Schedler 2

<u>Thm</u> If X has finitely many sympl. leaves then $M(X)$ is <u>holonomic</u>.

(in fact iff)

<u>Example of $M(X)$</u> X symplectic: $M(X) \simeq \Omega_X$ sheaf of volume forms

$X = \mathrm{af}$

The map $M(x) \longrightarrow \Omega_X$

$1 \longmapsto \mathrm{vol}(x)$

inj: it is so for gr.

If $Y \subseteq X$ a symplectic leaf,

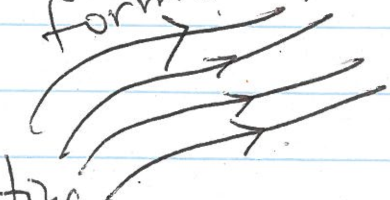
$i : \bar{Y} \hookrightarrow X$ closed Poisson embedding.

$M(X) \twoheadrightarrow i_* M(\bar{Y})$ but $M(\bar{Y})_X = \Omega_Y$

If you have ∞ many sympl. leaves: at some point

$\hat{X}_x \simeq V \times U$

formal sympl leaves



(perhaps after restricting to a subvariety...)

Proof of Thm

$X = \bigsqcup_i X_i$ sympl leaves

$SS(\mathcal{M}(X)) \subseteq \bigsqcup T^*_{X_i} X$

Conormal bdles

$$\boxed{\begin{matrix} H(x)|_x & \Rightarrow \text{Lagrangian} \\ \| & \\ T_x X_i & \text{if } x \in X_i \end{matrix}}$$

Structure of $\mathcal{M}(X)$:

By the proof, $\mathcal{M}(X)$ has a finite composition series.

Comp. factors are loc systs on symplectic leaves.

On open leaf, we get Ω_U.

Ex of computation of $\mathcal{M}(X)$

$X = \mathbb{C}^2 / \{\pm 1\} =$  $\subseteq \mathbb{C}^3$

$\operatorname{Spec} \mathbb{C}[x^2, xy, y^2]$

$\operatorname{Spec}[u, v, w]$

$uw = v^2$

$\xi_f = \{f, \bullet\}$

Explicitly:

$$M(X) = \mathcal{D}_{\mathbb{C}^3} \Big/ \langle \xi_u, \xi_v, \xi_w, uw - v^2 \rangle \mathcal{D}_{\mathbb{C}^3}$$

Prop $\quad M(X) \simeq IC(X) \oplus \delta_0$

$$IC(X) = j_{!*}\, \Omega_{X \setminus \{0\}}$$

$= $ irr $\mathcal{D}$-mod s.t. $IC(X)|_{X \setminus \{0\}} = \Omega_{X \setminus \{0\}}$

We use:

$\mathrm{Ext}(IC(X), \delta_0) = 0$

$\cong$

$H^1(X \setminus \{0\} / \mathbb{C}^*) = 0$

$$\mathrm{Hom}_{\mathcal{D}_X}\left(M(X), \delta_0\right) =$$

$$= (\delta_0)^{H(X)} = \left(\hat{\mathcal{O}}^*_{X,0}\right)^{H(X)}$$

$$= \left(\mathcal{O}^*_X\right)^{H(X)}$$

because $\mathcal{O}_X$ positively graded

$$\mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\} \simeq \mathbb{C}$$

$$HP_0(\mathcal{O}_X)^* = \langle \text{augmentation} \rangle$$

<u>Generalization to homogeneous hypersurfaces in $\mathbb{C}^3$.</u>

$Q = 0$ degree d

Naturally Poisson : $(\partial_u \wedge \partial_v \wedge \partial_w)\, dQ$

finitely many $\Leftrightarrow$ X has isolated singularity

(0-dim leaves = singular locus)

<u>Thm</u> If X has isolated singularities

$$M(X) \cong N \oplus \delta_0^{\mu - g}$$

$\mu =$ Milnor # $(= (d-1)^3)$

$$g = \frac{(d-1)(d-2)}{2}$$

$= \text{genus of } X \setminus \{0\} / \mathbb{C}^*$

$$0 \to j_! \Omega_{X \setminus \{0\}} \xrightarrow{inj} N \twoheadrightarrow \delta_0^g \to 0$$

$$\dim \mathrm{Ext}(IC(X), \delta_0) = \dim H^\circ(X \setminus \{0\})$$
$$= 2g$$

$$0 \to \delta_0^{2g} \to j_! \Omega_{X \setminus \{0\}} \twoheadrightarrow IC(X) \to 0$$

