

# NC theory

NC forms/multivectors, revisited

(Wai-kit Yeung)

$$B_{\cdot}^{sh}(A) : \Omega_A^{1,nc} \rightarrow A \otimes A$$

$$\begin{matrix} \parallel \\ B_1^{sh} \end{matrix} \longrightarrow \begin{matrix} \parallel \\ B_0^{sh} \end{matrix}$$

$$\uparrow = \quad \uparrow =$$

$$\dots \rightarrow B_2 \xrightarrow{\partial} B_1 \xrightarrow{\partial} B_0$$

$$\partial_{bas} = b' = \dots \quad \partial_{bas} = b'$$

$$B_{\cdot} = A \otimes \bar{A}^{\otimes n} \otimes A$$

$$\text{"L"}_{A/b}^{nc} \quad B_1^{sh} = B_1 / b' B_2 \xrightarrow{\sim} \Omega_A^{1,nc}$$

$$a_0 \cdot da_1 \cdot a_2 \mapsto a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2$$

$$\text{"T"}_{A/b}^{nc} \quad B_{\cdot}^{sh}(A)^{\vee} = \underset{A \otimes A^{\text{op}}}{\text{Hom}} \quad (B_{\cdot}^{sh}(A), A \otimes A)$$

$A \otimes A^{\text{op}}$  acts

still  $A \otimes A^{\text{op}}$ -module



$\wedge^k T_{A/b}^{nc}$

$$X^{(k)}(A) = \left[ \underset{A}{\underset{\wedge}{\underset{\wedge}{(B_{\cdot}^{sh}(A)^{\vee})}}} \otimes \dots \otimes \underset{A}{\underset{\wedge}{\underset{\wedge}{(B_{\cdot}^{sh}(A)^{\vee})}}} \right] \otimes_{A \otimes A^{\text{op}}} A \Big|_{C_k}$$

Shifted version:

$$\mathfrak{X}^{(k)}(A, d) = \mathfrak{X}^{(k)}(A)[-kd]$$

$$\mathfrak{X}^{(*)}(A, d) = \prod_{k=0}^{\infty} \mathfrak{X}^{(k)}(A, d)$$

("nc shifted multivectors")

Full version:  $B_*(A)$  instead of  $B_*^{sh}$ ;  
quis for semi-free dga.

Explicitly:  $A$  semi-free w/generators  
 $x^j$ ,  $j \in J$ .

$$\mathfrak{X}^{(*)}(A) = R \langle\langle t^*, x^j, \xi_j \rangle\rangle_{j \in J} / [ , ]$$

(take free algebra,  
factor out by commutators, complete).

$$B_1^{sh}(A) = \Omega_A^1 = \sum A \cdot d x^j \cdot A$$

$$(\text{Notation: } da = \sum_j \partial_j^{(1)} a \cdot dx^j \cdot \partial_j^{(2)} a)$$

$$B_1^{sh}(A)^v = \prod_j A \cdot \xi_j \cdot A \quad " \xi_j = \frac{\partial}{\partial x^j} "$$

$$t^* = 1 \otimes 1 \in A \otimes A = B_0^{sh}(A).$$

And the differential:

$$\partial_A(\xi_k) = \sum_j \pm \partial_k^{(2)}(\partial_A(x^j)) \cdot \xi_j \cdot \partial_k^{(1)}(\partial_A(x^j))$$

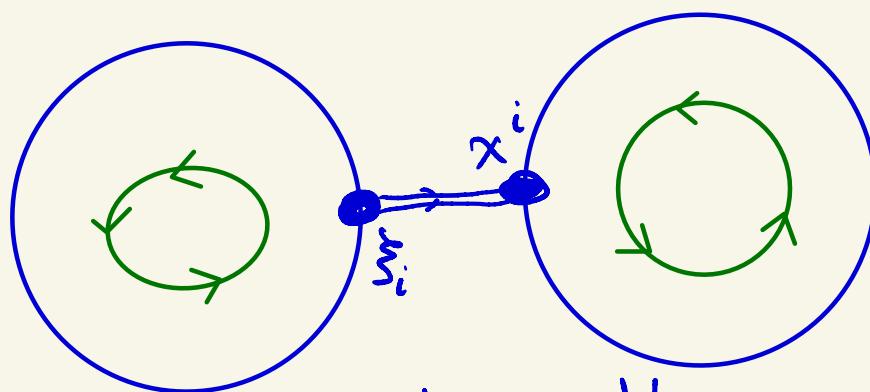
$$St^* = \sum_j [\xi_j, x^j]$$

Indeed:

$$\frac{\partial \mathcal{J}}{\partial x^j} \xrightarrow{\partial_A} \sum \partial_k^{(1)} (\partial_A x^j) \cdot dx^k \cdot \partial_k^{(2)} (\partial_A x^d)$$

↓  $\xi_i$       ↓  $\xi_i$       ←  $\xi_i$       ←  $\xi_i$       b/c  
 $\partial_u^{(2)}(-) \cdot s_i^k \cdot \partial_k^{(1)}(-)$   
 ↓  $s_i^j$       →  $\circ$   
 )  
 inner action  
 is used

## The (nc) Schouten bracket:



rotate both;

replace  $\xi_i \longleftrightarrow x^i$

by 1 every time.

2 cyclic words comprised  
of  $x_j, \bar{x}_j, t^*$ .

So: those are nc multivectors with the (nc) Schouten bracket.

Its MC elements are "nc shifted Poisson structures" on  $A$ .

But also: they are "nc volume elements".

Why? Because  $L\Omega_{A/k}^1$  (classical) is just one form, at least when  $A$  is smooth. But  $L\Omega_{A/k}^{1,nc}/[ , ]$  contains all HH, i.e. all nc forms.

