Friday, May 18, 2012
10:31 AM

A 2-category associated to a holomorphic symplectic manifold

(joint with A. Kapustin and N. Saulina)

Rough outline

A 2-category is a category such that

$\mathrm{Hom}(A, B)$ is a category

Holomorphic symplectic manifold (X, ω)

X - holomorphic manifold

$\omega \in \Omega^{2,0}(X)$, $d\omega = 0$, ω is nowhere degenerate

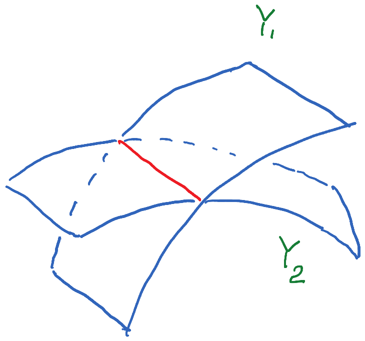
A 2-category $\ddot{L}(X, \omega)$

Simplest objects: $Y \subset X$ - lagrangian submanifolds

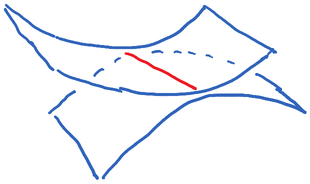
↑ automaticly holomorphic

A category of morphisms

A clean intersection



Not clean:



If $Y_1 \cap Y_2$ is clean, then

$$\mathrm{Hom}_{\ddot{L}}(Y_1, Y_2) = D^b(Y_1 \cap Y_2 ; \lambda)$$

↑ A_∞-deformation parameter

a composition of morphisms

$$\begin{tikzcd}
 & Y_1 \cap Y_2 \cap Y_3 \arrow[dl, "i_{12}"'] \arrow[d, "i_{23}"] \arrow[dr, "i_{13}"] & \\
Y_1 \cap Y_2 & Y_2 \cap Y_3 & Y_1 \cap Y_3
\end{tikzcd}$$

$$\mathcal{E}_{23} \circ \mathcal{E}_{12} = (i_{13})_* \left(i_{12}^* \mathcal{E}_{12} \otimes i_{23}^* \mathcal{E}_{23} \right)$$

More complicated objects

a holomorphic fibration

$$\begin{tikzcd}
\mathcal{Y} \arrow[d] \\
Y \subset X
\end{tikzcd}$$

← a real bundle, holomorphic projection

↑ lagrangian

$$\mathrm{Hom}(\mathcal{Y}_1, \mathcal{Y}_2) = D^b(\mathcal{Y}_1 \times_X \mathcal{Y}_2 ; \lambda)$$

TQFT motivation

a 3d B-model (L.R., E. Witten)

based on maps $M^3 \longrightarrow X$

$\ddot{L}(X)$ is the 2-category of its boundary conds

(similar to $D^b(X)$ being the category of boundary conds of a 2d B-model)

3d B-model implied an existence of non-standard monoidal structure on $D^b(X,\omega)$ (J. Roberts):

$$(\mathcal{E}_1 \otimes \mathcal{E}_2) \otimes \mathcal{E}_3 \xrightarrow{\substack{\text{Drinfeld} \\ \text{associator}}} \mathcal{E}_1 \otimes (\mathcal{E}_2 \otimes \mathcal{E}_3)$$

Reason: $D^b(X,\omega)$ is the Drinfeld center of $\ddot{L}(X,\omega)$

Algebraic model for $\ddot{L}(T^*\mathbb{C}^n)$

$\underline{x} = x_1,\dots,x_n$ - coordinates on $\mathbb{C}^n$

Simplest lagrangian submanifolds can be described by generating functions:

for $W \in \mathbb{C}[\underline{x}]$ define $Y_W = \{(\underline{x},\underline{p}) \mid \underline{p} = \partial W\} \subset T^*\mathbb{C}^n$

$$\boxed{\mathrm{Hom}(Y_1, Y_2) = MF(\underline{x}; W_2 - W_1)}$$

$MF(\underline{x}; W)$ is a category of matrix factorizations of $W(\underline{x})$

a crash course in matrix factorizations

Two approaches to the definition of $MF(\underline{x}, W)$

- A singular part of $D^b(\mathbb{C}[\underline{x}]/(W))$

  take a quotient over all finite-length resolutions

- A deformation of $D^b(\mathbb{C}[\underline{x}])$

Hochschild cohomology $HH^\bullet(\mathbb{C}[\underline{x}]) = \overbrace{\mathbb{C}[\underline{x}, \underset{\theta_1,\dots,\theta_n}{\underset{\shortparallel}{\underline{\theta}}}]}^{\text{polyvector fields}}$

$\lambda \in \mathbb{C}[\underline{x},\underline{\theta}]$ is a deformation parameter if $[\lambda,\lambda]=0$

$W(\underline{x}) \in \mathbb{C}[\underline{x}, \underline{\theta}]$

$MF(\underline{x}; W)$ as a deformation of $D^b(\mathbb{C}[\underline{x}])$

Objects

$D^b(\mathbb{C}[\underline{x}])$: $M \circlearrowleft D$, $\deg_{\mathbb{Z}} D = 1$, $D^2 = 0$

↳ $\mathbb{Z}$-graded free $\mathbb{C}[\underline{x}]$-module

$MF(\underline{x}, W)$: $M \circlearrowleft D$, $\deg_{\mathbb{Z}_2} D = 1$, $\boxed{D^2 = W \mathbb{1}_M}$

