Tamarkin

$X/\mathbb{F}_q$

Trig sums: analogs of oscillatory integrals

$$\int_X e^{itf(x)}\omega \qquad X \text{ smooth} \qquad t\to\infty$$

$\wr$

$\sum$ over critical pts of f

Analogue/ $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$X/\mathbb{F}_p \qquad f\in\Gamma(X,\mathcal{O}_X) \qquad f: X\to\mathbb{A}^1$

$x\in X(\mathbb{F}_p)$

$$\sum_{x\in X(\mathbb{F}_p)} e^{\frac{2\pi i f(x)}{p}}$$

Étale topology; sheaves Artin first: étale maps very important

Étale map: Analogue of local diffeos

Ex $\quad \mathbb{C}\setminus 0 \xrightarrow{z^n} \mathbb{C}\setminus 0$

$$\mathbb{C} \setminus 0 \xrightarrow{z^n} \mathbb{C}^n \qquad \text{still étale}$$

$$\begin{matrix} \mathbb{C} \setminus 0 \xrightarrow{z^n} \\ \sqcup \\ \mathbb{C} \setminus 0 \xrightarrow{z^{m+1}} \end{matrix} \mathbb{C}$$

Let A, B be rings (unital, comm)

A map $f: A \to B$ is <u>étale</u> if:

1) B of finite type $/A$

$\exists\, x_1, \dots, x_n \in B$:

$$\cancel{A/} \cdot A[x_1, \dots, x_n]/J \xrightarrow{\sim} B$$

2) Geometrically: $\operatorname{Spec} B \subset \operatorname{Spec} A \times \mathbb{A}^n$;

$\forall\, \mathfrak{p} \in \operatorname{Spec}(B)$ ($\mathfrak{p}$ is a prime ideal in $A[x_1, \dots, x_n]$; $\mathfrak{p} \supset J$)

$\exists\, f_1, \dots, f_n \in J$ s.t.

$$I_{\mathfrak{p}} = A[x_1, \dots, x_n]_{\mathfrak{p}} \cdot (f_1, \dots, f_n)$$

$$\left| \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \right| \notin \mathfrak{p}$$

SGA 4½

Equivalent dfns

- $f: A \to B$ of finite type
- B A-flat
- $\Omega^1_{B/A} = 0$

$f: X \to Y$ is étale if $\forall x \in X$ $\exists$ affine nbhd $V \subset Y$: $f(x) \in V$; nbhd $U \subset f^{-1}(V)$; $x \in U$; $U \to V$ étale

Ex $Y = \operatorname{Spec} k$ k a field

$X \xrightarrow{?} Y$

- finite Type: fin. many pts
- no nilpotents (Jacobian $\neq 0$)

Thm $X = \operatorname{Spec} L_1 \sqcup \operatorname{Spec} L_2 \sqcup \cdots \sqcup \operatorname{Spec} L_r$

$K \subset L_i$ is a finite separable extension

Ex $\mathbb{A}^1 \to \mathbb{A}^1$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$t \mapsto t^p - t$ $\quad \dfrac{\partial (t^p - t)}{\partial t} = p t^{p-1} - 1 = -1$

a nontrivial covering of the affine line
Artin-Schreier map

Artin-Schreier map is a Galois covering

$a \in \mathbb{F}_p$: $\quad (t+a)^p - (t+a) = (t^p - t) + (a^p - a) =$
$\equiv 1$
$$= t^p - t$$

$\mathbb{Z}/p\mathbb{Z}$ acts freely, transitively on $\mathbb{A}^1$

$$\begin{tikzcd} \mathbb{A}^1 \arrow[rr, "t^p - t"] \arrow[dr] & & \mathbb{A}^1 \\ & \mathbb{A}^1/\mathbb{F}_p \arrow[ur] & \end{tikzcd}$$

The covering is a principal $\mathbb{Z}/p$-bundle (a torsor...)