=

NC forms à la Waikit Yeung:

$$B_{\cdot}^{sh,(n)}(A) = \underset{A}{B_{\cdot}^{sh}(A)} \otimes \dots \otimes \underset{A}{B_{\cdot}^{sh}(A)}$$

$$\gamma_{\cdot}^{(n)}(A) = \left[ \underset{A \otimes A^P}{B_{\cdot}^{sh,(n)}(A)} \otimes \underset{A}{1} \right] c_n$$

" $\Omega^{\cdot nc}$ "

$$\gamma_{\cdot}^{(\omega)}(A) = A / \underset{n > 0}{[A, A]}$$

$$t_* = 1 \otimes 1$$

$$\gamma_{\cdot}^{(*)}(A) = \bigoplus \gamma_{\cdot}^{(n)}(A)$$

$[d, b] = [t_*, g]$

$$d: B_{\cdot}^{sh,(\pm)} \rightarrow B_{\cdot}^{sh,(\pm+1)}$$

" $d_{DR}^{nc}$ "

$$a \mapsto da \mapsto 0; t_* \mapsto 0$$

b/c:  
 $da \mapsto a t_* - t_* a$   
 $a \otimes 1 - 1 \otimes a$

So  $[d, b'] = 0$  on  $\mathcal{B}^{sh, (1)} / [ , ] = \gamma^{(*)}$

This is our nc DeRham complex.

Next: Poisson multivector  $\pi$ :

$$\begin{array}{c} \text{linear in } \omega \\ \text{(Multilinear} \\ \text{in } \pi \end{array} \quad \left( \begin{array}{ccc} \Omega_x & \xrightarrow{\pi} & \Lambda^* T_x \\ & & \\ \Lambda T_x^* & \xrightarrow{\pi} & \Lambda T \end{array} \right) \quad (\text{classically})$$

And  $\pi$  is in  $\Lambda^{>2} T_x$ : extra factors in  $T_x$ .

Intertwines  $d_{DR}$  with  $d_\pi = [\pi, -]_{Sch}$

nc version:

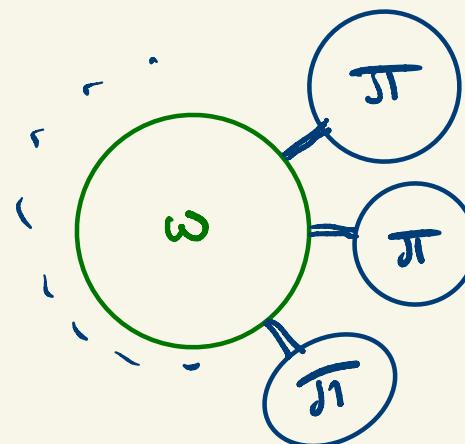
$\omega$  a cyclic word  
of  $x^j, dx^j, t_*$

$\pi$  a cyclic word  
of  $x^j, \xi_j, t^*$

Rotate all the words; when  
encounter  $dx^j$  or  $t^*$ :

$\xi_j$   $t^*$

replace with 1.



(\*)

$\gamma^{(*)} \pi_q \xrightarrow{(*)} \mathfrak{X}^{(*)}$

Another point: unlike  $\mathcal{X}^{(*)}(A)$ ,  $\mathcal{Y}^{(*)}(A)$  are all the same (for  $\epsilon > 0$ )

Reason:  $B_{\underset{A}{\otimes}} \dots \underset{A}{\otimes} B_{\cdot}$  is a free resolution of  $A$  over  $A \otimes A^{\otimes k}$ ;

and  $B_{\underset{A}{\otimes}}^{\text{sh}} \dots \underset{A}{\otimes} B_{\cdot}^{\text{sh}}$  too, when  $A$  is semi-free.

So: " $\Omega_A^{k, \text{cl}}$ " compute  $\text{HC}_{\cdot}(A)$  for all  $k > 0$ .

Remark When we use full  $B_{\cdot}(A)$  instead of  $B_{\cdot}^{\text{sh}}(A)$ : (same result for  $A$  semi-free)

Higher Hochschild cohomology

$$\text{HH}^{\cdot, (k)}(A) = \text{HH}^{\cdot}(A^{\otimes k}, {}_{\alpha}A^{\otimes k})$$

$\alpha \in \text{Aut}(A^{\otimes k})$   
cyclic shift

$$\text{HH}_{\cdot}^{(k)}(A) = \text{HH}_{\cdot}(A^{\otimes k}, {}_{\alpha}A^{\otimes k}) \cong \text{HH}_{\cdot}(A)$$

$\text{HH}^{\cdot}, \text{HH}_{\cdot}$  carry an interesting algebraic structure: an action of an operad consisting of operations multilinear in  $\text{HH}^{\cdot}$  and <sup>(at most)</sup> linear in  $\text{HH}_{\cdot}$ .

What is a more general algebraic structure  
on  $\mathrm{HH}^{\bullet}, {}^{(*)}$  and  $\mathrm{HH}_{\bullet}$  (or rather complexes  
computing them)?

Examples i) The Kontsevich-Vlassopoulos

(i.e. nc Schouten) bracket;

ii)  $(*)$ :  $\mathcal{Y}^{(\cdot)} \xrightarrow{T_{\mathcal{Y}}} \mathcal{X}^{(\cdot)}$  above; and also

iii) Kaledmin's Frobenius map based on:

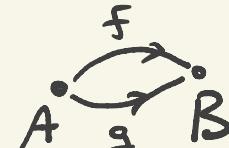
a) over  $\mathbb{Z}/p$ ,  $V \rightarrow (V^{\otimes p})^{t C_p}$

$$v \mapsto v^p$$

is additive;

b)  $\mathrm{HH}_{\bullet}^{(1)} \xrightarrow{\sim} \mathrm{HH}_{\bullet}^{(p)}$

Also, it should be animated:

i)  $\mathrm{HH}^{\bullet}, \mathrm{HH}_{\bullet}$ :   $\rightarrow C^{\bullet}(A, {}_f B)$ ;  $C_{\bullet}(A, {}_f B)$   
two morphisms of algs

$B$  is an  $A$ -bimod

$C^{\bullet}(A, B)$  dg category w/ objects  $f, g, \dots$  howw...  
 $C^{\bullet}(A, B) \otimes C^{\bullet}(B, C) \rightarrow C^{\bullet}(A, C)$  assoc up to

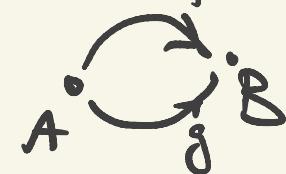
(Algebras form category in cocats)

↓  
category in categories up to...