↳ $\mathbb{Z}_2$-graded free $\mathbb{C}[\underline{x}]$-module

$$M \circlearrowleft D = \left(\begin{tikzcd} M_0 \arrow[r, bend left, "F"] & M_1 \arrow[l, bend left, "G"] \end{tikzcd} \right), \quad F, G \in Mat_{k \times k}(\mathbb{C}[\underline{x}])$$

$F \circ G = W \mathbb{1}_{M_1}$, $G \circ F = W \mathbb{1}_{M_0}$

Morphisms:

$\operatorname{Hom}_{\mathbb{C}[\underline{x}]}(M, M') \circlearrowleft d = [D, \cdot]$

$d^2 = [D, [D, \cdot]] = [D^2, \cdot] = 0$

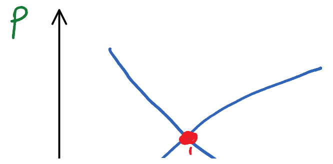
$\operatorname{Hom}(M, M') = H_d$

Localization to $Crit(W) = \{\partial W = 0\}$

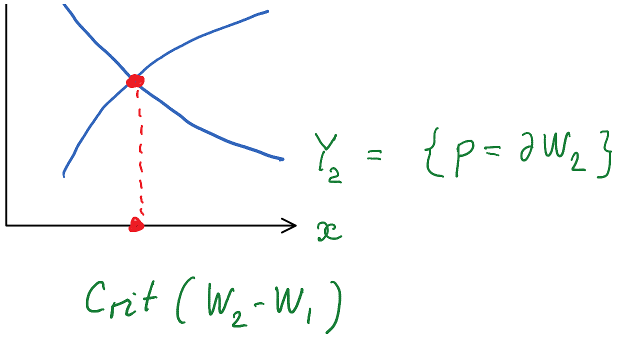
$D^2 = W \mathbb{1}_M \Rightarrow \partial W \cdot \mathbb{1}_M = [D, \partial D]$

↳ homotopically trivial

$$\mathbb{C}[\underline{x}] \longrightarrow \underbrace{\mathbb{C}[\underline{x}]/(\partial W)}_{\text{Jacobi algebra}} \to End_{MF}(M)$$

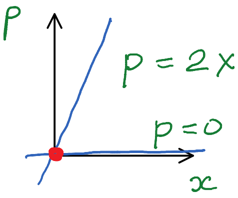


$Y_1 = \{p = \partial W_1\}$



Examples

1. $W = x^2$



A single indecomposable object

$$\begin{tikzcd} M = \Big(\mathbb{C}_0[x] \arrow[r, bend left, "x"] & \mathbb{C}_1[x] \arrow[l, bend left, "x"] \Big), \quad M[1] \cong M \end{tikzcd}$$

$$MF(x;\, x^2) \cong \text{Vect-}\mathbb{C}$$

not exactly $D^b(1\text{-pt}) \cong \mathbb{Z}$-graded Vect-$\mathbb{C}$

2. $W = xy$

Two indecomposable objects

$$\begin{tikzcd} M = \big(\mathbb{C}_0[x,y] \arrow[r, bend left, "x"] & \mathbb{C}_1[x,y] \arrow[l, bend left, "y"] \big) \end{tikzcd}$$

$$\begin{tikzcd} M[1] = \big(\mathbb{C}_0[x,y] \arrow[r, bend left, "y"] & \mathbb{C}_1[x,y] \arrow[l, bend left, "x"] \big) \end{tikzcd}$$

$$MF(x,y;\, xy) = \mathbb{Z}_2\text{-graded Vect-}\mathbb{C}$$

Knörrer periodicity

3. $W(\underline{x})$ such that $\mathrm{Crit}(W)$ is smooth and the matrix of second derivatives $\partial_i \partial_j W$ determines a non-degenerate pairing on $\underset{\uparrow\ \text{normal bundle}}{N}\mathrm{Crit}(W)$

Then $MF(\underline{x}; W) \underset{\uparrow\ \text{"almost"}}{\cong} \underset{\uparrow\ \mathbb{Z}_2\text{-graded}}{D^b}(\mathrm{Crit}(W))$

Also the intersection $\{\underline{p}=0\} \cap \{\underline{p} = \partial W\} \cong \mathrm{Crit}(W)$ is clean

Tensor product $\otimes_{\mathbb{C}[\underline{x}]}$ as a composition of morphism categories

Tensor product of matrix factorizations

$$M_1 \in MF(\underline{x}, \underline{a}; W_1(\underline{x}, \underline{a}))$$

extra variables

$$M_2 \in MF(\underline{x}, \underline{b}; W_2(\underline{x}, \underline{b}))$$

$$M_1 \otimes_{\mathbb{C}[\underline{x}]} M_2 \circlearrowleft D = \underbrace{D_1 \otimes 1_{M_2}}_{D_1} + \underbrace{(-1)^F \otimes D_2}_{D_2}$$

$$D^2 = D_1^2 + D_2^2 + \underbrace{\{D_1, D_2\}}_{=0} = (W_1 + W_2)\, 1_{M_1 \otimes M_2}$$

