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Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers

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Abstract

We give asymptotic formulas for the multiplicities of weights and irreducible summands in high-tensor powers $V_\lambda^{\otimes N}$ of an irreducible representation V_λ of a compact connected Lie group G. The weights are allowed to depend on N, and we obtain several regimes of pointwise asymptotics, ranging from a central limit region to a large deviations region. We use a complex steepest descent method that applies to general asymptotic counting problems for lattice paths with steps in a convex polytope.

Keywords: ∎; ∎; ∎

0. Introduction

This article is concerned with the interplay between combinatorics of lattice paths with steps in a convex polytope and asymptotics of weight multiplicities (and multiplicities of irreducible representations) in high tensor powers $V_\lambda^{\otimes N}$ of irreducible representations V_λ of a compact connected Lie group G. Our main results give asymptotic formulae for

- multiplicities $m_N(\lambda; \nu)$ of weights ν in $V_\lambda^{\otimes N}$;

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- multiplicities $a_N(\lambda;\nu)$ of irreducible representations V_ν with highest weight ν in $V_\lambda^{\otimes N}$.
- multiplicities of lattice paths with steps in a convex lattice polytope P from 0 to an N-dependent lattice point $\alpha\in NP$.

Asymptotic analysis of multiplicities in high tensor powers are of interest because the known formulae for multiplicities of weights and irreducibles in tensor products (Steinberg's formula, Racah formula, Littlewood–Richardson rule and others [FH,BD]) rapidly become complicated as the number of factors increases.

Our analysis of multiplicities is based on the simple and well-know fact [S] that the multiplicities of lattice paths can be obtained as Fourier coefficients of powers $k(w)^N$ of a complex exponential sum of the form

$$k(w) = \sum_{\beta\in P} c(\beta)e^{\langle\beta,w\rangle}, \quad w\in\mathbb{C}^n \tag{1}$$

with positive coefficients $c(\beta)$, where P is a convex lattice polytope. One can obtain the precise asymptotics of the Fourier coefficients of $k(w)^N$ by a complex stationary phase (or steepest descent) argument. It is necessary to deform the contour of the Fourier integral to pick up the relevant complex critical points and to study the geometry of the complexified phase, which is closely related to the moment map for a toric variety. In fact, it was the analysis of this latter problem in [TSZ1,SZ] which led to the present article.

When P is the convex hull of a Weyl orbit of the weight λ, the Fourier coefficients are weights of $V_\lambda^{\otimes N}$. When $P = p\Sigma$ with the simplex Σ and a positive integer p, and $c(\beta) = \binom{p}{\beta}$ $(|\beta|\leqslant p)$, then the Fourier coefficients are, of course, multinomial coefficients of the form $\binom{Np}{\gamma}$ with $|\gamma|\leqslant Np$. Thus, lattice path multiplicities in general behave much like multinomial coefficients, whose asymptotics (obtained form Stirling's formula) have been studied since Boltzmann in probability theory and statistical mechanics (cf. [E,F]). In view of the rather basic nature of the lattice path counting problem and its applications, it might seem surprising that a pointwise asymptotic analysis has not been carried out before (at least, to our knowledge). The closest prior result appears to be Biane's central limit asymptotics for multiplicities of irreducibles in tensor products [B], which does not make use of the connection to lattice path counting.

To state our results, we need some notation. We fix a maximal torus $T\subset G$ and denote by g and t the corresponding Lie algebras. Their duals are denoted by g^* and t^*. We fix an open Weyl chamber C in t^*, and denote the set of dominant weights by $I^*\cap\bar{C}$ where I^* is the lattice of integral forms in t^*. For $\lambda\in I^*\cap\bar{C}$, we denote by V_λ the irreducible representation of G with the highest weight λ, and denote its character by χ_{V_λ} or more simply by χ_λ. We further denote by $Q(\lambda)\subset t^*$ the convex hull of the orbit of λ under the action of the Weyl group W. The multiplicity of a weight μ in V_λ is denoted by $m_1(\lambda;\mu)$. We set $M_\lambda = \{\mu; m_1(\lambda;\mu)\neq 0\}\subset Q(\lambda)$.

It is well known that the weights (and highest weights of irreducibles) occurring in $V_\lambda^{\otimes N}$ all lie within $Q(N\lambda)$. Our aim is to obtain pointwise asymptotic formulae for the multiplicities for all possible weights. As will be seen, the asymptotics fall into several regimes. We begin with some simple results on the bulk properties of weight asymptotics and progress to our main results giving individual asymptotic formulae.

The simplest problem is to determine the asymptotic distribution of multiplicities of weights in $V_\lambda^{\otimes N}$. Let us define a probability measure on $Q(\lambda)$ as follows:

$$dm_{\lambda,N} := \frac{1}{\dim V_\lambda^{\otimes N}} \sum_{v \in Q(N\lambda)} m_N(\lambda, v)\delta_{N^{-1}v}. \tag{2}$$

This measure charges each possible weight v of $V_\lambda^{\otimes N}$ with its relative multiplicity $\frac{m_N(\lambda,v)}{\dim V_\lambda^{\otimes N}}$ and then dilates the weight back to $Q(\lambda)$. As $N \to \infty$, the dilated weights become denser in $Q(\lambda)$ and we may ask how they become distributed. In particular, which are the most probable weights?

Theorem 1. *Assume that λ is a dominant weight in the open Weyl chamber. Then, we have*

$$m_{\lambda,N} \to \delta_{Q^*(\lambda)}$$

weakly as $N \to \infty$, where $\delta_{Q^*(\lambda)}$ is the Dirac measure at the (Euclidean) center of mass $Q^*(\lambda)$ of the polytope $Q(\lambda)$ given by

$$Q^*(\lambda) = \frac{1}{\dim V_\lambda} \sum_{v \in M_\lambda} m_1(\lambda; v)v. \tag{3}$$

This is an elementary result because

$$\chi_{V_\lambda^{\otimes N}} = \chi_{V_\lambda}^N \Rightarrow dm_{\lambda,N} = D_{\frac{1}{N}} dm_\lambda * \cdots * dm_\lambda, \tag{4}$$

where $dm_\lambda = dm_{\lambda,1}$ and where $D_{\frac{1}{N}}$ is the dilation operator by $\frac{1}{N}$ on the dual Cartan subalgebra t^*. Hence, the sequence of measures $\{dm_{\lambda,N}\}$ satisfies the central limit theorem and the (Laplace) large deviations principle. In the central limit theorem, we translate the center of mass to 0 and dilate by $(D_{\sqrt{N}} : X^* \ni x \mapsto \sqrt{N}x \in X^*)$ so that the support spreads out to all of X^*.