<u>Finite</u> étale maps

Now B is fin. gen. as an <u>A-module</u>
(Open embeddings do not satisfy that)

<u>Claim</u> Let $f: \mathcal{E} \to \mathbb{A}^1$ be a finite (connected) étale map. Suppose $|f^{-1}(0)|$ indivisible by p.
Then f is an isomorphism.
(over an algebraically closed field!)

$$\mathbb{A}^1 \underset{\mathbb{F}_p}{\times} \mathbb{F}_{p^n} \to \mathbb{A}^1_{\mathbb{F}_{p^n}}$$ ~~always~~

$\mathbb{P}^1$ is simply connected...

<u>Fundamental group</u> Paths are unavailable.

<u>Coverings:</u> problem #1: $k \hookrightarrow \bar{k}_s$ <u>not</u> étale

(pro ...)

Will use categorical approach.

X alg variety; let x be geometric point:

$x_0 \in X$ ~~$(x_0 \in \mathrm{Spec}\, X)$~~

k_{x_0} = residue field of x_0

$\mathrm{Spec}\, K \to \mathrm{Spec}\, k_{x_0} \to X$

$K = \bar{k}_{x_0}$ (<u>separable</u> closure)

<u>this</u> is a geometric point

Why?. because $\mathrm{Spec}\, k_{x_0}$ not simply-conn.

Let us define $\pi_1(X, x_0)$

Let Cov_X be the category of all finite étale maps to X $\qquad Y \to X$

maps:
$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, bend right] & X \\ Y_2 \arrow[ur] & \end{tikzcd}$$
(automatically étale)

Given $Y \to X$ Set $\mathrm{Fiber}_x(Y) =$

$$\begin{tikzcd} Y \arrow[r] & X \\ \mathrm{Fiber}_x \arrow[u] \arrow[r] & x \arrow[u] \end{tikzcd}$$
$$= Y \times_X x$$

$$\mathcal{F}_x = \hom_x(X, \mathrm{Fiber}_x)$$

Get $\mathcal{F}_x : \mathrm{Cov}_X \to \mathrm{Sets}$

$$\pi_1(X, x) = \mathrm{Aut}\, \mathcal{F}_x$$

$\forall Y$ $\quad \pi_1(X, x)$ acts on its <u>fiber</u> $\mathcal{F}_x(Y)$

look at the <u>kernel</u> of this action.

Decl~~ar~~are these subgroups to be <u>open</u> (actually they will be <u>clopen</u>)

<u>Example</u> $\pi_1(\operatorname{Spec} k_0, x) = \operatorname{Gal}(\bar{k}_s/k)$

$$F_x : \quad \operatorname{Cov}_X \to \text{finite groups with continuous } \pi_1(X,x) \text{ action}$$

Now, F_x is an equivalence of categories.

$$\mathbb{F}_{p^n} \subset \overline{\mathbb{F}}_{p^n} \qquad \varphi(x) = x^{p^n}$$

$\varphi \in \operatorname{Gal}(\overline{\mathbb{F}}_{p^n}/\mathbb{F}_{p^n})$

$$\varphi^{-1} := \operatorname{Frob}_{p^n} \in \operatorname{Gal}$$

$$\mathbb{A}^1(\overline{\mathbb{F}}_p) = \operatorname{Spec}(\overline{\mathbb{F}}_p[x]) \qquad E \to \mathbb{A}^1$$

$$[\overline{\mathbb{F}}_p(x) : k(E)] < \infty$$

$$\pi_1(\mathbb{A}^1, 1) \to \operatorname{Gal}(k(E)/\overline{\mathbb{F}}_p(x))$$

$$k(E) \subset \overline{\overline{\mathbb{F}}_p(x)}$$

No ramification in $\mathbb{A}^1$. <u>Any</u> ramification at $\infty \in \mathbb{P}^1$, including <u>wild</u> ramification.

$$k(x) \subset k(x)[t]/(t^p - t = x)$$

Wild ramification at ∞.

Wild ram. always contributes a p-divisible group.

$\pi_1(\mathbb{A}^1, 1)$ is p-divisible.

(An extension of $k(x)$ must be ramified <u>somewhere</u>; so $\pi_1(\mathbb{P}^1) = \phi$).

=

Let X be <u>connected</u>.

$$H^1(X, A) = \hom(\pi_1(X, x); A)$$

A an abelian group

$H^1(\mathbb{A}^1_{/\overline{\mathbb{F}}_p}; A) = 0$ if p is invertible in A

$A = \mathbb{Z}/\ell^n\mathbb{Z}$ ℓ prime $\neq p$

<u>And</u> $H^0(X, A) = \{\text{loc const fns}\}$
no need for étale top.

