x_0} \simeq \mathcal{F})|_{\partial\Delta_{x_0}}$

$\widetilde{X}(D)$

Deligne: define the completion of $\widetilde{X}(D)$ by filling the discs. These discs are "formal" discs $\hat{\Delta}_{x_0}$.

Stokes perverse sheaf: $(\mathcal{F}, \mathcal{F}_\bullet)$ Stokes filtered local system on $\widetilde{X}(D)$.

$\mathcal{G}_{x_0}$ (usual) perverse sheaves $\hat{\Delta}_{x_0}$ iso: $\mathcal{G}_{x_0}\Big|_{\partial\hat{\Delta}_{x_0}} \simeq gr_0 \mathcal{F}$

1) Abelian category (can be obtained from a t-structure of Stokes-constructible sheaves)
2) Duality functor
3) RH correspondence

$[(\mathcal{F}, \mathcal{F}_\bullet), (\mathcal{G}_{x_0})_D, \text{iso}]$ Stokes perverse sheaf

Define a true sheaf on $\widetilde{X}(D)$

$\mathcal{F}_{\leq 0}$ on $\widetilde{X}(D)$; $(\mathcal{G}_{x_0})$ iso $\mathcal{G}_{x_0}\big|_{\partial\hat{\Delta}_{x_0}} \simeq gr_0 \mathcal{F}$

only depends on $\mathcal{F}_{\leq 0}$

complex of sheaves on
$$\begin{array}{ccc} \widetilde{X}(D) & & \widehat{\Delta}_{x_0} \\ \downarrow \pi & & \downarrow \\ X & & x_0 \end{array}$$

$R\pi_*$ of the glued sheaf

is the perverse sheaf $\leftrightarrow DR(\mathcal{M})$

This is the theory in dimension one.
Completed in the 80s by Deligne-Malgrange.

Q: why don't Stokes <u>lines</u> appear?
A: they appear if you want to integrate smth

<u>Sabbah 3</u>

<u>Stokes structures in higher dimensions</u>

By the '80s: for Riemann surfaces, understood. Why stopped <u>there</u>?

<u>Pb</u> How to generalize the Levelt-Turrittin?

<u>dim 1</u>: $\widehat{M}$ $\mathbb{C}((t))$ - vector space $+ \nabla$

$$\widehat{\mathcal{M}} = \bigoplus El(\rho, \varphi, R_\varphi)$$

$\mathcal{M}$ loc free $\mathcal{O}_X(*D)$ - ~~bundles~~ mods, fin rank, flat ∇

$x_0 \in D$ $\qquad \mathcal{M}_{\widehat{X|x_0}} = \widehat{\mathcal{O}}_{X,x_0} \otimes_{\mathcal{O}_{X,x_0}} \mathcal{M}_{x_0}$.

Can we understand <u>these</u>?

<u>Generically</u>, yes. Bad points: not only singular points of D, but some smooth points of the divisor (Y. André: <u>turning</u> points)

<u>Conj</u> Given $\mathcal{M}$, $\exists \pi : X' \to X$, $D' \to D$ proper modification s.t. $\pi^*(\mathcal{M}, D)$ has "formal structure at each $x_0' \in D_0'$

Y. André, Inventiones '07

<u>Proof</u> C.S. small rank case; ... 2000

2008-2010: Two proofs:
- T. Mochizuki (alg setting)
- K. Kedlaya also works in loc anal setting.

Mochizuki: a) surfaces. b) general case

Reduces to char $p \gg 0$.

In char p: ∂ vector field $\Rightarrow \partial^p$ vector field.

$$(D_\partial)^p - D_{\partial^p} = p\text{-curvature of } D$$

Plays a central role in Mochizuki's proof for surfaces.

Similar role in arbitrary dimension: Higgs field.

This opens the door to studying Stokes structures in higher dimensions.

Stokes filtered local systems.

X complex manifold.
D strict normal crossings divisor.

$\widetilde{X}(D) \xrightarrow{\varpi} X$ oriented real blowup of X along all components of D.

Locally: $D = (t_1 \cdots t_\ell = 0)$

$$\widetilde{X}(D) = (S^1)^\ell \times [0,\varepsilon)^\ell \times \underset{\text{polydisc}}{\Delta_{n-\ell}}$$

Sheaf $\mathcal{J} = \varinjlim_{d} \mathcal{J}_d$

for simplicity work with $\mathcal{J}_1 = \varpi\, \mathcal{O}_X(*D) / \mathcal{O}_X$

$$\mathcal{J}_1^{\text{ét}} = \left(\mathcal{O}_X(*D)/\mathcal{O}_X \right)^{\text{ét}} \times_X \tilde{X}(D)$$

NOT Hausdorff near crossing points of D

$D = \{t_1 t_2 = 0\}$ $\quad \frac{c}{t_1}$ is very near to 0

$$D_I^0 = \bigcap_{i \in I} D_i \setminus \bigcup_{j \notin I} D_j$$

Order $\underline{\theta} = (\underline{\theta}, \underline{0}, x_0')$

$$\eta \in \mathcal{O}_X(*D)/\mathcal{O}_X$$

$$\eta \cancel{\leq \underline{\theta}\ \theta}$$

$$\eta \leq_{\underline{\theta}} 0 \iff e^{\eta} \ldots$$

$\underline{Ex}$ $\quad \eta = \frac{1}{t_1} + \frac{1}{t_2} = \frac{t_1 + t_2}{t_1 t_2}$