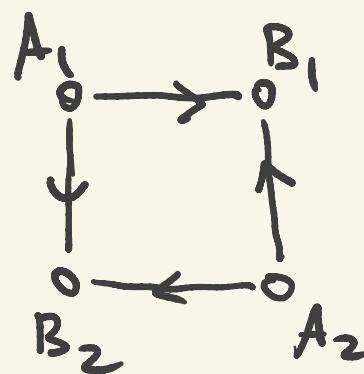
When include chains: some sort of interesting structure, part of which:

2-category with a trace functor  
(up to...) f

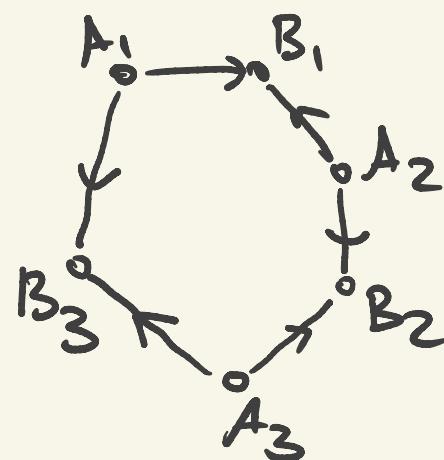
Adding  $k > 1$ : together with



there are



$$\rightsquigarrow C^{\bullet}(A_1 \otimes A_2, B_1 \otimes B_2)$$



$$\rightsquigarrow C^{\bullet}(\otimes A_i, \otimes B_i)$$

---

(end of Remark)

Next: nc multivectors/forms vs  
equivariant multivectors / forms

Recall:  $\text{Rep}_N(A)$ :  $\mathcal{O}_{\text{Rep}_N(A)}$  dg comalg;

generators:  $p_{ij}(a)$ ,  $a \in A$ ; linear in  $A$

relations:  $p_{ij}(ab) = \sum_k p_{ik}(a)p_{kj}(b)$

$$p_{ij}(1) = \delta_{ij}$$

differential:  $\partial p_{ij}(a) = p_{ij}(a_A a)$

$GL_N \otimes \mathcal{O}_{\text{Rep}_N(A)}$  :  $g \begin{bmatrix} p_{ij}(a) \\ \vdots \\ p_{ij}(a) \end{bmatrix} g^{-1}$

Ginzburg  
Ginzburg-Schedler:

$$\begin{array}{ccc}
 a \cdot db \cdot c & \Omega^{1, nc}_A & \xrightarrow{b'} A \otimes A \\
 \downarrow & \downarrow & \downarrow \\
 \text{tr } \rho(ca \cdot db) & \Omega^1_{\text{Rep}_N(A)} & \rightarrow [gl_N^* \otimes \mathcal{O}_{\text{Rep}_N(A)}]
 \end{array}$$

$a \otimes c$   
 $GL_N$   
 $\sum_{j,k} E_{jk}^* \otimes \rho_{jk}(ca)$

(recall:  $M^\vee = \text{Hom}_{A \otimes A^g}(M, A \underset{\sim}{\otimes} A)$ )

$$\begin{array}{ccccc}
 a \otimes c & A \otimes A & \xrightarrow{\delta} & (\Omega_A^1)^\vee & D \\
 \downarrow & \downarrow & & \downarrow & \downarrow \\
 \sum_{j,k} \rho_{jk}(a) \in (gl_N \otimes 0)_{Rep_N(A)}^{GL_N} & \xrightarrow{\quad} & T_{Rep_N(A)} & & 
 \end{array}$$

$(\rho_{jk}(a) \mapsto \sum \rho_{jk}(D'(a)) D'(a))$

where:  $D: A \rightarrow A \underset{\sim}{\otimes} A$

derivation  $D(a) = \sum D'(a) \otimes D''(a)$

$$\begin{aligned}
 X^{(*)}(A, d) &\rightarrow \left[ \text{Sym } g[-] \otimes \text{Sym}_0^m T_{Rep_N(A)} \right]^{GL_N} \\
 Y^{(*)}(A) &\rightarrow \left[ \text{Sym } g^*[-2] \otimes \Omega_{Rep_N(A)} \right]^N
 \end{aligned}$$

a dgla morphism.

Therefore: MC element  $\mapsto$  MC element.

Pre-CY str. on  $A \mapsto$  derived Poisson str. on  $[Rep_N(A)]_{GL_N}$

# CY structures, I

## Motivation/weaker dfn

$D \in HH^m(A)$ ,  $\alpha \in HH_n(A)$ :

$$\gamma_D \alpha \in HH_{n-m}(A)$$

$$\gamma_D(a_0 \otimes \dots \otimes a_n) = \pm h_0 D(a_1, \dots, a_m) \otimes a_{m+1} \otimes \dots \otimes a_n$$

(or cohomologous:

$$\pm D(a_{n+1}, \dots, a_n) a_0 \otimes a_1 \otimes \dots \otimes a_m$$

$$\cap: C^\circ(A, A) \otimes C_-(A, A) \rightarrow C_-(A, A)$$

Actually extends to

$$C^\circ(A, A) \otimes HC_-(A) \rightarrow HC_-(A)$$

(Key to Ezra's GM connection)

Therefore: a)  $\omega \in \text{HH}_m(A, A)$

(1)  $\iota, \omega: \text{HH}^*(A, A) \rightarrow \text{HH}_{*+m}(A, A)$

b)  $\tau: \text{HH}_m(A) \rightarrow k$

(2)  $\text{HH}^{*-m}(A, A) \rightarrow \text{HH}_*(A)^*$

(linear dual)

$D \longmapsto \langle \tau, \iota_D(-) \rangle$

Dfn (weaker than the actual one,  
probably enough)

a left CY structure on  $A$ :

$\omega \in \text{HC}_m^-(A)$  whose projection to

$\text{HH}_m(A)$  makes (1) an isomorphism

a right CY structure on  $A$ :

$\tau \in \text{HC}^m(A)$  whose image in  $\text{HH}_m(A)^\vee$   
makes (2) an isom.