Theorem 2. *Assume that the dominant weight λ is in the open Weyl chamber. We define the measure $d\mu_N^\lambda$ by*

$$d\mu_N^\lambda := \frac{1}{\dim V_\lambda^{\otimes N}} \sum_{v \in Q(N\lambda)} m_N(\lambda; v)\delta_{\frac{1}{\sqrt{N}}(v - NQ^*(\lambda))}, \tag{5}$$

which is considered as a measure on the subspace X^* in t^* spanned by the simple roots. Then, as a measure on X^*, $d\mu_N^\lambda$ satisfies the following formula:

$$\text{w-}\lim_{N\to\infty} d\mu_N^\lambda = \frac{e^{-\langle A_\lambda^{-1}x,\, x\rangle/2}}{(2\pi)^m\sqrt{\det A_\lambda}}, \tag{6}$$

where $m = \dim X^*$, *and the positive definite linear transform* $A_\lambda : X \to X^*$ *is defined by*

$$A_\lambda = \frac{1}{\dim V_\lambda}\sum_{\mu\in M_\lambda} m_1(\lambda;\mu)\mu\otimes\mu - Q^*(\lambda)\otimes Q^*(\lambda). \tag{7}$$

For more precise description for the matrix A_λ, see (17), (18) and Theorem 2.8. When G is semisimple, then $X^* = t^*$, and the center of mass $Q^*(\lambda)$ is the origin (Lemma 2.5). Hence, in this case, $d\mu_N^\lambda = (D_{\sqrt{N}})_* dm_{\lambda,N}$.

Next, we consider the large deviations principle. Let us recall the definitions: Let m_N $(N = 1, 2, \dots)$ be a sequence of probability measures on a closed set $E \subset \mathbb{R}^n$. Let $I : E \to [0, \infty]$ be a lower semicontinuous function. Then, the sequence m_N is said to satisfy the *large deviation principle with the rate function I* (and with the speed N) if the following conditions are satisfied:

(a) The level set $I^{-1}[0, c]$ is compact for every $c \in \mathbb{R}$.
(b) For each closed set F in E,

$$\limsup_{N\to\infty} \frac{1}{N}\log m_N(F) \leqslant -\inf_{x\in F} I(x).$$

(c) For each open set U in E,

$$\liminf_{N\to\infty} \frac{1}{N}\log m_N(U) \geqslant -\inf_{x\in U} I(x).$$

The following is a consequence of Cramér's theorem [DZ, Theorems 2.2.3, 2.2.30]:

Theorem 3. *Assume that G is semisimple. Then, the sequence* $\{dm_{\lambda,N}\}$ *of measures on* $Q(\lambda)$ *satisfies a large deviations principle with speed N and rate function*:

$$I_\lambda(x) = \sup_{\tau\in t}\left\{\langle\tau, x\rangle - \log\left(\frac{\chi_\lambda(\tau/(2\pi i))}{\dim V_\lambda}\right)\right\}, \quad x \in t^*, \tag{8}$$

where $\chi_\lambda(\tau/(2\pi i)) = \sum_{\nu\in M_\lambda} m_1(\lambda;\nu)e^{\langle\nu,\tau\rangle}$ *denotes the character of* V_λ *extended on* $t\otimes\mathbb{C}$.

The assumption that G is semisimple is not necessary. However, in general case, the definition of the rate function is slightly modified. See Section 2 for details.

Before stating our more refined results on weights, we note that there exist analogous laws of large numbers, central limit theorems and large deviations principles for multiplicities of irreducibles. In place of $dm_{\lambda,N}$, we now weight $\mu \in Q(N\lambda)$ by the multiplicity of the irreducible representation V_μ in $V_\lambda^{\otimes N}$. We thus define

$$dM_{\lambda,N} \coloneqq \frac{1}{B_N(\lambda)} \sum_{v \in Q(N\lambda)} a_N(\lambda, v)\delta_{N^{-1}v}, \qquad \left(B_N(\lambda) = \sum_v a_N(\lambda; v)\right). \tag{9}$$

The measures $dM_{\lambda,N}$ are measures on the closed positive Weyl chamber $\bar{C}$. They also satisfies the Laplace large deviations principle, but the proof is not quite as simple as for $dm_{\lambda,N}$. The measures $dM_{\lambda,N}$ and $dm_{\lambda,N}$ are related by an alternating sum over the Weyl group (see Proposition 2.4 and Lemma 2.7).

$$dM_{\lambda,N}(\mu) = \frac{(\dim V_\lambda)^N}{B_N(\lambda)} \sum_{w \in W} \operatorname{sgn}(w) dm_{\lambda,N}(\mu + \rho - w\rho). \tag{10}$$

We can thus deduce the upper-bound half (b) in the definition of the large deviation principle for the measure $dM_{\lambda,N}$ from that for $dm_{\lambda,N}$. It follows from Theorem 3 that:

Corollary 4. *Assume that G is semisimple. The sequence* $\{dM_{\lambda,N}\}$ *of measures on* $Q(\lambda)$ *satisfies the upper-bound in a large deviations principle with speed N and rate function* $I_\lambda(x)$ *given by* (8).

The lower bound will follow from our pointwise asymptotics. We should note the large deviations principle with the rate function (8) has already been proved by Duffield [D] for $dM_{\lambda,N}$ by a different method.

These results give the bulk properties of the measures $dm_{\lambda,N}, dM_{\lambda,N}$ in that they give the exponents of the measures of N-independent closed/open sets. Our main results give apparently optimal refinements, in which we give pointwise asymptotics for multiplicities of (N-dependent) weights. As mentioned above, they are based on the combinatorics of lattice paths rather than on large deviations theory, which does not seem capable of seeing the finer details of the asymptotics.

To introduce our results, we recall one of the first and most basic results of a similar kind, namely Boltzmann's analysis of the asymptotics of multinomial coefficients (see [E] for historical background and the relation to the present problem):

$$\begin{cases} m_N : \{k = (k_1, \ldots, k_m) \in \mathbb{N}^m : |k| \coloneqq k_1 + \cdots + k_m \leqslant N\} \to \mathbb{R}^+, \\ m_N(k) = \dbinom{N}{k} = \dfrac{N!}{(N - |k|)! k_1! \cdots k_m!}. \end{cases}$$