$$\boxed{MF(W_1) \otimes MF(W_2) \xrightarrow{\otimes} MF(W_1 + W_2)}$$

Composition of morphism categories

$$\underset{\shortparallel}{Hom(W_1(\underline{x}), W_2(\underline{x}))} \times \underset{\shortparallel}{Hom(W_2(\underline{x}), W_3(\underline{x}))} \longrightarrow \underset{\shortparallel}{Hom(W_1(\underline{x}), W_3(\underline{x}))}$$

$$\begin{tikzcd}
\mathrm{Hom}(W_1(\underline{x}), W_2(\underline{x})) \times \mathrm{Hom}(W_2(\underline{x}), W_3(\underline{x})) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(W_1(\underline{x}), W_3(\underline{x})) \arrow[d, equal] \\
MF(\underline{x}; W_2 - W_1) \times MF(\underline{x}; W_3 - W_2) \arrow[r, "\otimes \mathbb{C}[\underline{x}]"] & MF(\underline{x}; W_3 - W_1)
\end{tikzcd}$$

because $(W_2 - W_1) + (W_3 - W_2) = W_3 - W_1$

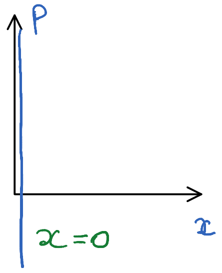
General objects and their morphisms

A general object of $\ddot{M}F(\underline{x})$ is a pair $(\underbrace{\underline{a}}_{\text{extra variables}} ; \overset{\in \mathbb{C}[\underline{x},\underline{a}]}{W(\underline{x},\underline{a})})$

It describes a (generalized) lagrangian submanifold or a fibration with a lagrangian base:

$$Y_W = \{ (\underline{x}, \underline{p}, \underline{a}) \mid \underline{p} = \partial_{\underline{x}} W , \ \partial_{\underline{a}} W = 0 \}$$

Example 1 $W = ax$ describes



Example 2 $W = ax^2$ desribes $\{ p = 2ax , \ x^2 = 0 \}$

supported at $x = 0 , p = 0$

$$\mathrm{Hom}\big((\underline{a} ; W_1(\underline{x}, \underline{a})), (\underline{b} ; W_2(\underline{x}, \underline{b})) \big) = MF(\underline{a}, \underline{b}, \underline{x} ; W_2 - W_1)$$

Composition of morphisms

$$\begin{tikzcd}
\mathrm{Hom}((\underline{a};W_1),(\underline{b};W_2)) \times \mathrm{Hom}((\underline{b};W_2),(\underline{c},W_3)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}((\underline{a};W_1),(\underline{c};W_3)) \\
MF(\underline{x},\underline{a},\underline{b};W_2-W_1) \times MF(\underline{x},\underline{b},\underline{c};W_3-W_2) \arrow[r, "\otimes \mathbb{C}[\underline{x}]"] & MF(\underline{x},\underline{a},\underline{c};W_3-W_1)
\end{tikzcd}$$

$\uparrow$ forget $\mathbb{C}[\underline{b}]$ - structure

Lagrangian correspondences

A lagrangian correspondence is an object $(\underline{a}; W) \in \ddot{MF}(\underline{x}, \underline{y})$

$x_1, \dots, x_n$ $\quad$ $y_1, \dots y_m$

It determines a functor

$$\ddot{MF}(\underline{x}) \xrightarrow{(\underline{a}; W)} \ddot{MF}(\underline{y})$$

$$(\underline{b}; W_1(\underline{x}, \underline{b})) \longmapsto (\underline{a}, \underline{b}, \underline{x}; W_1 + W)$$

Action of a lagrangian correspondence on a category of morphisms

$$\ddot{MF}(x) \xrightarrow{W(x,y)} \ddot{MF}(y)$$

$$W_1(x) \longmapsto W_1(a) + W(a,y)$$

the name does not matter

$$W_2(x) \longmapsto W_2(b) + W(b,y)$$

We need a functor

$$MF(x; W_2(x) - W_1(x)) \longrightarrow MF(a, b, y; W(b,y) - W(a,y) + W_2(b) - W_1(a))$$

Describe it as a "bimodule": a matrix factorization of

$$MF\Big(x, y, a, b; \underbrace{\big(W(b,y) - W(a,y)\big) + \big(W_2(b) - W_2(x)\big) - \big(W_1(a) - W_1(x)\big)}_{W^{tot}(a,b,x,y)}\Big)$$

Since $W^{tot} \in \underbrace{(a-x, b-x)}_{\text{a regular sequence in } \mathbb{C}[a,b,x,y]}$, there is a canonical choice

Koszul matrix factorizations

Koszul complex : for $p \in \mathbb{C}[\underline{x}]$, $K(p) = \left(\mathbb{C}_0[\underline{x}] \xrightarrow{p} \mathbb{C}_1[\underline{x}]\right)$

for $\underline{p} = p_1, \dots, p_k \in \mathbb{C}[\underline{x}]$, $K(\underline{p}) = \left(\bigotimes_{i=1}^{k} K(p_i)\right)$

Koszul matrix factorization:

for $p, q \in \mathbb{C}[\underline{x}]$, $K(p;q) = \left(\mathbb{C}_0[\underline{x}] \underset{q}{\overset{p}{\rightleftarrows}} \mathbb{C}_1[\underline{x}]\right) \in MF(\underline{x}; pq)$

for $\underline{p}, \underline{q} \in \mathbb{C}[\underline{x}]$, $K(\underline{p};\underline{q}) = \left(\bigotimes_{i=1}^{k} K(p_i, q_i)\right) \in MF(\underline{x}; \underbrace{\underline{p} \cdot \underline{q}}_{\sum_{i=1}^{k} p_i q_i})$

If $W \in (\underline{p})$, then $\exists\, \underline{q}$ s.t. $W = \underline{p} \cdot \underline{q}$

Theorem If $\underline{p}$ is a regular sequence, then $K(\underline{p};\underline{q}) \in MF(\underline{x}; W)$ does not depend on the choice of $\underline{q}$.