The construction of  $\tau_D$  shows that the  $X$  and  $Y$  complexes enter naturally:

$P \xrightarrow{\quad} P \otimes_P P \xrightarrow{D \otimes \text{id}} A \otimes_P P = P$   
 $\downarrow 2 \qquad \qquad \qquad \downarrow 2$   
 $A \xrightarrow{\quad} A \otimes_P P = P$   
 $\downarrow 2 \qquad \qquad \qquad \downarrow 2$   
 $A \xrightarrow{\quad} A$

bimodule  
resolution

(another)  
bimodule  
resolution

$$(a_0 \otimes \dots \otimes a_m) \mapsto (a_0 \otimes \dots \otimes 1) \underset{A}{\otimes} (1 \otimes a_{m+1} \otimes \dots \otimes a_{n+1})$$

$a_0$        $\int$   
 $\pm a_0 D(a_1, \dots, a_m) \otimes a_{m+1} \otimes \dots \otimes a_{n+1}$

Next: (left) CY structures from the point of view of  $\mathcal{X}$  and  $\mathcal{Y}$  complexes.

As in the classical case: try

$$\mathcal{X}^{(1)} \xrightarrow{\quad} \mathcal{Y}^{(1)}$$

given  $w \in \mathcal{Y}^{(2)}$

(analog of  $T \xrightarrow[\omega]{} T^*$ )

$$w \in \left[ \left( B_{\underset{A}{\otimes}}^{sh} \otimes B_{\underset{A}{\otimes}}^{sh} \right) \otimes_{A \otimes A^{op}} A \right] / C_2$$

When  $A$  semi-free:  $\cong \left[ \left( A_{\underset{A}{\otimes}}^L \otimes_A A \right) \otimes_{A \otimes A^{op}}^L A \right] / C_2$

(but  $C_2$  acts trivially)

$$HH(A) \simeq A \otimes_{A \otimes A^{op}}^L A$$

$A!$

!!

$$\mathcal{X}^{(1)}(A) = \left[ R\text{Hom}_{A \otimes A^{op}} \left( A, A \underset{\text{in}}{\otimes} A \right) \right] \otimes_{A \otimes A^{op}}^L A$$

$A$  smooth if this is  $\approx$

$$R\text{Hom}_{A \otimes A^{op}}(A, A) = R\text{Hom}_{A \otimes A^{op}}(A, (A \underset{\text{in}}{\otimes} A) \otimes_{A \otimes A^{op}}^L A)$$

$$\text{Again: } \mathcal{X}^{(*)} = X^{(*)}/[X, X] \quad (\text{completed})$$

$$\mathcal{Y}^{(*)} = Y^{(*)}/[ , ]$$

$$A^! = X^{(1)} = R\text{Hom}_{A \otimes A^\text{op}}(A, A \otimes_m A)$$

$$R\text{Hom}(A \overset{L}{\otimes}_{A \otimes A^\text{op}} A, A \otimes_m A \overset{L}{\otimes}_{A \otimes A^\text{op}} A)$$

$$R\text{Hom}(A \overset{L}{\otimes}_{A \otimes A^\text{op}} A, A)$$

Therefore: given  $\omega \in A \overset{L}{\otimes}_{A \otimes A^\text{op}} A [N]$

get  $[\omega]: A^! \rightarrow A [N]$

A  $CY_N$  structure on a smooth  $A$   
is a negative cyclic class  $\omega$  in  $HC_N(A)$

s.t.

$$[\omega_{\text{Hoch}}]: A^! \xrightarrow{\sim} A [N]$$

if  $\omega_{\text{Hoch}}$  is the image of  $\omega$  in  
 $HH_N(A)$ .

Therefore  $X^{(1)} \otimes Y^{(2)} \xrightarrow{\omega} X^{(1)}$  involving  
 $X^{(1)} \cong Y^{(1)}$  (up to shift) comes from a left CY str.

Next: more systematically about CY.

$(A, B)$ -bimod  $M \longmapsto (B, A)$ -bimod  $M^\vee$

$$M^\vee = R\text{Hom}_B(M, B)$$

B | B  
right action of A      left action of B

Dfn  $M$  right-dualizable if

$$N \mathop{\otimes}\limits_B^L M^\vee \rightarrow R\text{Hom}_B(M, N)$$

for any (cofibrant) right bimodule  $N$

Ex:  $f: A \rightarrow B$   $(f_B)^\vee = B_f$

$(A, B)$  bimod  $M \longmapsto (B, A)$  bimod  $M^\vee$

$$M^\vee = R\text{Hom}_A(M, A)$$

A | A  
left action of B      right action of A

$A^M_B$  is left dualizable if

$$\check{M} \stackrel{L}{\otimes}_A N \xrightarrow{\sim} R\text{Hom}_A(M, N)$$

for any (cofibrant) left module  $N$ .