Let us consider the case $m = 1$ of binomial coefficients. It is easy to see that the binomial coefficient $b_N(k) = \binom{N}{k}$ peaks at the center $k = \frac{N}{2}$ and by Stirling's formula $r! \sim \sqrt{2\pi} r^{r+\frac{1}{2}} e^{-r}$, $b_N(\frac{N}{2}) \sim N^{-1/2} 2^N$. We measure distance from the center by $d_N(k) = k - \frac{N}{2}$. We then have (see [F, Chapter 7] for the first two lines):

$$b_N(k) \sim \begin{cases} (\mathrm{CL})\, C N^{-1/2} 2^N e^{-\frac{2d_N(k)^2}{N}} & \text{if } d_N(k) = o(N^{\frac{2}{3}}) \\ (\mathrm{MD})\, C N^{-1/2} 2^N e^{-\frac{2d_N(k)^2}{N} - Nf\left(\frac{2d_N(k)}{N}\right)} & \text{if } d_N(k) = o(N), \\ \quad \text{with } f(x) = \sum_{n=2}^{\infty} \frac{x^{2n}}{(2n)(2n-1)} & \\ (\mathrm{SD})\, \frac{1}{\sqrt{2\pi N a(1-a)}} a^{-aN} (1-a)^{-(1-a)N}, & k \sim aN,\ a < 1; \\ (\mathrm{RE})\, C_0 N^{k_0}, & k = k_0,\ N - k_0, \end{cases}$$

where we note that the function $f(x)$ has more simple form:

$$f(x) + \frac{x^2}{2} = \frac{1}{2}[(1+x)\log(1+x) + (1-x)\log(1-x)].$$

We refer to the first region as the central limit region (CL), where the asymptotics are normal (i.e. have the form $N^{-1/2} 2^N \phi(\frac{d_N(k)}{\sqrt{N}})$, where ϕ is the Gaussian). The exponential growth is fixed at $\log 2$ as long as $d_N(k) = O(\sqrt{N})$. In the next region (MD) of moderate deviations, the exponent is decreased by the function f. In the next regime (SD) of strong deviations, the growth exponent is $a \log \frac{1}{a} + (1 - a) \log \frac{1}{1-a} < \log 2$. In the final boundary (RE) region of rare events, the exponent vanishes and the growth rate is algebraic.

In a somewhat similar way, multiplicities peak at weights near the center of gravity $Q^*(\lambda)$ of $Q(N\lambda)$, have a common exponential rate for weights in a ball of radius $O(\sqrt{N})$ around the center of mass, and then the exponential rate declines as the weight moves from a moderate to a strong deviations region towards the boundary of $Q(N\lambda)$. At the boundary point $N\lambda$ of $Q(N\lambda)$, the multiplicity equals one.

0.1. Statements of results on weight multiplicities

To state our results precisely, we will need further notation. Let $X^* \subset t^*$ denote the subspace spanned by the simple roots, and let $X = (X^*)^*$ be its dual space. Using an inner product which is invariant under the action of the Weyl group, the space X is identified with the subspace of t spanned by the inverse roots. As is shown in Section 2, the polytope $Q(\lambda) - \lambda$ is contained in X^*. In the following, the interior of $Q(\lambda)$ means the interior of $Q(\lambda)$ in the affine subspace $X^* + \lambda$. Let ρ denote half the sum of the positive roots. Let L^* be the lattice of weights in X^*. Since all the roots is in L^*, the lattice L^* is of maximal rank in X^*. Let Λ^* be the root lattice in X^*, i.e., Λ^* is the linear span of all the roots over $\mathbb{Z}$, which satisfies $\Lambda^* \subset L^*$. The both lattices Λ^* and L^* are of maximal rank. Their duals are denoted by Λ and L respectively. Then we have $L \subset \Lambda$, and hence the quotient $\Pi(G) := \Lambda/L$ is a finite abelian group.

0.1.1. Central limit region

Our first result concerns the 'central limit region' of weights which are within a ball of radius $O(\sqrt{N})$ around the center of mass in Theorem 1. For the sake of simplicity we will assume that G is semisimple. In this case, we have $X^* = t^*$, and we can use the (negative) Killing form for the inner product invariant under the action of the Weyl group.

Theorem 5. *Assume that G is semisimple. Fix a dominant weight λ in the open Weyl chamber C. Let v_N be a sequence of weights such that $|v_N| = O(N^{1/2})$. Assume that $m_N(\lambda; v_N) \neq 0$ for every sufficiently large N. Then, we have*

$$m_N(\lambda; v_N) = (2\pi N)^{-m/2} |\Pi(G)| (\dim V_\lambda)^N \left(\frac{e^{-\langle A_\lambda^{-1} v_N, v_N \rangle/(2N)}}{\sqrt{\det A_\lambda}} + O(N^{-1/2}) \right), \tag{11}$$

where $|\Pi(G)|$ is the order of the finite group $\Pi(G) = \Lambda/L$, $m = \dim t$ is the rank of G and the positive definite linear transform $A_\lambda : t \to t^*$ is given by

$$A_\lambda = \frac{1}{\dim V_\lambda} \sum_{\mu \in M_\lambda} m_1(\lambda; \mu) \mu \otimes \mu. \tag{12}$$

We note that in this regime, the exponent of growth of multiplicities is the constant $\log \dim V_\lambda$. The assumption that $m_N(\lambda; v_N) \neq 0$ for every sufficiently large N can be replaced by that $m_{N_0}(\lambda; v_N) \neq 0$ for some N_0 if 0 is a weight in V_λ. In Section 2, we prove a stronger result, Theorem 2.8, which extends the central limit regime to weights $v_N \in NQ(\lambda)$ of the form

$$v_N = NQ^*(\lambda) + d_N(v_N), \quad |d_N(v_N)| = o(N^s) \tag{13}$$

with $0 \leqslant s \leqslant 2/3$. Here, as in the case of binomial coefficients, $d_N(v_N)$ represents the distance to the center of gravity of $Q(\lambda)$.

0.1.2. Large deviations region

We now consider the moderate and strong deviations regimes. As suggested by the behavior of multinomial coefficients, the exponent must decrease as we move away from the center of gravity of $Q(N\lambda)$. A key role in the exponent correction will be played by the map

$$\mu_\lambda : X \to Q(\lambda), \mu_\lambda(x) := \frac{1}{\sum_{\mu \in M_\lambda} m_1(\lambda; \mu) e^{\langle \mu, x \rangle}} \sum_{\mu \in M_\lambda} m_1(\lambda; \mu) e^{\langle \mu, x \rangle} \mu. \tag{14}$$

This map is a homeomorphism from X to the interior of $Q(\lambda)$ (see e.g. [Fu]), and resembles the moment map of a toric variety, restricted to the real torus in $(\mathbb{C}^*)^m$. We define a function δ_λ on the interior $Q(\lambda)^o$ of the polytope $Q(\lambda)$ by

$$\delta_\lambda(x) = \log\left(\sum_{\mu\in M_\lambda} m_1(\lambda;\mu)e^{\langle \mu - x, \tau_\lambda(x)\rangle}\right), \tag{15}$$

where $\tau_\lambda = \mu_\lambda^{-1} : Q(\lambda)^o \to X$. It is clear that $\delta_\lambda(v) > 0$ for $v \in Q(\lambda)^o \cap M_\lambda$. When G is semisimple, the function δ_λ is related to the rate function I_λ given by (8) by the formula

$$\delta_\lambda(x) = \log(\dim V_\lambda) - I_\lambda(x), \quad x \in Q(\lambda)^o. \tag{16}$$

For $v \in Q(\lambda)^o$, we further define the linear map $A^0_\lambda(v) : t \to t^*$ by

$$A^0_\lambda(v) = \sum_{\mu\in M_\lambda} \frac{m_1(\lambda;\mu)e^{\langle \mu, \tau_\lambda(v)\rangle}}{\sum_{\mu'\in M_\lambda} m_1(\lambda;\mu')e^{\langle \mu', \tau_\lambda(v)\rangle}} \mu\otimes\mu - v\otimes v. \tag{17}$$

In general, the linear transform $A^0_\lambda(v)$ defined above has a zero eigenvalue. However, its restriction to the subspace X, which is denoted by

$$A_\lambda(v) := A^0_\lambda(v)|_X \tag{18}$$

is shown to be positive definite as a linear map from $X \to X^*$.