Fourier-Legendre transform

$$\ddot{M}F(\underline{x}) \xrightarrow{\underline{x} \cdot \underline{y}} \ddot{M}F(\underline{y})$$

This is an equivalence of categories, the inverse transform is

$$\ddot{M}F(\cdot\underline{y}) \xrightarrow{-\underline{x} \cdot \underline{y}} \ddot{M}F(\underline{x})$$

The composition is

$$\ddot{M}F(\underline{x}) \xrightarrow{\underline{a} \cdot (\underline{x} - \underline{y})} MF(\underline{y})$$

$$W(\underline{x}) \longmapsto \underline{a} \cdot (\underline{b} - \underline{x}) + W(\underline{b}) \cong W(\underline{x})$$

$$W(\underline{x}) \longmapsto \underbrace{\underline{a}\cdot(\underline{b}-\underline{x}) + W(\underline{b})}_{} \cong W(\underline{x})$$

$$= \underbrace{\left(a + \frac{W(b)-W(x)}{b-x}\right)}_{=\tilde{a}} \underbrace{(b-x)}_{\tilde{b}} + W(x)$$

$$= \tilde{a}\,\tilde{b} + W(x) \cong W(x)$$

↑ by Knörrer periodicity

Drinfeld center of $\ddot{MF}(\underline{x})$

Endomorphism category of the identity functor

$$MF(\underline{x}) \xrightarrow{\underline{a}\cdot(\underline{y}-\underline{x})} MF(\underline{y})$$

↑ renamed $\underline{x}$

$$(\underline{a}\,;\, \underline{a}\cdot(\underline{y}-\underline{x})) \in \ddot{MF}(\underline{x},\underline{y})$$

$$End(\underline{a}\cdot(\underline{y}-\underline{x})) = MF(\underline{x}, y, \underline{a}, b; (\underline{b}-\underline{a})\cdot(\underline{x}-\underline{y}))$$

↑ renamed $\underline{a}$

Knorrer periodicity

$$\cong D^b(\mathbb{C}[\underline{x},\underline{a}])$$

↑ momenta $\underline{p}$

Monoidal structure of the Drinfeld center coincides with tensor product $\otimes_{\mathbb{C}[\underline{x},\underline{p}]}$

$\ddot{MF}(\underline{x})$ and derived algebraic geometry

A - abelian algebra (e.g. $\mathbb{C}[x]$) ⇝ 2-category $\ddot{A}$

an object B – an algebra over A (e.g. $\mathbb{C}[x]$, $\mathbb{C}[x]/(p)$)

$\mathrm{Hom}_{\ddot{A}}(B_1, B_2)$ - bimodules, that is $D^b(B_1 \underset{\substack{A \\ \uparrow \text{ derived}}}{\otimes} B_2)$

Composition - usual composition of bimodules, that is, tensor product over the intermediate algebra

Conjecture $\boxed{\ddot{\mathbb{C}[\underline{x}]} \cong \ddot{MF}(\underline{x})}$

$$\mathbb{C}[\underline{x}]/(\underline{p}) \leftrightsquigarrow \underline{a}\cdot\underline{p}$$

Example 1 $B = \mathbb{C}[x]/(x) \leftrightsquigarrow (a; ax)$

$\mathrm{End}_{\ddot{A}} B$: resolution of B : $\mathbb{C}[x,\underset{\substack{\uparrow \\ \text{odd variable}}}{\theta}] \circlearrowright x\,\partial_\theta$

$$B \overset{L}{\otimes} B = \mathbb{C}[\theta]$$

$$\mathrm{End}_{\ddot{A}} B = D^b(\mathbb{C}[\theta])$$

$$\mathrm{End}_{\ddot{MF}(x)}(ax) = MF(a,b,x;\ x\underbrace{(b-a)}_{\tilde{b}})$$

$$= MF(a;0) = D^b(\mathbb{C}[a]) \underset{\substack{\uparrow \\ \text{Koszul} \\ \text{duality}}}{=} D^b(\mathbb{C}[\theta])$$

Example 2 $B = \mathbb{C}[x]/(p) \leftrightsquigarrow (a, ap(x))$

$\mathrm{End}_{\ddot{A}} B$ resolution of B : $\mathbb{C}[x,\theta] \circlearrowright p\,\partial_\theta$

$$B \overset{L}{\otimes} B = \mathbb{C}[\theta] \otimes \mathbb{C}[x]/(p)$$

$$\mathrm{End}_{\ddot{MF}}(ap) = MF(a,b,x;\ \underbrace{(a-b)}_{\tilde{a}}p)$$

$$\begin{array}{cccc} = & \mathbb{C}[b] & \otimes & MF(\tilde{a}, x ; \tilde{a}p) \\ & \cong & & \cong \\ & \mathbb{C}[\theta] & & \mathbb{C}[x]/(p) \end{array}$$