$H^1(\mathbb{G}_m, A)$ $\quad \mathbb{G}_m = \mathbb{A}^1 \setminus 0$

over $\overline{\mathbb{F}_p}$

$\mathbb{A}^1 \setminus \{0\} \xrightarrow{z^n} \mathbb{A}^1 \setminus \{0\}$, $n \neq p$ $\quad (n,p)=1.$

$\pi_1(\mathbb{G}_m, 1)$ will act on the fiber

$= \text{preim of } 1 = \mu_n \subset \overline{\mathbb{F}_p}$

$$\mu_{Nn} \xrightarrow{z^N} \mu_n$$

$$\mu_n = \{ w \in \overline{\mathbb{F}_p} \mid w^n = 1 \}$$

$\varprojlim \mu_N \simeq \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell$ $\quad$ not canonically

But $\prod_{\ell \neq p} \mathbb{Z}_\ell^{\times}$ acts on $\mu_\infty = \varprojlim \mu_N$ $\quad$ $\boxed{\text{and these are all cont. isoms}}$

So you must get

$$\pi_1(\mathbb{G}_m, 1) \to \prod_{\ell \neq p} \mathbb{Z}_\ell^{\times} \;\; (\simeq \mu_\infty)$$

Claim This is isom up to p-torsion.

Cor $H^1(\mathbb{G}_m, \mathbb{Z}/\ell^n\mathbb{Z}) \simeq \hom(\mathbb{Z}_\ell^{\mathbb{Z}}, \mathbb{Z}/\ell^n\mathbb{Z})$

$$\simeq \mathbb{Z}/\ell^n\mathbb{Z}$$

$$\mu_\infty = \varprojlim_n \mu_n \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell$$

non canonically

$$\boxed{\begin{array}{l} \mathrm{Gal}(\mathbb{G}_m, 1) \simeq \mu_\infty \text{ up to } p\text{-tors} \\ H^1(\mathbb{G}_m, \mathbb{Z}/\ell^n\mathbb{Z}) \simeq \mathbb{Z}/\ell^n\mathbb{Z} \end{array}}$$

Étale sheaves

'Open subset of X' is by def

any étale map $U \xrightarrow{f_U} X$

$$\begin{tikzcd} U_1 \arrow[r, "f_{U_1}"] \arrow[d, "\varphi"] & X \\ U_2 \arrow[ur, leftrightarrow, "f_{U_2}"'] & \end{tikzcd}$$

Category $\bar{E}t_X$; Ētale Presheaf on X:

contravariant functor $\bar{E}t_X \to Ab$

$U_1 \xrightarrow{\text{ētale}} U_2$ a <u>cover</u> if onto

$$\mathcal{F}(U_2) \xrightarrow{\varphi^*} \mathcal{F}(U_1) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{F}(U_1 \underset{U_2}{\times} U_1)$$

$\mathcal{F}(U_2)$ must be <u>equalizer</u> for p_1^*, p_2^* to make $\mathcal{F}$ a <u>sheaf</u>

<u>Constant sheaf</u> A_X A ab grp

$$\mathcal{F}(U) := \hom(\pi_0(U); A)$$

<u>or</u>: <u>sheafification</u> of the constant presheaf

$$H^i(X, A) = R^i\Gamma(X, A_X) = \mathrm{Ext}^i(\mathbb{Z}_X, A_X)$$

<u>Fact</u> Loc const sheaves = reps of $\pi_1(X, x)$

<u>If we believe this</u>: $H^1(X, A_X) \simeq$

~~$\simeq \mathrm{Ext}^1_{\text{Reps } \pi_1} = \text{Exts of}$~~

$\simeq \{\text{Extensions of } A_X \text{ by } \mathbb{Z}_X\}$

__But__ exts of loc const sheaves
= loc const sheaves

So $\mathrm{Ext}^1 = \mathrm{Ext}^1_{\text{Cont reps of } \pi_1}(\mathbb{Z}, A) =$