First, we consider the 'strong deviations' regime where the weight in question has the form $v = Nv_0 + f$.

Theorem 6. *Let $\lambda \in C \cap I^*$ be a dominant weight, and let $v_0 \in M_\lambda$ be a weight of V_λ which lies in the interior $Q(\lambda)^o$ of the polytope $Q(\lambda)$. We fix a weight f in the root lattice Λ^*. Then, we have the following asymptotic formula:*

$$m_N(\lambda; Nv_0 + f) = (2\pi N)^{-m/2} \frac{|\Pi(G)| e^{N\delta_\lambda(v_0) - \langle f, \tau_\lambda(v_0)\rangle}}{\sqrt{\det A_\lambda(v_0)}} (1 + O(N^{-1})),$$

where m is the number of the simple roots, $|\Pi(G)|$ is the order of the finite group $\Pi(G) = \Lambda/L$, and $\tau_\lambda(v_0) = \mu_\lambda^{-1}(v_0) \in X$.

Next, we consider a general weight v. We have just handled the case where $d_N(v) \sim Nv_0$, so now we assume that $|d_N(v)| = o(N)$, i.e. the weight lies in the moderate deviations region. All of the objects in the previous result continue to make sense in this regime, but now depend on N.

Theorem 7. *Let $\lambda \in C \cap I^*$ be a dominant weight, and let $v_N \in NQ(\lambda)$ be a weight of the form*

$$v_N = Nx + d_N(v_N), \quad |d_N(v_N)| = o(N),$$

where $|d_N(v_N)|$ *denotes the norm of the vector* $d_N(v_N)$ *with respect to the fixed* W*-invariant inner product on* t^*, *and where* $x \in Q(\lambda)^o$ *is not necessarily a weight.*

Assume that $m_N(\lambda; v_N) \neq 0$ *for every sufficiently large* N. *Then, in the notation above, we have*:

$$m_N(\lambda; v_N) = (2\pi N)^{-m/2} \frac{|\Pi(G)| e^{N\delta_\lambda(v_N/N)}}{\sqrt{\det A_\lambda(v_N/N)}}(1 + O(N^{-1})).$$

Furthermore, we have the following formula:

$$\lim_{N \to \infty} \frac{1}{N} \log m_N(\lambda; v_N) = \delta_\lambda(x).$$

Note that, in Theorem 7, the point v_N/N is in the interior $Q(\lambda)^o$ of the polytope $Q(\lambda)$ for sufficiently large N, since the vector $d_N(v_N)$ is assumed to be of order $o(N)$. Theorem 7 is regarded as an "interpolation" between the central limit region and the region of moderate deviation discussed in the beginning of this section. In fact, one can deduce Theorem 5 from Theorem 7. See Theorem 10 and Proposition 1.7 in Section 1.

0.2. Statement of results on irreducible multiplicities

As we will discuss below, the multiplicities of irreducibles in $V_\lambda^{\otimes N}$ can be expressed as an alternating sum of weight multiplicities. Thus, it would be natural to expect that one might obtain asymptotics of irreducible multiplicities from our theorems on weight multiplicities stated above. Before stating our result, we should mention the following result, due to Biane [B], which gives the asymptotics of irreducible multiplicities in the central limit region. To our knowledge, this is the only prior result on asymptotics on pointwise multiplicities in high tensor products.

Theorem 8 (Biane [B, Théorème 2.2]). *Assume that* G *is semisimple. For every positive integer* N, *let* NM_λ *be the set of weights of the form* $v_1 + \cdots + v_N$ *with* $v_j \in M_\lambda$.

Then, for $\mu \notin NM_\lambda$, $a_N(\lambda; \mu) = 0$. *For*, $\mu \in NM_\lambda$ *with* $|\mu| \leqslant C\sqrt{N}$, *we have*

$$a_N(\lambda; \mu) = \frac{|\Pi(G)|(\dim V_\lambda)^N (\dim V_\mu) \prod_{\alpha \in \Phi_+} \langle A_\lambda^{-1}\alpha, \rho \rangle}{\sqrt{\det A_\lambda}(2\pi)^{m/2} N^{(\dim G)/2}} \times (e^{-\langle A_\lambda^{-1}(\mu+\rho), \mu+\rho \rangle/(2N)} + O(N^{-1/2})),$$

where the matrix A *is defined in* (12), m *is the rank of* G *and the inner product* $\langle \cdot, \cdot \rangle$ *is the Killing form.*

To be more precise, in [B] G is the Lie group with Lie algebra g (which is assumed to be simple in [B]) such that the integral lattice of a maximal torus is identified with the dual of the lattice I_λ^* generated by M_λ. His quadratic form q is the same as our A_λ. Thus, for example, the term $k(E)/\mathrm{Vol}_q(t/\check{Q})$ in [B] is equal to our $|\Pi(G)|/\sqrt{\det A}$ when $E = V_\lambda$.

The two theorems can be formally related by expressing the multiplicity $a_N(\lambda;\mu)$ as an alternating sum of the weight multiplicities (see Proposition 2.4). By (11) one has

$$m_N(\lambda;\mu+\rho-w\rho) = \frac{|\Pi(G)|(\dim V_\lambda)^N}{(2\pi N)^{m/2}\sqrt{\det A_\lambda}}(c_{w,N}(\lambda;\mu) + O(N^{-1/2})),$$

$$c_{w,N}(\lambda;\mu) = e^{-\langle A_\lambda^{-1}(\mu+\rho-w\rho),(\mu+\rho-w\rho)\rangle/(2N)}.$$

Since the matrix A_λ is W-invariant if the Lie algebra is simple, it follows that the quadratic form $\langle A_\lambda^{-1}v, v\rangle$ is a multiple of the Killing form by some positive constant. Thus, we have

$$\sum_{w\in W}\mathrm{sgn}(w)c_{w,N}(\lambda;\mu) = \frac{(\dim V_\mu)\prod_{\alpha\in\Phi_+}\langle A_\lambda^{-1}\alpha,\mu\rangle}{N^d}(1+O(N^{-1})),$$

where d is the number of the positive roots. Therefore the alternating sum above agrees with the leading term of Biane's formula, since $\dim G = m + 2d$. However, to prove Theorem 8 in this way, one would need to prove that the remainder similarly cancels to order N^{-d} when summed over the Weyl group, and that would be harder than the (relatively simple) direct proof of Biane.