Part 2

Reminder: 2-category $\dot{M}\dot{F}(\underline{x})$

$\uparrow x_1, \dots, x_n$

An object $(\underline{a} ; W(\underline{x}, \underline{a}))$

$$\underset{\uparrow \text{ extra variables}}{\underline{a}} \qquad W(\underline{x}, \underline{a}) \in \mathbb{C}[\underline{x}, \underline{a}]$$

A category of morphisms:

$$\mathrm{Hom}\big((\underline{a} ; W_1), (\underline{b}, W_2)\big) = MF(\underline{a}, \underline{b}, \underline{x} ; W_2 - W_1)$$

$\underline{a}$, $\underline{b}$: always distinct

Composition of morphisms: $\otimes_{\mathbb{C}[\underline{x}, \underline{b}]}$

$\uparrow$ intermediate extra variables

Equivalence with derived algebraic geometry

"Algebraization" of an additive category $\mathcal{C}$

A magic object $A \rightsquigarrow$ algebra $\mathcal{A} = \mathrm{End}_{\mathcal{C}}(A)$

Any object $B \rightsquigarrow$ an $\mathcal{A}$-module $\mathrm{Hom}_{\mathcal{C}}(A, B)$

Sometimes $\mathcal{C} \to D^b(\mathcal{A}\text{-mod})$ is an isom. of categories

A 2-category version of this construction

A magic object of $\ddot{MF}$: $(W \equiv 0)$

Monoidal category of endomorphisms: $\mathrm{End}_{\ddot{MF}}(0) = MF(\underline{x}; 0) = D^b(\mathbb{C}[\underline{x}])$

An object of $\ddot{MF}$ $\longmapsto$ A "module" category over $D^b(\mathbb{C}[\underline{x}])$

$(\underline{a}; W(\underline{x}, \underline{a}))$ $\qquad \mathrm{Hom}_{\ddot{MF}}(0, (a, W)) = MF(\underline{a}, \underline{x}; W)$

Conj: $MF(a, \underline{x}; ap(x)) \cong D^b(\mathbb{C}[\underline{x}]/(p))$

not only as categories but also as "module"-categories over $D^b(\mathbb{C}[\underline{x}])$

Comment $D^b(a, \underline{x}) \overset{\uparrow}{\cong} D^b(\theta, \underline{x})$

Koszul duality

Apply deformation: $W = ap(\underline{x}) \rightsquigarrow d = p(x)\partial_\theta$

(X, ω) - holomorphic symplectic manifold

$\ddot{L}(X, \omega)$ is hard to define, but it is local.

Locality means that all constructions involving an object $Y \subset X$

↑ lagrangian

are determined by tubular neighborhood $\mathrm{Tub}(Y) \subset X$

Generally, $\begin{matrix}\mathrm{Tub}(Y)\\ \cap \\ X\end{matrix} \not\cong \begin{matrix}\mathrm{Tub}(Y)\\ \cap \\ T^*Y\end{matrix}$ ← zero-section

but there should be a deformation $(T^*Y)_{\underline{x}}$ of

deformation parameter

the holomorphic symplectic manifold T^*Y such that

$$\begin{tikzcd} \mathrm{Tub}(Y) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cap"] & \mathrm{Tub}(Y) \arrow[d, phantom, "\cap"] \\ X & (T^*Y)_{\ae} \end{tikzcd}$$

Hence if we understand $\ddot{L}(T^*Y;\ae)$, then we know the part of $\ddot{L}(X)$ which involves Y

$\ddot{L}(T^*U)$ as $\ddot{D}_{\mathbb{Z}_2}(U)$

U - complex manifold

$\Omega^{\bullet}(U) = \Omega^{\hat{0}} + \Omega^{\hat{1}}$ - Dolbeault $((0,\cdot))$ forms on U

with $\mathbb{Z}_2$-grading $\quad (\mathbb{Z}_2 = \{\hat{0}, \hat{1}\})$

Let $W \in \Omega^{\hat{0}}(U)$, $\bar{\partial} W = 0$

↳ even Dolbeault forms $(0,\bullet)$

A category of matrix factorizations $D^{\cdot}_{\mathbb{Z}_2}(U;W)$

$$\begin{tikzcd} E \arrow[d] \\ U \end{tikzcd}$$

Def A matrix factorization of W is a $\mathbb{Z}_2$-graded vector bundle with the differential

$$\bar{\nabla} : \Omega^{\bullet}(E) \longrightarrow \Omega^{\bullet}(E)$$

such that

(1) $\deg_{\mathbb{Z}_2} \bar{\nabla} = \hat{1}$

(2) $\bar{\nabla}(\alpha\sigma) = (\bar{\partial}\alpha)\sigma + (-1)^{|\alpha|}\alpha\,\bar{\partial}\sigma$

for any $\alpha \in \Omega^\bullet(U)$, $\sigma \in \Omega^\bullet(E)$

that is, locally $\boxed{\bar\nabla = \bar\partial + A}$, $\underset{\uparrow\ \text{odd}}{A} \in \Omega^\bullet(\text{End}\, E)$

(3) $\boxed{\bar\nabla^2 = W \mathbb{1}_E}$

Remark Even if $W=0$, then $D_{\mathbb{Z}_2}(U;0)$ is a bit bigger than $D^b(U)$:

(1) $\mathbb{Z}_2$-grading instead of $\mathbb{Z}$-grading

(2) allow A to contain Dolbeault degree more than 1

a 2-category $\ddot{D}_{\mathbb{Z}_2}(U)$ $\left(= \ddot{L}(T^*U)\right)$

Simplest object: $W \in \Omega^{\hat{0}}(U)$. (adding $\bar\partial w$ creates an isomorphic object)

Morphisms: $\underset{\dot{MF}(U)}{\text{Hom}}(W_1, W_2) = MF(U; W_2 - W_1)$

Deformation

Deformation of the holomorphic symlectic structure of (X, ω)

(without changing $[\omega] \in H_{DR}(X)$)

a complex structure of X is deformed by Beltrami differential

$$\mu \in \Omega^1(TX), \quad \underbrace{\bar\partial\mu + \frac{1}{2}\underbrace{[\mu,\mu]}_{\text{Lie Bracket}}}_{\text{Cartan-Maurer eq-n}} = 0$$

so that $\bar\partial \rightsquigarrow \bar\partial' = \bar\partial + \mu \lrcorner \partial$

A holomorphic symplectic structure is deformed by a hamiltonian Betrami differential

$$\bar\partial(\omega \llcorner \mu) = 0$$

or for simplicity by Hamilton differential

could use 1 ↑

$$\varkappa \in \Omega^1(X), \quad \boxed{\underbrace{\bar\partial\varkappa + \tfrac{1}{2}\underbrace{\{\varkappa,\varkappa\}}_{\text{Poisson bracket}} = 0}_{\text{Maurer-Cartan eq-n}}}$$

so that $\mu = \omega^{-1}(\partial\varkappa)$, $\omega \rightsquigarrow \omega' = \omega + \underbrace{d\varkappa}_{\partial\varkappa + \bar\partial\varkappa}$

Deformation of $T^*\mathcal{U}$

$$\Omega^1(\underbrace{T^*\mathcal{U}}_{\text{total space}}) \rightsquigarrow \Omega^1(\underbrace{S^\bullet T\mathcal{U}}_{\text{vector Bundle}}) \ni \varkappa$$

$$\bar\partial\varkappa + \frac{1}{2}\underbrace{[\varkappa,\varkappa]}_{\text{Schouten Bracket}} = 0$$

Components of $\varkappa$

$\varkappa$ is a $(0,1)$-form taking values in polynomials of fibers of $T^*\mathcal{U}$

$$\varkappa = \varkappa_0 + \varkappa_1 + \underset{\beta}{\overset{\shortparallel}{\varkappa_2}} + \underset{\gamma}{\overset{\shortparallel}{\varkappa_3}} + \ldots, \quad \varkappa_i \in \Omega^1(\underbrace{S^i T\mathcal{U}}_{\text{polynomials of degree } i})$$

irrelevant → $\varkappa_0$

↑ $\varkappa_1$ deforms complex structure of $\mathcal{U}$

zero section in $T^*\mathcal{U}$ ↓

$$\beta \in H^1_{\bar\partial}(S^2 T\mathcal{U}) \hookrightarrow \mathrm{Ext}^1(N\mathcal{U}_0, T\mathcal{U})$$

$\uparrow$

normal bundle to zero section
in the total space $T^*\mathcal{U}$

$\beta \neq 0$ means that $T\mathcal{U}_0 \to T(T^*\mathcal{U})_{\varkappa}\big|_{\mathcal{U}_0} \to N\mathcal{U}_0$ does not split

Consider the diagonal $\Delta_X \subset \overset{\swarrow -\omega}{X^{op}} \times X$

$\Lsh$ lagrangian

For $\Delta_X \subset X^{op} \times X$, $\beta = 0$, $\gamma = R \in H^1_{\bar\partial}(S^3TX) \hookrightarrow \mathrm{Ext}^1(S^2TX, TX)$

Atiyah class of TX
represented by Riemann curvature

Describe a lagrangian (with respect to $\omega' = \omega + d\varkappa$)
submanifold $Y \subset T^*\mathcal{U}$ as a graph of ∂W:

$$Y = \{(\underline{x}, \underline{p}) \,/\, \underline{p} = \partial W\}$$

This time $\boxed{\bar\partial W = \varkappa(\partial W)}$

$\Lsh$ polynomial function
on fibers of $T^*\mathcal{U}$

Explanation: we want $\alpha = p\,\partial x + \varkappa$ to be exact on Y:

$$\alpha\Big|_{p=\partial W} = \frac{\partial W}{\partial x}\partial x + \varkappa(\partial W) = \partial W + \bar\partial W = dW$$

Category of morphisms $\mathrm{Hom}_{\dot{MF}(\mathcal{U})}(W_1, W_2)$

The old choice $\mathrm{Hom}(W_1, W_2) = MF(\mathcal{U}; \underbrace{W_2 - W_1}_{\text{not holomorphic}})$ does NOT work

$$\bar\partial(W_2 - W_1) = \varkappa(\partial W_2) - \varkappa(\partial W_1) \neq 0$$

A_∞- deformations of $D_{\mathbb{Z}_2}(\mathcal{U})$

$$\Omega^{\bullet}(\wedge^{\bullet} T\mathcal{U}),\ \bar\partial,\ \underbrace{[-,-]}_{\text{Schouten-Nijenhuis bracket}}$$