Ex.  $A \xleftarrow{g} B \quad (\check{A}_g) = {}_g A$

Lemma On  $(A, B)$ -bimod:

right dualizable  $\Rightarrow$  perfect as a right  $B$ -mod  
left  $\sim$  as a left  $A$ -mod

Lemma

$M$  left dualizable:  $\check{V}_B L \quad V_A N$

$$R\text{Hom}_B(L, \check{M} \otimes N) \xrightarrow{\sim} R\text{Hom}_A(M \otimes L, N)$$

$M$  right dualizable:  $V_L A \quad \check{V}_N B$  [mispelt in book]

$$R\text{Hom}_A(L, N \otimes M) \xrightarrow{\sim} R\text{Hom}_B(L \otimes M, N)$$

$A_{\text{diag}} = A$  viewed as  $(A \otimes A^{\text{op}}, k)$ -bimod

$A$  smooth:  $A_{\text{diag}}$  left dualizable

$A$  proper:  $A_{\text{diag}}$  right dualizable

$$A' = \check{A}_{\text{diag}} = \underset{A \otimes A^{\text{op}}}{R\text{Hom}}_{\text{in}}(A, A \otimes A)$$

$$A^* = \check{A}_{\text{diag}} = R\text{Hom}_k(A, k)$$

Lemma For a smooth  $A$ :

$$R\text{Hom}_{A \otimes A^{\text{op}}}(A, A) \cong A' \otimes_{A \otimes A^{\text{op}}}^L A$$

$$\text{HH}^\bullet(A, A) \cong \text{HH}_*(A, A')$$

For a proper  $A$ :  $R\text{Hom}_{A \otimes A^{\text{op}}}(A, A^*) \cong$

$$\text{HH}^\bullet(A, A^*) \cong \text{HH}_*(A, A)^* \cong R\text{Hom}(A \otimes_{A \otimes A^{\text{op}}}^L A, k)$$

Left CY:

$$\omega \in HC_n(A)$$

$\downarrow$

$$\omega_0 \in HH_n(A)$$

$$A' = R\text{Hom}(A, A \otimes A) \xrightarrow{\omega_0} A$$

$\cong$

$$X^{(1)}(A)$$

$\downarrow$

$$IR\text{Hom}\left(\begin{array}{c} A \otimes A \\ \downarrow \\ A \otimes A^{\text{op}} \end{array}, \begin{array}{c} (A \otimes A) \otimes A \\ \downarrow \\ A \otimes A^{\text{op}} \end{array}\right)$$

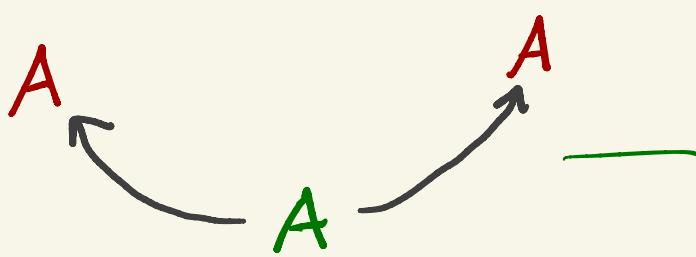
$\downarrow$

$$ev_{\omega_0}$$

$\cong$

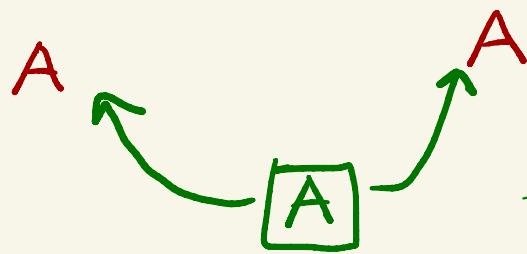
$$A$$

In pictures:



stands for  
 $R\text{Hom}_{A \otimes A^{\text{op}}}(A, A \otimes A)$

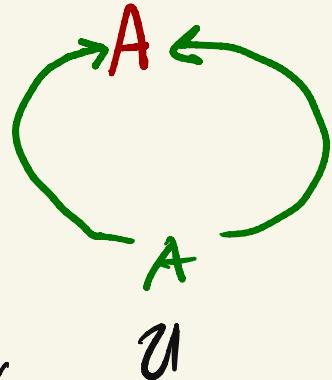
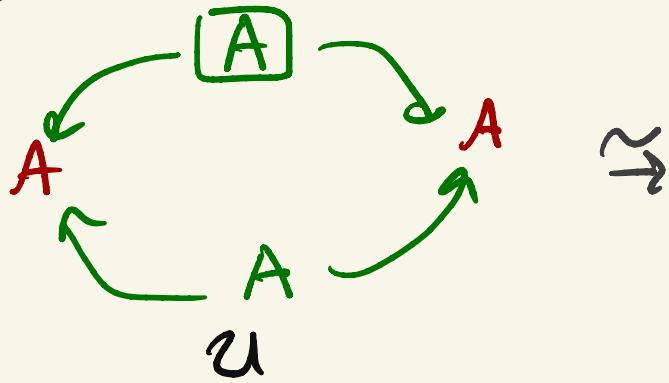
$$X^{(1)}(A)$$



stands for  
 $A \otimes A$   
 $\downarrow$   
 $A \otimes A^{\text{op}}$

$$(A \otimes A) \simeq A$$

A smooth:



$$A \xrightarrow[L]{A \otimes A^{\text{op}}} R_{\text{Hom}}(A, A) \xrightarrow[A \otimes A^{\text{op}}]{A \otimes A^{\text{op}}} (A, A)$$

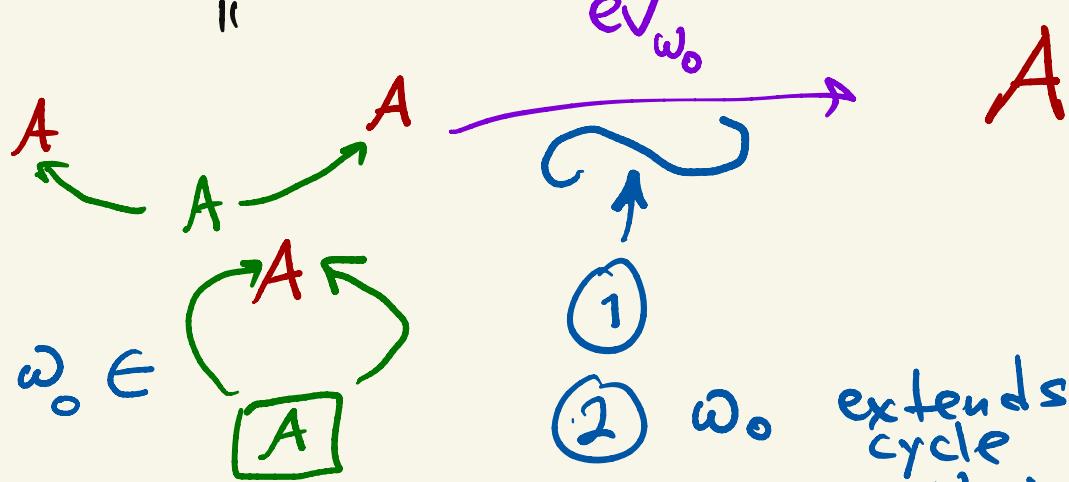
$$A \xrightarrow[L]{A \otimes A^{\text{op}}} A^!$$

$$[X^{(1)} \otimes_{A^{\otimes A^{\text{op}}}} Y^{(1)}] \xrightarrow[A \otimes A^{\text{op}}]{A \otimes A^{\text{op}}} A$$