Although the alternating sum approach to the irreducible multiplicities does not seem to be optimal in the central limit region as explained above, we can deduce an asymptotic formula for the irreducible multiplicities from Theorem 6 in the region of the strong deviations under some assumptions on the dominant weight.

Theorem 9. *Let V_λ be an irreducible representation of G with the highest weight $\lambda\in C$. Let $v\in M_\lambda\cap\bar{C}$ be a dominant weight which occurs in V_λ as a weight and is assumed to lie in the interior of the polytope $Q(\lambda)$. Then we have the following asymptotic formula for the multiplicity $a_N(\lambda;Nv)$:*

$$a_N(\lambda;Nv) = (2\pi N)^{-m/2}e^{N\delta_\lambda(v)}\left(\frac{|\Pi(G)|\Delta(\tau_\lambda(v)/(2\pi i))e^{-\langle\rho,\tau_\lambda(v)\rangle}}{\sqrt{\det A_\lambda(v)}} + O(N^{-1})\right), \quad (19)$$

where m is the number of simple roots, $|G_\lambda|$ is the order of the finite group $G_\lambda = L_\lambda/L$. The positive constant $\delta_\lambda(v)>0$, the vector $\tau_\lambda(v)\in X$ and the real positive matrix $A_\lambda(v)$ are given in (15), *in the text after* (15) *and* (18), *and Δ is the Weyl denominator extended to the complexification $t^{\mathbb{C}}$.*

Remarks.

- The constant $\delta_\lambda(v)$ and the matrix $A_\lambda(v)$ are determined by the irreducible representation V_λ itself. In particular, they can be computed by the logarithmic differential of the character of the irreducible representation V_λ.
- The constant $\delta_\lambda(v)$ is positive under the assumptions in Theorems 6, 7 and 9. Hence, the multiplicities $a_N(\lambda; v)$ have an exponential growth with respect to N in the regions under consideration.
- It follows from Theorem 9 that the term $\Delta(\tau_\lambda(v)/2\pi i)$ in (19) is non-negative for such a v as in Theorem 9. We prove this fact directly for $G = U(2)$ in Section 3. As the example in Section 3 suggests, if the dominant weight v is in a wall of a Weyl chamber, then the leading term in (19) might vanish.

0.2.1. Rare events

It should be possible to obtain further results on rare events reminiscent of the Poisson limit law for the multinomial distribution. Recall that the binomial distribution with parameter p tends to a Poisson distribution if $p \to 0$ as $N \to \infty$ with $p/N \to C$. Because our results allow for general coefficient weights c on S, we believe there are analogous results on multiplicities of weights near the boundary of $Q(N\lambda)$. However, for the sake of brevity we do not carry out the analysis of this case.

0.2.2. Joint asymptotics

The asymptotics of tensor products $V_\lambda^{\otimes N}$ as $N \to \infty$ may be regarded as a thermodynamic limit. As recalled in Section 4.2, the asymptotics as the highest weight $\lambda \to \infty$ is a semiclassical limit studied by Heckman, Guillemin–Sternberg and others. By combining the methods of this paper and those of Heckman et al., one could probably obtain joint asymptotics as $N \to \infty, \lambda \to \infty$ of multiplicities of $V_\lambda^{\otimes N}$. This again is motivated by the complexity of multiplicity formulae when either N or λ is large.

0.2.3. Log concavity

Our results give some evidence for the log concavity conjectures of Okounkov [O]. In the case of unitary groups $U(k)$, the multiplicty of V_μ in $V_\lambda \otimes V_\gamma$ is given by the Littlewood–Richardson coefficient $m^\mu_{\lambda\gamma}$. Okounkov has conjectured that these multiplicities are log-concave in (λ, γ, μ), and more generally that the representation valued function $V : \lambda \to V_\lambda$ is log-concave with respect to the natural ordering and tensor multiplication. Here, concavity is defined as follows: Let $F : \mathbf{A} \to \mathbf{O}$ be a function from an abelian semi-group (e.g. dominant weights) to an ordered abelian semi-group (e.g. representations). Then F is concave if

$$(p+q)F(C) \geqslant pF(A) + qF(B)$$

for any $A, B, C \in \mathbf{A}$ satisfying

$$(p+q)C = pA + qB, \quad p, q \in \mathbb{N}.$$

Our results indicate that at least the multiplicities of V_μ in $V_\lambda^{\otimes N}$ are asymptotically log concave. Indeed, since a rate function is convex, it follows that the exponent $\delta_\lambda(x)$ in (16) is concave as a function of x. Regarding the λ aspect, Okounkov notes that dim V_λ is a concave function of λ (by the Weyl dimension formula). So it is plausible that $\delta_\lambda(x)$ is asymptotically log-concave in (λ, x).

0.3. Statement of results on lattice path multiplicities

As mentioned above, our results on multiplicities of weights and irreducibles are special cases of results on asymptotic counting of lattice paths with steps in a convex lattice polytope. Relations between lattice paths and representations have been studied for some time, and one is proved by Grabiner–Magyar [GM]. We include a proof of an adequate relation for our purposes in Proposition 2.4 (see also Proposition 2.3 for the case of weights). General and conceptually clear relations can be derived from the path discussed in Littelmann's expository article [Lit]. We add some further comments in Section 4.

Let us now recall what the combinatorics of lattice paths is about: Given a set $S \subset \mathbf{N}^m$ of *allowed steps*, an S-lattice path of length N from 0 to β is a sequence $(v_1, \ldots, v_N) \in S^N$ such that $v_1 + \cdots + v_N = \beta$. We define the multiplicity (or partition) function of the lattice path problem by

$$P_N(\gamma) = \#\{(v_1, \ldots, v_N) \in S^N : v_1 + \cdots + v_N = \gamma\}. \tag{20}$$

The set of possible endpoints of an S-path of length N forms a set $P_{S,N}$, and we may ask how the numbers $P_N(\gamma)$ are distributed as γ varies over $P_{S,N}$.

It is useful (and requires no more work) to consider a somewhat more general problem: Let X be a real vector space and let and $L \subset X$ be a lattice. Also, let X^* and L^* be their duals. Let $S \subset L^*$ ($\#S \geqslant 2$) be a finite set which satisfies the following condition:

$$\text{The set } \{\beta - \beta'; \beta, \beta' \in S\} \quad \text{spans } X^*.$$

Let P be the convex hull of the finite set S. Let $L(S)^*$ be the lattice in X^* spanned by $\{\beta - \beta'; \beta, \beta' \in S\}$ over $\mathbb{Z}$, and let $L(S)$ be its dual lattice. By the above assumptions, we have $L \subset L(S)$, and the quotient $\Pi(S) := L(S)/L$ is a finite group. For a strictly positive function c on S, we define the *weighted multiplicity of lattice paths P_N^c of length N with weight c* and *the set of the allowed steps S* by

$$P_N^c(\gamma) = \sum_{\beta_1, \ldots, \beta_N \in S;\, \gamma = \beta_1 + \cdots + \beta_N} c(\beta_1) \cdots c(\beta_N), \quad \gamma \in (NP) \cap L^*. \tag{21}$$

If $c \equiv 1$, then $P_N^c(\gamma) = P_N(\gamma)$.