Maurer-Cartan el-t $\lambda \in \Omega^{\bullet}(\wedge^{\bullet} T\mathcal{U})$, $\deg_{\mathbb{Z}_2} \lambda = \hat{0}$

$$\boxed{\bar\partial\lambda + \frac{1}{2}[\lambda,\lambda] = 0}$$

<u>Conjecture</u> There exists a unique "universal" MC element

$$\lambda = \overset{\text{curving}}{\overset{\downarrow}{\lambda_0}} + \overset{\text{Beltrami differential}}{\overset{\downarrow}{\lambda_1}} + \overbrace{\lambda_2 + \lambda_3 + \dots}^{A_\infty\text{-deformation}}$$

(1) $\lambda_i \in \Omega^{\overset{\text{same}}{i}}(\wedge^{i} T\mathcal{U})$ (relatively balanced)

(2) $\lambda_0 = W_2 - W_1$

Substitute $\lambda = (W_2 - W_1) + \lambda_1 + \dots$ into MC equation:

$$[\lambda_n, W_2 - W_1] = -\underbrace{\bar\partial\lambda_{n-1} - \frac{1}{2}\sum_{i=1}^{n-1}[\lambda_i, \lambda_{n-i}]}_{\text{depends on } \lambda_0, \lambda_1, \dots, \lambda_{n-1}}$$

Find λ perturbatively:

compute the r.h.s. and recognize it as $[\underset{\Omega^n(\wedge^n T\mathcal{U})}{\underset{\cap}{\lambda_n}}, W_2 - W_1]$

First step: $[\lambda_1, W_2 - W_1] = -\bar\partial(W_2 - W_1)$

$$= \underset{\uparrow}{\ae}(\partial W_1) - \underset{\uparrow}{\ae}(\partial W_2)$$

polynomial function
on fibers of $T^*\mathcal{U}$

$$= \underbrace{\varkappa'(\partial W_1, \partial W_2)}_{-\lambda_1} \llcorner (\partial W_1 - \partial W_2)$$

where divided difference: $p(x) - p(y) = p'(x,y)\,(x-y)$

The first term which does not depend on ∂W_1 and ∂W_2 appears in λ_3:

$$\frac{1}{3}\beta\beta R \text{ or explicitly } \frac{1}{3}\beta^{IL}\beta^{JM} R^{K}_{LM}\, \partial_I \wedge \partial_J \wedge \partial_K$$

Its presence implies a deformation of even $\mathrm{End}_{\ddot{D}_{\mathbb{Z}_2}(\mathcal{U})}(0)$

<u>Claim</u> If $\beta = 0$ and $W_1 = W_2 = 0$, then $\lambda = 0$

that is, $\mathrm{End}_{\ddot{D}_{\mathbb{Z}_2}}(0)$ is not deformed

(but its monoidal structure may still be deformed!)

Deformation of composition of morphisms

For simplicity work perturbatively over $\varkappa$: ignore all terms of quadratic and higher order

Then MC equation is simply $\bar{\partial}\varkappa = 0$

and, most importantly, the deformation parameter for

$$\mathrm{Hom}_{\ddot{D}_{\mathbb{Z}_2}(\mathcal{U};\varkappa)}(W_1, W_2) = D_{\mathbb{Z}_2}(\mathcal{U}; \overset{\downarrow}{\lambda})$$

has only two terms: $\lambda = \underset{\underset{W_2 - W_1}{\parallel}}{\lambda_0} + \underset{\underset{\mu_{12} = -\varkappa'(\partial W_1, \partial W_2)}{\parallel}}{\lambda_1}$

hence deformations are limited to "curving" and infinitesimal change of complex structure

A morphism $\mathcal{E}_{12} \in \mathrm{Hom}(W_1, W_2)$ can be described as a $\mathbb{Z}_2$-graded vector bundle $E \to \mathcal{U}$. with deformed $\bar{\nabla}$ denoted as $\tilde{\nabla}$

$$\tilde{\nabla} = \underset{\substack{\uparrow \\ \text{compatible with } \bar{\partial}}}{\bar{\nabla}} + \mu \lrcorner \underset{\substack{\uparrow \\ (1,0)\text{- connection}}}{\nabla}$$

such that $\tilde{\nabla}^2 = \bar{\nabla}^2 + \underbrace{[\bar{\nabla}, \mu \lrcorner \nabla]}_{= \mu \lrcorner \underset{\uparrow \text{ Atiyah "class"}}{F}} = (W_2 - W_1)\,\mathbb{1}_E$

Note: $F = [\bar{\nabla}, \nabla]$, hence $\bar{\nabla} F = [\underset{\substack{\shortparallel \\ W_2 - W_1}}{\bar{\nabla}^2}, \nabla] = \partial(W_2 - W_1)$