Consider polar parts

$$\eta = t^{-\underline{q}} u_q(t) \qquad u_q(t) \text{ hol}$$
$$u_q(0) \neq 0$$

$$\eta \leq_{\underline{\theta}} 0 \iff \arg u_q(0)$$

<u>Definition</u> (of <u>good</u>)

Given $\Phi \subset \mathcal{O}_{X,x_0}(*D) / \mathcal{O}_{X,x_0}$

finite subset

Φ is good if $\forall \varphi, \psi \in \Phi$,

$\varphi - \psi$ is monomial (or zero)

<u>Lemma</u> $\quad \Phi$ good at $x_0 \Rightarrow \Phi$ good on nbhd (x_0)

goodness is an open property

<u>Def</u> Stokes-filtered loc sys

$(\mathcal{F}, \mathcal{F}_{\leq})$

$\mathcal{F}$ is a local system on $X^* = X \setminus D$

(extend by $\tilde{j}_* \mathcal{F}$ to $\tilde{X}(D)$).

$$R\tilde{j}_* \mathcal{F} \qquad \begin{array}{c} \tilde{X}(D) - \partial\tilde{X}(0) \\ \downarrow \\ \tilde{X}(D) \end{array}$$

$\mathcal{F}_{\leq} \subset \mu^{-1} \mathcal{F}$

$\mu : \quad \mathcal{J}^{\text{ét}} \to \tilde{X}(D)$

- filtration condition
- local splitting condition

need $\mathcal{F}_{<}$

$< \iff \leq$ and $\neq$

<u>But</u>: × the étale space is non-Hausdorff
× the $=$ condition is not closed
× the $<$ condition is not open.

So this is <u>not</u> a sheaf at crossing pts

$gr\,\mathcal{F}$ only defined over each stratum

$\Rightarrow$

$gr\,\mathcal{F}|_{D_I^\circ}$ is a local system on its support

$\Sigma_I \to \partial\widetilde{X}(D)|_{D_I^\circ}$

And you have compatibility condition as in Sasha Beilinson's talk.

Fix $\Sigma = \bigcup \Sigma_I$ a good stratified covering of $\partial\widetilde{X}(D)$

$\bigcap$ jet

fiber is I.e. good at any point

Thus the category of stokes filtered local systems whose support is contained in Σ is an Abelian category.

Morphisms are strict.

D. Kaledin: relation to log geometry?

Thm (RH) $\mathcal{M}$ a loc free $\mathcal{O}_X(*D)$-module finite rank, integrable D

Assume that at each $x_0 \in D$

$\mathcal{M}_{\widehat{X, x_0}}$ has a formal structure which is GOOD

then $\mathcal{H}^j DR(\mathcal{A}^{mod}_{\tilde{X}(D)} \otimes \mathcal{M}) = 0, \; j > 0$

and

$$\mathcal{M} \mapsto \mathcal{H}^0\left(DR(\mathcal{A}^{mod}_{\tilde{X}(D)} \otimes \mathcal{M}, D + D\varphi\right)$$
$$= \mathcal{F}_{\leq \varphi}$$

Stokes filtered local system

is an equivalence of categories.

E^1-degeneration $\quad X \xrightarrow{f} \mathbb{C}$

$f^{-1}(0) = D$ strict normal crossings

$\mathcal{M}$ meromorphic on (X, D)

Assume it is <u>good</u>. If not, resolve by Kedlaya–Mochizuki thm).

$f_+(\mathcal{M}, \nabla)$ Gauss–Manin connection.

Can be irregular. Has Levelt–Turrittin decomposition. Corresponds to a Stokes-filtered local system on S^1.

$(\mathcal{F}, \mathcal{F}_{\leq})$ attached to $\mathcal{M}$ by RH corr.

$f : X \to \mathbb{C}$ can be lifted to

$\tilde{f} : \tilde{X}(D) \to \tilde{\mathbb{C}}$ (f in polar coords)

Given any $\varphi \in \mathbb{C}((t))/\mathbb{C}[[t]]$: $\varphi = t^{-q} u_\varphi(t)$

$$f^*\varphi = f^{-q}(u_\varphi \circ f)$$

Can define $\mathcal{F}_{\leq f^*\varphi} \subset \mathcal{F}|_{\partial \tilde{X}(D)}$

$R^j f_* \mathcal{F}_{\leq f^*\varphi}$ sheaf on S^1

"E_1 degeneration of the Stokes filtration":

The map

$$R^j f_* \mathcal{F}|_{\partial\widetilde{X}(0)} \quad \text{loc syst on } S^1$$
$$\uparrow$$
$$R^j f_* \mathcal{F}_{\leq f^*\varphi} \quad \text{sheaf on } S^1$$

inj; defines a Stokes filtered loc syst.

<u>Thm</u> In this case, E_1-degeneration holds

<u>Proof</u>: Mochizuki. Goes through the RH correspondence.

DR compatible w/ proper pushforward.

not easy [Commutation of $DR(\mathcal{A}^{mod}_{\widetilde{X}})$ w/ proper pushforward)

Through analysis.