If we take $S = p\Sigma \cap \mathbb{N}^m$, where Σ is the standard simplex and p is a positive integer, and if we take the weight function $c(\beta) = \frac{p!}{\beta!(p-|\beta|)!} = \binom{p}{\beta}$, the corresponding weighted multiplicity function $P_N^c(\gamma)$ is given by $P_N^c(\gamma) = \binom{Np}{\gamma}$, and in general one may regard P_N^c as a generalized multinomial coefficient. In Proposition 2.3, we prove that weight multiplicities can be equated with weighted multiplicities of certain lattice paths, specifically

$$m_N(\lambda;\mu) = P_N^{c_\lambda}(\mu - N\lambda), \tag{22}$$

where $P_N^{c_\lambda}$ is a certain weighted lattice path partition function.In Proposition 2.4, we further prove that

$$a_N(\lambda;\mu) = \sum_{w\in W} \operatorname{sgn}(w) P_N^{c_\lambda}(\mu - N\lambda + \rho - w\rho). \tag{23}$$

We now state our results on multiplicities of lattice paths, following the same outline as for weight multiplicities. As in the case of group representations, the simplest question to consider is the weak limit of the measure

$$d\mu_{S,N} := \frac{1}{(\#S)^N} \sum_{\beta\in P_{S,N}} P_N(\beta)\delta_{\frac{\beta}{N}}. \tag{24}$$

It is well-known and easy to prove (see Proposition 1.1) that

$$d\mu_{S,N} \to \delta_{m_S^*}, \quad \text{where } m_S^* = \frac{1}{\#S}\sum_{\beta\in S}\beta \tag{25}$$

is the center of mass of the set S. In the more general case of weighted lattice paths, the center of mass $m_S^* \in P^o$ is given by

$$m_S^* = \frac{1}{V(S)}\sum_{\beta\in S} c(\beta)\beta, \quad V(S) = \sum_{\beta\in S} c(\beta). \tag{26}$$

We then consider the asymptotic distribution of multiplicities of lattice paths in regions around the center point.

These refined results involve the 'moments maps',

$$\mu_P : X \to P^o, \quad \mu_P(\tau) = \sum_{\beta\in S} \frac{c(\beta)e^{\langle\beta,\tau\rangle}}{\sum_{\beta'\in S} c(\beta')e^{\langle\beta,\tau\rangle}}\beta. \tag{27}$$

For $x \in P^o$, the interior of the polytope P, we define the function $\delta_c(S,x)$

$$\delta_c(S,x) = \log\left(\sum_{\beta\in S} c(\beta)e^{\langle\beta - x, \tau_P(x)\rangle}\right), \tag{28}$$

and the positive definite linear map $A_c(S,x) : X \to X^*$ by

$$A_c(S,x) = \sum_{\beta \in S} \left(\frac{c(\beta) e^{\langle \beta, \tau_P(x) \rangle}}{\sum_{\beta' \in S} c(\beta') e^{\langle \beta', \tau_P(x) \rangle}} \right) \beta \otimes \beta - x \otimes x, \quad A = A_c(S, m_S^*), \tag{29}$$

where the diffeomorphism $\tau_P : P^o \to X$ is the inverse of the 'moment map' μ_P.

Remarks. It should be noted that the constant $\delta_c(S,\alpha)$ defined in (28) depends on the choice of the weight function c. In fact, this constant can be negative if we choose the weight function c small enough. However, if c takes positive integer values, then it turns out that the constant $\delta_c(S,\alpha)$ is positive. See *Remark* after the proof of Theorem 11 in Section 1.

0.3.1. Central limit region

Our first result on lattice paths concerns the central limit region where $\gamma_N = Nm_S^* + d_N(\gamma_N)$, where $d_N(\gamma_N) = O(N^s)$ for a variety of $s<1$.

Theorem 10. *Let $0 \leqslant s < 1$. Let γ_N be a sequence of lattice points such that $P_N^c(\gamma_N) \neq 0$ for every sufficiently large N, and assume also that γ_N has the form*

$$\gamma_N = Nm_S^* + d_N(\gamma_N), \quad d_N(\gamma_N) = O(N^s). \tag{30}$$

Then we have

$$P_N^c(\gamma_N) = (2\pi N)^{-m/2} \frac{|\Pi(S)| e^{N\delta_c(S, \frac{\gamma_N}{N})}}{\sqrt{\det A}} (1 + O(N^{-(1-s)})). \tag{31}$$

Furthermore, if $0 \leqslant s \leqslant 2/3$ and $d_N(\gamma_N) = o(N^s)$, we have

$$P_N^c(\gamma_N) = (2\pi N)^{-m/2} \frac{|\Pi(S)| V(S)^N e^{-\langle A^{-1} d_N(\gamma_N), d_N(\gamma_N)/(2N) \rangle}}{\sqrt{\det A}} (1 + \varepsilon_N), \tag{32}$$

where

$$\varepsilon_N = \begin{cases} O(N^{-(1-s)}) & \text{for } 0 \leqslant s \leqslant 1/2, \\ o(N^{3s-2}) & \text{for } 1/2 < s \leqslant 2/3. \end{cases}$$

0.3.2. Large deviations region

We now assume that d_N is of order N.

Theorem 11. *Let α be a lattice point in S which is assumed to lie in the interior of the polytope P. Then, for every $f \in L(S)^*$, we have*

$$P_N^c(N\alpha + f) = (2\pi N)^{-m/2} \frac{|\Pi(S)| e^{-\langle f, \tau_P(\alpha) \rangle + N\delta_c(S,\alpha)}}{\sqrt{\det A_c(S,\alpha)}} (1 + O(N^{-1})), \tag{33}$$

where $|\Pi(S)|$ *denotes the order of the finite group* $\Pi(S) = L(S)/L$. *The exponent* $\delta_c(S, \alpha)$ *is positive if* $c(\alpha) \geqslant 1$.

Our analysis starts from the fact that

$$P_N(\gamma) = \chi_S(u)^N|_{u^\gamma},$$

where $\chi_S(u)^N|_{u^\gamma}$ denotes the coefficient of the monomial u^γ in the Nth power of the *admissible step character*,

$$\chi_S(u) = \sum_{\alpha \in S} u^\alpha. \tag{34}$$

We apply a steepest descent argument to an integral representation of $P_N^c(\gamma)$ (see (39) in Section 1). Our basic reference for the stationary phase for complex phase functions is [Hö].