Composition of morphisms as a deformed tensor product

$$\underset{\substack{\cap \\ \mathrm{Hom}(W_2, W_3)}}{\mathcal{E}_{23}} \circ \underset{\substack{\cap \\ \mathrm{Hom}(W_1, W_2)}}{\mathcal{E}_{12}} = \left(E_{12} \otimes E_{23} ;\ \tilde{\nabla}_{12} + \tilde{\nabla}_{23} + \overset{??}{\text{deformation}}\right)$$

$$\tilde{\nabla}_{12} + \tilde{\nabla}_{23} = \underbrace{\underbrace{\bar{\nabla}_{12} + \bar{\nabla}_{23}}_{\bar{\nabla}_{13}} + \mu_{13} \lrcorner (\underbrace{\nabla_{12} + \nabla_{23}}_{\nabla_{13}})}_{\substack{\text{Compatible with } \tilde{\partial} = \bar{\partial} + \mu \lrcorner \partial \\ \text{deformed compl. str.}}} + \underbrace{(\mu_{12} - \mu_{13}) \lrcorner \nabla_{12} + (\mu_{23} - \mu_{13}) \lrcorner \nabla_{23}}_{\delta \text{ - has to be removed}}$$

deformation $= -\delta + a$, $a \in \Omega^{\cdot}(\mathrm{End}\,E)$

Condition on a : $[\bar{\nabla} \quad , a]$

Condition on a: $\left[\bar{\nabla}_{13}, -\delta + a\right] = 0$

$$[\bar{\nabla}_{13}, \delta] = (\mu_{12} - \mu_{13}) \lrcorner F_{12} + (\mu_{23} - \mu_{13}) \lrcorner F_{23}$$

$$\mu_{12} - \mu_{13} = \varkappa'(\partial W_1, \partial W_2) - \varkappa'(\partial W_1, \partial W_3)$$

$$= \underbrace{\varkappa''(\partial W_1, \partial W_2, \partial W_3)}_{\text{second divided difference}} \lrcorner\, \partial(W_2 - W_3)$$

By the symmetry of the second divided difference

$$\mu_{23} - \mu_{13} = \varkappa''(\partial W_1, \partial W_2, \partial W_3) \lrcorner\, \partial(W_1 - W_3)$$

$$b = \varkappa''(\partial W_1, \partial W_2, \partial W_3) \lrcorner (F_{12} F_{23})$$

$$= \boxed{\beta \lrcorner F_{12} F_{23}} + O(W)$$

is non-trivial at $W_1 = W_2 = W_3 = 0$.

Deformation of the monoidal structure of $\mathrm{End}_{\ddot{D}}(0)$

$$\mathcal{E}_1 \circ \mathcal{E}_2 = \Big(E_1 \otimes E_2 ;\ \bar{\nabla}_1 + \bar{\nabla}_2 + \underbrace{\beta \lrcorner F_1 F_2}_{\substack{\text{non-commutativity} \\ \text{of monoidal structure}}}\Big)$$

Associators in $\mathrm{End}_{\ddot{D}}(0)$

Suppose that $R = 0$ on $\mathcal{U}$

↑ Atiyah class of $T\mathcal{U}$

Then $\mathrm{End}_{\ddot{D}}(0) = D_{\mathbb{Z}_2}(\mathcal{U})$ is undeformed even for a general $\varkappa \in \Omega^{\cdot}(S^{\cdot} T\mathcal{U})$

However monoidal structure is deformed

$$(E_1;\bar{\nabla}_1)\circ(E_2;\bar{\nabla}_2)=(E_1\otimes E_2;\bar{\nabla}_1+\bar{\nabla}_2+a_{12})$$

and associativity requires associator:

$$(E_1\circ E_2)\circ E_3=(\overbrace{E_{123}}^{E_1\otimes E_2\otimes E_3};\overbrace{\bar{\nabla}_{123}}^{\bar{\nabla}_1+\bar{\nabla}_2+\bar{\nabla}_3}+a_{123})$$

$\downarrow b_{123}$ – associator (gauge transformation)

$$E_1\circ(E_2\circ E_3)=(E_{123};\bar{\nabla}_{123}+a'_{123})$$

$$\bar{\nabla}_{123}b_{123}+a'_{123}b_{123}-b_{123}a_{123}=0$$

Solve perturbatively over Dolbeault degree starting with $a=\beta\llcorner(F_1F_2)$

The first term in associator is $\frac{2}{3}\gamma\llcorner F_1F_2F_3$ non-zero even if $\beta=0$

Categorified Riemann-Roch-Hirzebruch

General idea

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C} \arrow[r, "ch"] \arrow[d, "intersect"'] & Z(\mathcal{C})\times Z(\mathcal{C}) \arrow[d, "intersect"] \\
\mathcal{C}' \arrow[r, "ch"] & Z(\mathcal{C}')
\end{tikzcd}$$

$$D^b(U)\times D^b(U)\xrightarrow{ch\times ch}HH(\mathcal{C})\times HH(\mathcal{C})$$

$$\begin{tikzcd}
{} \arrow[d, "\mathrm{Ext}"'] & {} \arrow[d, "\cap"] \\
\mathbb{C}\text{-}\mathrm{Vect} \arrow[r, "\dim"] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
Y_1, Y_2 \subset X \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{O}_{Y_1}, \mathcal{O}_{Y_2} \in D^b(X) \arrow[d, mapsto, "\mathrm{Ext}, \mathrm{Tor}"] \\
D^b(Y_1 \cap Y_2; \lambda) \arrow[r, mapsto, "HH_\bullet, HH^\bullet"] & \mathbb{C}\text{-}\mathrm{Vect}
\end{tikzcd}$$

$$\mathrm{Ext}_X(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \cong HH_\bullet\left(\mathrm{Hom}_{L^{\cdot\cdot}}(Y_1, Y_2) \right)$$

$$\mathrm{Tor}_X(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) \cong HH^\bullet\left(\mathrm{Hom}_{L^{\cdot\cdot}}(Y_1, Y_2) \right)$$