$= H^1(\pi_1(X, x), A) = \mathrm{Hom}(\pi_1^{ab}(X, x), A)$

$\equiv$

__Problem__: π_1 __profinite__; $H^1(\text{finite}, \mathbb{Q}_{\cdot\cdot}) = 0$;

__how__ do we get $H^{\cdot}(X, \text{grps of char } 0)$?

$$\varprojlim_n H^i(X; \mathbb{Z}_{\ell^n}) = \text{`}H^i(X, \mathbb{Z}_\ell)\text{'}$$

$\text{`}H^i(X, \mathbb{Z}_\ell)\text{'} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ — vector space/$\mathbb{Q}_\ell$

No natural rational lattice.

$\mathbb{Z}_\ell$-sheaf on X is the following data:

- A collection of sheaves $\mathcal{F}_n$, $n = 1, 2, \ldots$
- isomorphisms $\underset{\underset{\text{Cokernel in the Abelian category}}{\parallel}}{\mathcal{F}_n / \ell^{n-1}} \xrightarrow{\sim} \mathcal{F}_{n-1}$

$\mathbb{Q}_\ell$-sheaf: same object as above;

$\hom(\mathcal{F}, \mathcal{G}) = \hom_{\mathbb{Z}_\ell\text{-sheaves}}(\ ,\ ) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

(Miraculously, still an Abelian category; Serre quotient by the Serre subcategory of torsion sheaves.

$\overline{\mathbb{Q}}_\ell$-sheaves: i) $K \leftarrow \mathbb{Q}_\ell$ finite ext

$\mathcal{O}_K$ - ~~adic~~ sheaves exactly as before

<u>Do not</u> form an abelian categories.

The best you can do:
Work with derived categories of $\mathbb{Z}/\ell^n\mathbb{Z}$ - sheaves separately;
they form some system of triang cats...

Anyway, there <u>is</u> $\mathcal{D}(\overline{\mathbb{Q}}_\ell\text{-sheaves})$.

[$f_*, f^*, f_!, \ldots$ <u>all</u> defined at the level of derived categories.

$$X_0/\mathbb{F}_q \qquad X_0(\mathbb{F}_q) = ? \qquad X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$$

$$\mathrm{Frob}_q : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$$

Frob_q acts on X

Let $\mathcal{F}$ be any sheaf on X_0

$$\mathcal{F} := p^{-1}\mathcal{F}_0 \qquad X \to X_0$$

$$\mathrm{Frob}_p^* \longrightarrow \mathcal{F}$$

$x \in \{\text{geom pts of } X\} \Rightarrow$

$\Rightarrow \mathcal{F}_x \in Sh_x \simeq Ab$

$$\overline{\mathbb{F}_p} \longrightarrow X \longrightarrow \mathrm{Spec}\ \overline{\mathbb{F}_p}$$

$X(\overline{\mathbb{F}_p})$ consists of geom points

$\wr\wr$

$X_0(\overline{\mathbb{F}_p})$ Frob acts on X, also acts on $X(\overline{\mathbb{F}_p})$

$\{\text{Fixed points of } \mathrm{Frob}^n\} \simeq X_0(\mathbb{F}_{q^n})$

<u>In usual topology</u>:

$f: X \to X$ cont

$\#(\text{fixed pts with mult.}) = \sum (-1)^i \mathrm{Tr}\, f^* | H_c^i$

Amazing to see: all intersections in the ētale case are transversal. All multiplicities $=1$.

Verdier-Lefschetz:

$$\mathrm{STr}\left(\mathrm{Frob}\left(H_c^*(X, \mathbb{Q}_\ell)\right)\right) = |X_0(\mathbb{F}_q)|$$

$\mathcal{F}$ a $\overline{\mathbb{Q}}_\ell$-sheaf on X_0:

$$\mathrm{Str}\,\mathrm{Frob}^n\Big|H_c^*(X, \mathcal{F}) = \sum_{x \in X_0(\mathbb{F}_{p^n})} \mathrm{Str}\,\mathrm{Frob}_x^n\Big|H^\bullet \mathcal{F}_x$$

$\mathcal{F}_x$ - complex of $\overline{\mathbb{Q}}_\ell$-vector