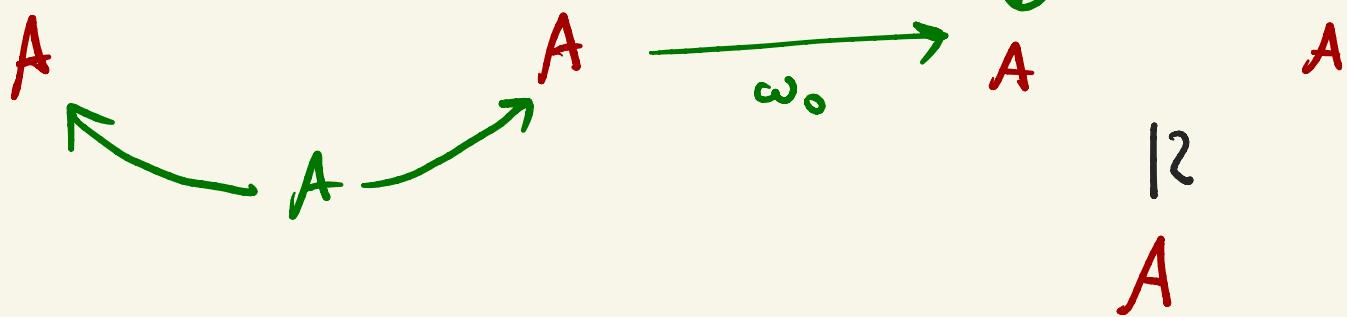
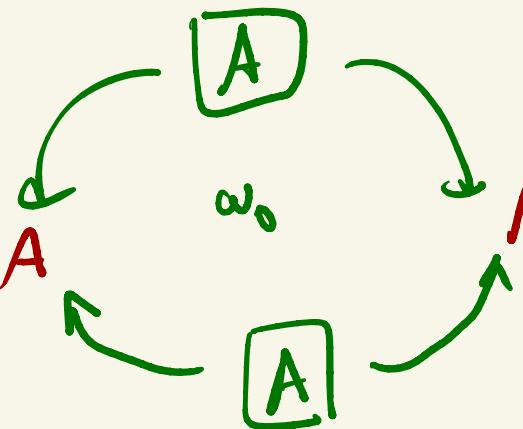
\*

A left CY:

$$\begin{matrix} A' \\ \parallel \\ X^{(1)} \\ \parallel \end{matrix}$$



OR: represent  $\omega_0$  in  $\mathcal{Y}^{(2)}$



↓

$\omega_0: \mathcal{X}^{(1)} \longrightarrow \mathcal{Y}^{(1)}$  as above

Example Ginzburg's algebras  $A_{\Phi}$   
 (nc versions of derived critical loci)

$$F = k[x_1, \dots, x_n] \quad \Phi \in F/[F, F]$$

Recall:  $F/[F, F] \rightarrow \Omega^1_F/[F, -]$

$$\Phi \mapsto \sum \frac{\partial \Phi}{\partial x_j} \cdot dx_j \quad \bigoplus_j F \cdot dx_j$$

$$[x_j, \frac{\partial \Phi}{\partial x_j}] = 0$$

$$A_{\Phi} = F / \left\langle \frac{\partial \Phi}{\partial x_1}, \dots, \frac{\partial \Phi}{\partial x_n} \right\rangle$$

An attempt at a resolution:

Classical case:  $\Phi \in k[x_1, \dots, x_n]$

$$A_{\Phi} = k[x_1, \dots, x_n] / \langle \partial \Phi / \partial x_j \rangle \quad \boxed{\Phi = \sum \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j}}$$

$$\mathcal{X}(A^n), [\varPhi, -]_{\text{Sch}} = k[x_1, \dots, x_n, \tilde{x}_1, \dots, \tilde{x}_n].$$

Define:  $k\langle x_1, \dots, x_n, \xi^1, \dots, \xi^n, t^* \rangle$

$$\partial_{\frac{\partial \phi}{\partial x_j}}: \xi^j \mapsto \frac{\partial \phi}{\partial x_j} \quad x_j \mapsto 0$$

$$t^* \mapsto \sum_j [\xi^j, x_j]$$

$$\partial_{\frac{\partial \phi}{\partial t^*}} = 0$$

$$\sum_{i=1}^n u_i = 3 \quad \phi = [x, y] \cdot z = xy^7 - yx^7$$

$$\frac{\partial \phi}{\partial x} = y^7 - 7y \quad \frac{\partial \phi}{\partial y} = 7x - x^7 \quad \frac{\partial \phi}{\partial z} = xy - yx$$

$$A_{\frac{\partial \phi}{\partial t^*}} = k[x_1, x_2, x_3]$$

$$\phi = xy^7 - yx^7$$

$$\frac{\partial \phi}{\partial x} = yz - 7y \quad \frac{\partial \phi}{\partial y} = zx - 7xz \quad \frac{\partial \phi}{\partial z} = xy - 7yx$$

$$xy^7 = 7yz - 7yx = y^7x = 7xy \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}$$

$$7xz = zx - 7xy = 7xy - 7yx$$

- consistent

Try to emulate the classical case:

$$\omega = \sum d\xi^j dx_j \quad \partial_\phi \omega = \sum \frac{\partial \phi}{\partial x_j \partial x_a} dx_j dx_a$$