Note $\quad HP_0(\mathcal{O}_X) \simeq \mathbb{C}^{\mu}$

with polynomial grading, get Jacobi ring

Thm (Alev, Lambre) $\quad HP_0(\mathcal{O}_X) \simeq$ Jacobi ring in quasi hom setting

One more example

Thm (ES) $\quad HP_0(\mathcal{O}_{V/G}) \simeq \bigoplus_{\substack{K<G \\ \text{parabolic}}} IC\left(V^K \big/ (N(K)/K)\right) \otimes HP_0\left((V^K)^* / K\right)$

Conj (Etingof - S)

if sympl res: $\quad HP_0(\mathcal{O}_X) \simeq H^{\dim X}(\tilde{X})$

$$\boxed{p_* \Omega_{\tilde{X}} \simeq M(X)}$$

($\exists$ sympl res $\Rightarrow$ fin. many sympl. leaves)

Conj's proved in cases: $X = \mathbb{C}^2/\Gamma \twoheadleftarrow \widetilde{\mathbb{C}^2/\Gamma}$

$\mathrm{Sym}^m(\mathbb{C}^2/\Gamma) \twoheadleftarrow \mathrm{Hilb}^m \widetilde{\mathbb{C}^2/\Gamma}$

In all cases: explicit computation that works in these cases

$N = \mathrm{Nil}(\mathfrak{g}) \overset{\twoheadleftarrow}{\underset{p}{}} T^*(G/B)$

$S_e \cap N \twoheadleftarrow p^{-1}(S_e \cap N)$

$\mathrm{Sym}^m Y \twoheadleftarrow \mathrm{Hilb}^m Y$

Y - smooth surface symplectic

$\equiv$

X affine Poisson (nonneg graded)

A filtered quantization $A = \bigcup_{m \geq 0} A_{\leq m}$

$(\mathrm{gr} A, \{,\})$

$$HP_0(\mathcal{O}_X) \twoheadrightarrow \mathrm{gr}\, HH_0(A)$$

$\rho : A \to \mathrm{End}(V)$ $\quad \mathrm{tr}(\rho) \in HH_0(A)^*$

<u>Thm</u> $\rho_1, \dots, \rho_m$ distinct $\Rightarrow$ $\mathrm{tr}\rho_1, \dots, \mathrm{tr}\rho_m$ lin. independent

$$\# \text{ f. d. irreps } A \leq \dim HH_0(A)^* \leq \dim HP_0(\mathcal{O}_X)^*$$

In particular if X has finitely many symplectic leaves then A has finitely many fin dim representations

Ex $\mathcal{U}(sl_2)/(C-\lambda)$ $\quad C = ef + fe + \frac{1}{2}h^2$

thm (Alev-F-Lambre-Solotar)

$$\dim HH_0\left(\mathrm{Weyl}(V)^G\right) = \# \text{ conj classes of } g \in G: g - I \text{ invertible}$$

$$HH_0\left(\mathrm{Weyl}(V)^G\right) \cong HP_0(\mathcal{O}_{V/G})$$

Conj If $\tilde{X} \to X$ symplectic resolution then $HP_0(\mathcal{O}_X) \xrightarrow{\sim} HH_0(A)$ is an isomorphism

Dfn $HP_*^{DR}(X) = \pi_* \mathcal{M}(X)$ derived $\quad \pi: X \to pt$

There is another D-module $\mathcal{M}(X)_A$ using any quantization A of X

$$HH_*^{DR} = \pi_* \mathcal{M}(X)_A$$

Conjecture $HH_*^{DR}(X) \simeq HP_*^{DR}(X)$

Would like: If $f: Y \twoheadrightarrow X$ proper, birational, semi-small:

$$f_* \mathcal{M}(Y) \simeq \mathcal{M}(X)$$