0.4. Organization

We first prove the results on lattice paths, Theorems 10 and 11, in Section 1. We then deduce the main results on multiplicities, Theorems 5–9, in Section 2. In that section, we also review the relation between multiplicities of weights and lattice paths. In Section 3, we illustrate the results for some representations of $U(m)$ with $m = 2$. In Section 4, we make some final comments on the connections between lattice paths and weight multiplicities and on the symplectic model for tensor product multiplicities.

1. Asymptotics of the number of Lattice paths

Let X be a finite-dimensional real vector space of dimension m, and let L be a lattice in X. Let X^* and L^* be, respectively, the dual vector space of X and the dual lattice of L. Let $S \subset L^*$ be a finite set such that $\#S \geqslant 2$, and set

$$D(S) := \{\beta - \beta' \in L^*; \beta, \beta' \in S\}. \tag{35}$$

We assume that

$$\mathrm{span}_{\mathbb{R}} D(S) = X^*. \tag{36}$$

Let $P = P_S$ be the convex hull of S, which is an integral polytope in X^*. Let

$$c : S \to \mathbb{R}_{>0}$$

be a strictly positive function on S. Our aim in this section is to investigate the asymptotics of the number of the lattice paths $P_N^c(\gamma)$ for the lattice point γ in various regions (central limit region, regions of moderate and strong deviations discussed in the Introduction) as $N \to \infty$.

We introduce the *weighted character* (or the *weighted S-character*) with the weight function c defined by

$$k(w) := \sum_{\beta \in S} c(\beta) e^{\langle \beta, w \rangle}, \quad w \in X^{\mathbb{C}} := X \otimes \mathbb{C}, \tag{37}$$

which is considered as a function on $X^{\mathbb{C}} = X \otimes \mathbb{C}$. Here, and in what follows, a functional $f \in X^*$ is considered as a $\mathbb{C}$-linear functional on $X^{\mathbb{C}}$. We fix a primitive basis for the lattice L, which is also considered as a fixed basis for X. Note that, for $\tau \in X$, the function $\varphi \mapsto k(\tau + i\varphi)$ is a smooth function on the torus $\mathbf{T}^m := X/(2\pi L)$, since we have assumed $S \subset L^*$. The fixed basis in L defines a Lebesgue measure on X, and hence on $X^{\mathbb{C}}$, normalized so that $\mathrm{Vol}(\mathbf{T}^m) = (2\pi)^m$. We also fix an inner product on X which has the fixed basis for L as an orthonormal basis, and we denote by $|\varphi|$ the norm of $\varphi \in X$ with respect to this inner product.

It is clear that the Nth power of the function $k(w)$ is given by

$$k(w)^N = \sum_{\gamma \in (NP) \cap L^*} P_N^c(\gamma) e^{\langle \gamma, w \rangle}. \tag{38}$$

Therefore, the lattice paths counting function P_N^c has the following integral expression:

$$P_N^c(\gamma) = \frac{1}{(2\pi)^m} \int_{\mathbf{T}^m} e^{-i\langle \gamma, \varphi \rangle} k(i\varphi)^N \, d\varphi. \tag{39}$$

To begin with, we shall consider the simplest case, that is, consider the problem how the numbers of lattice paths with endpoints varying in $NP \cap L^*$ distributes. This would be expressed as the weak limit of the measure defined by the following:

$$m_{S,N} := \frac{1}{V(S)^N} \sum_{\gamma \in NP \cap L^*} P_N^c(\gamma) \delta_{\gamma/N}, \quad V(S) := k(0) = \sum_{\beta \in S} c(\beta). \tag{40}$$

Noting that $P_1^c(\gamma) = c(\gamma)(\gamma \in S)$, we have

$$V(S)^N = \sum_{\gamma \in NP \cap L^*} P_N^c(\gamma),$$

which shows that the measure $m_{S,N}$ is a probability measure. The following proposition will be used to prove Theorem 1 in the next section.

Proposition 1.1. *The probability measure $m_{S,N}$ tends weakly to the Dirac measure $\delta_{m_S^*}$ at the point $m_S^* \in P$ given in* (26).

Proof. It suffices to show that the Fourier transform (characteristic function) $\widehat{m_{S,N}}(\varphi)$ of the probability measure $m_{S,N}$ converges to the Fourier transform of the

Dirac measure $\delta_{m_S^*}$ at the point m_S^* for every $\varphi \in X$. The Fourier transform of $\delta_{m_S^*}$ is given by $\varphi \mapsto e^{-i\langle m_S^*, \varphi\rangle}$. By (39), the Fourier transform of $m_{S,N}$ is given by

$$\widehat{m_{S,N}}(\varphi) = \left[\frac{k(-i\varphi/N)}{V(S)}\right]^N, \quad \varphi \in X.$$

Thus we need to show that $\widehat{m_{S,N}}(\varphi) \to e^{-i\langle m_S^*, \varphi\rangle}$ as $N \to \infty$. Since $\widehat{m_{S,N}}(0) = 1$, we can choose a compact neighborhood U of the origin in X such that a branch of the logarithm $\log \widehat{m_{S,N}}(\varphi)$ exists for $\varphi \in U$. For any $\varphi \in X$ we take N large enough so that $\varphi/N \in U$. Then, a Taylor expansion at the origin gives

$$e^{N \log \widehat{m_{S,N}}(\varphi/N)} = e^{-i\langle m_S^*, \varphi\rangle + N^{-1} R_N(\varphi)},$$

where $R_N(\varphi)$ is bounded on compact sets uniformly in N. Therefore, we have $\widehat{m_{S,N}}(\varphi) \to e^{-i\langle m_S^*, \varphi\rangle}$ as $N \to \infty$. □

Our next result is a central limit theorem for the sequence of probability measures.

Proposition 1.2. *We define the measure $d\mu_N$ by*

$$d\mu_N := (D_{\sqrt{N}})_*(\varphi_S)_* dm_{S,N} = \frac{1}{V(S)^N} \sum_{\gamma \in (NP) \cap L^*} P_N^c(\gamma) \delta_{\frac{1}{\sqrt{N}}(\gamma - N m_S^*)}, \tag{41}$$

where $D_{\sqrt{N}} : X^* \to X^*$ denotes the dilation $D_{\sqrt{N}}(x) = \sqrt{N} x$ and $\varphi_S : X^* \to X^*$ denotes the translation $\varphi_S(x) = x - m_S^*$ by the center of mass m_S^*. Then, we have

$$\text{w-}\lim_{N \to \infty} d\mu_N = \frac{e^{-\langle A^{-1}x, x\rangle/2}}{(2\pi)^{m/2}\sqrt{\det A}} dx. \tag{42}$$

where the positive definite symmetric matrix $A = A_c(S; m_S^*)$ is defined in (29).