The negative 2-cocycle:

$$\omega = \sum f \otimes (\xi^j \otimes x_j + x_j \otimes \xi^j - t^*)$$

$\xrightarrow{uB} 0$

$\partial_\phi \downarrow$

$$\sum (1 \otimes \partial_\phi \otimes x_j - 1 \otimes x_j \otimes \partial_\phi)$$

$\downarrow J_B$

$$+ \cancel{\xi^j \otimes x_j} + \cancel{x_j \otimes \xi^j} - 1 \otimes \xi^j x_j - 1 \otimes x_j \xi^j - \cancel{\cancel{\xi^j \otimes x_j}}$$

Now, the image of  $\sum 1 \otimes (\partial_j \phi \otimes x_j - \dots)$   
 in  $\Omega^2_{A^n}$  is zero, so it is homologous to

zero in  $CC_0^-(k[x_1, \dots, x_n])$  BY ADDING  
EXTRA TERMS.

We get an element in  $HC_3^-(A_{\Phi})$ .

Example:  $\phi = [x, y] \neq ?$

$$\sum_1^3 1 \otimes (z^j \otimes x_j - x_j \otimes z^j)$$

$$+ \underbrace{1 \otimes ([y, z] \otimes x - x \otimes [y, z])}_{Q}$$

$$\underbrace{1 \otimes Ht(x \otimes y \otimes z)}_{\text{the first extra term}} \xrightarrow{\text{HKR}}$$

$$dx dy dz \in \Omega^3_{A^3}$$

The only chance for it to be nondeg. -  $n=3$ .

# The obstruction complex

## Notation

$$A_\phi = F / \langle \frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n} \rangle$$

$$R_\phi = k \langle x_i, \xi^i; t^* \rangle \text{ as above}$$

$$A_\phi \otimes_{R_\phi} B_{\cdot}^{sh}(R_\phi) \otimes_{R_\phi} A_\phi$$

Explicitly:

Common motive:  
chains of cochains

$$A_\phi \cdot dt^* \cdot A_\phi \xrightarrow{-3} \bigoplus A_\phi \cdot d\xi^j \cdot A_\phi \xrightarrow{-2} \bigoplus A_\phi \cdot dx_j \cdot A_\phi \xrightarrow{-1} A_\phi \cdot t_* A_\phi$$

$$dt^* \mapsto \sum [x_j, d\xi^j]$$

$$\pm \sum \cancel{[dx_j, \xi^j]}$$

$$0 \in A_\phi$$

$$d\xi^j \xrightarrow{\quad} d \frac{\partial \phi}{\partial x_j}$$

Note/compare:

$$A_\phi \otimes_F \left[ F \#^* F \xrightarrow{\quad} \bigoplus F \cdot \theta^j \cdot F \xrightarrow{\quad} \bigoplus F \cdot dx_j \cdot F \xrightarrow{\quad} F \#_{t_*} F \right] \otimes_A$$

(vanden Berg)

connecting diff!

$$\times(F)$$

$$Y(F)$$

$$\begin{array}{ccccc}
 A_\phi & \otimes_{R_\phi} & B_{\cdot}^{sh}(R_\phi) & \otimes_{R_\phi} & A_\phi \\
 & & \downarrow & & \\
 & & B_{\cdot}^{sh}(F) & \xrightarrow{-2} & B_{\cdot}^{sh}(F) \\
 & -3 & & & -1 \quad 0
 \end{array}$$

Beautiful generalization of the 3D grad-div-curl picture. Indeed, for  $\phi = [x, y]^T$ :

$$\begin{array}{ccccc}
 & [x, d\zeta] & & \downarrow & \\
 dt^* \nearrow & \nearrow & \nearrow & \nearrow & \\
 [y, d\eta] & & [z, dy] & & [y, t_*] \\
 \searrow & \searrow & \searrow & \searrow & \\
 [z, d\zeta] & & [y, dz] & & [z, t_*]
 \end{array}$$

T

$$\frac{\partial \phi}{\partial x} = y_T - z_y$$

$$\frac{\partial \phi}{\partial x} = [dy, z] - [dz, y] \quad \text{etc.}$$

This has a chance to be a resolution of  $A_\phi$ . In fact:

Thm (Ginzburg)  $A_\phi \otimes_{R_\phi} B^{\text{sh}}(R_\phi) \otimes_{R_\phi} A_\phi$

$\stackrel{\cong}{\rightarrow}$  a resolution of  $A_\phi \iff$

$\iff A_\phi$  is a 3-CY algebra  
 (left)

## Relation to Rep Schemes

Lemma  $\text{Rep}_d(A_\phi) = \text{Crit}(\text{Tr } \rho(\phi))$

$$\text{Tr } \rho(\phi) = \sum_{j=1}^d \rho(\phi_{jj})$$

Pf  $\phi$  is sum of monoidals

$$x_{j_1} \dots x_{j_N} \quad p_{pq}^{(x_k)} \text{ polyn. generators}$$

$$\text{Tr } \rho(\phi) = \sum p_{i_1 j_1}^{(x_1)} \dots p_{i_N j_N}^{(x_N)}$$

$$\frac{\partial \tau \rho(\phi)}{\partial p_{pq}(x_k)} = \sum \rho_{i_1 i_2}(x_{j_1}) \dots \rho_{i_s i_p}(x_{j_{s-1}}) \cdot \cancel{\rho_{pq}(x_k)} \rightarrow \rho_{q i_{s+1}}(x_{j_{s+1}}) \dots \rho_{i_n i_l}(x_{j_N})$$

whenever  $x_k$  enters  
the word:  $x_{i_s} = x_k$

$$= \rho_{qp} \left( \frac{\partial \phi}{\partial x_k} \right)$$

Given a scheme  $X$  and a function  $\phi$ :

$$T_X \longrightarrow T_X^* \\ v \longmapsto \downarrow v(\phi) = d \in_{\phi}(v)$$

descends to

$$T_X|_{C_{\text{crit}}(\phi)} \longrightarrow T_X^*|_{C_{\text{crit}}(\phi)}$$

"

$$\mathcal{O}_{C_{\text{crit}}(\phi)} \otimes_{\mathcal{O}_X} T_X \rightarrow \mathcal{O}_{C_{\text{crit}}(\phi)} \otimes_{\mathcal{O}_X} T_X^*$$

In fact:  $f v \mapsto d(f v(\phi)) = \cancel{d f \cdot v(\phi)} + f \cdot d v(\phi)$

## The four-term complex (for $G \otimes X^{\text{if}} \otimes G^{-\text{inv}}$ )

$$\begin{array}{c}
 \left(g \otimes 0_{C_{\text{crit}} + \phi}\right)^G \rightarrow T_x \mid \xrightarrow[\text{crit}(\phi)]{\frac{d \circ \iota}{d\phi}} T_x^* \mid \xrightarrow[\text{crit}(\phi)]{} \left(\begin{smallmatrix} * & g \otimes 0 \\ & C_{\text{crit}} \end{smallmatrix}\right) \\
 \downarrow \text{a. k. a. Hessian}(\phi) \Big|_{\text{crit}(\phi)}
 \end{array}$$

$$\underline{\text{Fact}}: A_\phi \otimes_{R_\phi} B^{\text{sh}}(R_\phi) \otimes_{R_\phi} A_\phi$$

$$B_{\cdot}^{sh}(F) \xrightarrow{\text{"Hess } (\phi) \text{"}} B_{\cdot}^{sh}(F)$$

$$(g|_d \otimes \mathcal{O}_{\text{Rep}(A_\phi)})^{GL_d} \rightarrow T_{\text{Rep}(F)} \Big|_{\text{Rep}(A_\phi)}$$

Hess (Trp(φ))

$$\begin{array}{ccc} \text{Rep}(F) & \xrightarrow{\quad \text{Rep}(A_\phi) \quad} & \left\{ \begin{array}{c} gl_d^+ \otimes 0 \\ \text{Rep}(A_\phi) \end{array} \right\}^{GL_d} \end{array}$$

# Summary (of what is left)

pre-CY  
structure on  
 $A$   
= MC element of  
 $\hat{\mathcal{X}}^{(*)}(A, d)$

① Left CY structure  $\rightsquigarrow$  pre-CY str.

Sketch of the construction:

In (comu.) shifted Poisson geometry:

Shifted symplectic structures

↑  
Pridham

Nondegenerate shifted Poisson  
structures

nc version (Wailei Young):

Same construction, replacing  
multivectors with  $\hat{\mathcal{X}}^{(*)}(d, t)$  and  
forms with  $\gamma^{(\cdot)}(A)$ .

② Right CY structure  $\rightarrow$  pre CY str.:

Partial case: Frobenius algebra

First, defn/interpretation of the KV bracket.  
Let  $A$  be any algebra.

Let  $\dim_k(A) < \infty$ . Start with

$$A \oplus A^*[d-1]$$

Product: on  $A$ , the usual;

$a \cdot : A^*[d] \rightarrow$  adjoint to  $\cdot a : A \rightarrow$

(\*) Get a Frobenius algebra.

Cyclic  $A_\infty$  algebra:

$A$  with nondeg pairing of degree ...

$$\langle a_0, m_n(a_1, \dots, a_n) \rangle$$

cyclically invariant.

Example: any Frobenius algebra.

A pre-CY structure on  $A$ : a new

cyclic  $A_\infty$  algebra structure on  $A \oplus A[-]$

(\*) + (new terms)

( $A$  stays a subalgebra)

Example: for a Frobenius algebra,  
the new term is the differential  $m_1$

$$m_1: A^*[-d] \xrightarrow{\sim} A$$

(of degree 1)  
induced by  $\langle , \rangle$

Rel of pre-CY to Hochschild:

$m_n$  operations, via  $\langle , \rangle$ , are:

$$(A \oplus A^*[-d])^{\otimes n} \rightarrow k$$

cyclically invariant

$$m \in \cap \left( \left[ A^{\otimes n_1} \otimes A^* \otimes A^{\otimes n_2} \otimes A^* \otimes \dots \otimes A^{\otimes n_k} \otimes A^* \right]^* \right)_{\text{cyclic}}^{n \text{ invariant}}$$

$$\prod \left[ (A^{\otimes n_1})^* \otimes \underline{A} \otimes \dots \otimes (A^{\otimes n_k})^* \otimes \underline{A} \right]_{\text{cyclic}}^{n \text{ invariant}}$$

$$\simeq \prod_{k=1}^{\infty} \text{Hom}_{A^{\otimes k} \otimes A^{\otimes k, \text{op}}} ((B_*(A))^{\otimes k}, {}_2 A^{\otimes k})_{\text{cyclic}}$$

where  $d : a_1 \otimes \dots \otimes a_n \mapsto a_k \otimes a_1 \otimes \dots \otimes a_{k-1}$

### Conclusion:

- 1) The cyclic complex of  $A \oplus A^*[d-1]$  ( $\dim_k A < \infty$ ) is a dgla whose MC elements are pre-CY str. on  $A$ . A Frobenius algebra structure on  $A$  produces one.

- 2). This dgla can be generalized for any  $A$ :

$$\prod_k \text{Hom}_{A^{\otimes k} \otimes A^{\otimes k, \text{op}}} \left( B_*^{\otimes k}(A), \alpha^{A^{\otimes k}} \right)^{C_k}$$

(in the smooth case: same as

$$\prod_u B_*^{\vee}(A)^{\otimes k} \otimes_{A^{\otimes u} \otimes A^{\otimes u, \text{op}}} \left( \alpha^{A^{\otimes k}} \right)^{C_u}$$

MC elements of that are pre-CY str.

- 3) When  $A$  is quasi-free, can replace  $B_*^{\text{sh}}$  by  $B$ ; Witten's argument produces a MC element from a LEFT CY.

3) Perhaps (K-Takeda-V.?) construct a  
pre-CY structure for a right CY  
structure.