Proof. We use the central limit theorem [Hö, Theorem 7.6.7] for the measure $d\mu := (\varphi_S)_* dm_{S,1}$. Note that we need the translation φ_S because

$$\int_{X^*} x \, dm_{S,1}(x) = m_S^*,$$

which is, in general, not the origin. Then, we dilate the measure $d\mu$ to get $d\mu_N$ defined in (41). Clearly, the probability measure $d\mu$ satisfies the following properties.

$$\int |x|^2 \, d\mu < +\infty, \quad \int x \, d\mu = 0, \quad A = (A_{jk}), \quad A_{jk} = \int x_j x_k \, d\mu,$$

where we identify X^* with $\mathbb{R}^m$ with respect to the fixed basis. As in the proof of the following Proposition 1.3, if E denotes the infinite product space of P, $d\rho$ denotes the infinite product measure of $dm_{S,1}$ and $X_j : E \to P$ denotes the projection for the jth component, then it is easy to show that

$$(D_{1/N})_*\left(\sum_{j=1}^{N} X_j\right)_* d\rho = (D_{1/N})_*(dm_{S,1} * \cdots * dm_{S,1}) = dm_{S,N},$$

and hence we have

$$d\mu_N = (D_{1/\sqrt{N}})_*(d\mu * \cdots * d\mu),$$

which is precisely the measures described in [Hö]. (Note that, in [Hö], the pull-back of distribution (measure) is used instead of push-forward.) Therefore, the assertion follows from directly from Theorem 7.6.7 in [Hö]. □

Further, we note the large deviations principle for the measures $m_{S,N}$.

Proposition 1.3. *The sequence of measures* $\{m_{S,N}\}$ *satisfies the large deviation principle with the rate function given by*

$$I_S(x) = \sup_{\tau \in X}\{\langle \tau, x\rangle - \log(k(\tau)/V(S))\}. \tag{43}$$

Proof. We apply Cramér's theorem [DZ, Theorem 2.2.30]. We shall recall the setting-up for the Cramér's theorem. Let X_j $(j = 1, 2, \ldots)$ be a sequence of independent identically distributed m-dimensional random vectors on a probability space with X_1 distributed according to the probability measure μ on $\mathbb{R}^m$. Let m_N be the distribution (probability measure) for the empirical means $S_N := \frac{1}{N}\sum_{j=1}^{N} X_j$. Then, Cramer's theorem states that the sequence of measures $\{m_N\}$ satisfies the LDP with the rate function

$$I(x) = \sup_{\tau \in \mathbb{R}^m}\{\langle \tau, x\rangle - \Lambda(\tau)\}, \quad \Lambda(\tau) = \log \mathbf{E}(e^{\langle \tau, X_1\rangle})$$

if $\Lambda(\tau) < \infty$ for every $\tau \in \mathbb{R}^m$. In our case, We take the probability space $E := P \times \cdots$ (infinite product of the polytope P), and the probability measure $m_S \times \cdots$ on E. The random variable X_j is the projection onto the jth factor. Then, it is easy to see that $\Lambda(\tau) = \log(k(\tau)/V(S))$, and the push-forward of the measure $m_S \times \cdots$ by the empirical mean $S_N = \frac{1}{N}\sum_{j=1}^{N} X_j$ is nothing but $m_{S,N}$. Therefore, the assertion is a direct consequence of Cramér's theorem stated above. □

Proposition 1.1 suggests that the number of lattice paths would have a 'peak' at the center of mass (although, in general, the center of mass might not be in the lattice L^*). Thus, it is natural to ask that how the lattice paths counting function $P_N^c(\gamma)$ behave with the distance between γ and the center of mass getting large. But, when N becomes large, the possible end points of the S-lattice paths is in the polytope NP, and the center of mass of NP is Nm_S^* where m_S^* is the center of mass of P defined in

(26). Thus it is natural to consider the behavior of $P_N^c(\gamma)$ when the distance between γ and Nm_S^* varies.

Our next aim in this section is to prove Theorems 10 and 11 which corresponds respectively the the case where γ is in the central limit region (and the region of moderate deviations) and the region of the strong deviations.

1.1. Proof of Theorem 11

First we shall prove Theorem 11. To prove Theorem 11, we need to prepare notation.

Let $\exp : X \to \mathbf{T}^m := X/(2\pi L)$ be the exponential map, i.e., the canonical projection. Since the set of differences $D(S)$ defined in (35) spans X^*, it spans a lattice, $L(S)^*$, in X^* of maximal rank over $\mathbb{Z}$:

$$L(S)^* = \mathrm{span}_{\mathbb{Z}}\, D(S) \subset L^*, \tag{44}$$

and its dual lattice in X is denoted by $L(S)$. We have $L(S)^* \subset L^*$, and hence $L \subset L(S)$. Both of the lattices is of maximal rank. Thus, the quotient group $\Pi(S) := L(S)/L$ is a finite group.

The finite group $\Pi(S)$ is naturally identified with the kernel of the surjective homomorphism

$$\pi_S : \mathbf{T}^m \to T(S) := X/(2\pi L(S)), \quad \pi_S(\exp\varphi) = \exp_S(\varphi), \tag{45}$$

where $\exp_S : X \to T(S)$ denotes the canonical projection.

Remarks. If we begin with a polytope P, the function c above should be a non-negative function on $P \cap L^*$. In this case, the corresponding finite set S should be the support of the function c. Thus, the support S of the function c is assumed to satisfy (36). If the set $D(S)$ defined in (35) spans the lattice L^* over $\mathbb{Z}$, then the corresponding torus $T(S)$ coincides with the original torus $\mathbf{T}^m$, and hence $\Pi(S) = \{1\}$.

Lemma 1.4. *For any fixed vector $\tau \in X$, we denote $k_\tau(\exp\varphi) := k(\tau + i\varphi)$, which is considered as a function on $\mathbf{T}^m$, where the function k on $X^{\mathbb{C}}$ is given in* (37). *Then we have $|k_\tau(\exp\varphi)| \leqslant k(\tau)$. The equality holds exactly on the kernel of the homomorphism $\pi_S : \mathbf{T}^m \to T(S)$:*

$$\{t \in \mathbf{T}^m; |k_\tau(t)| = k(\tau)\} = \ker \pi_S \cong \Pi(S).$$

In particular, the set in the left hand side is finite.

Proof. The inequality $|k_\tau(\exp\varphi)| \leqslant k(\tau)$ follows from the Cauchy–Schwarz inequality. It is easy to see that the condition $|k_\tau(\exp\varphi)| = k(\tau)$ on $\varphi \in X$ is equivalent to the following:

$$\langle \beta - \beta', \varphi \rangle \in 2\pi\mathbb{Z}, \quad \beta, \beta' \in S.$$

Since $L(S)^* = \text{span}_{\mathbb{Z}} D(S)$, this condition is equivalent to say that $\varphi \in 2\pi L(S)$. This completes the proof. □

Note that the function $k(w) = k(\tau + i\varphi)$ is holomorphic in $w = \tau + i\varphi \in X^{\mathbb{C}}$, and is $2\pi L$-periodic with respect to the variable $\varphi \in X$. Therefore, we can deform the contour of the integral in (39), and hence, by setting $\gamma = N\alpha + f$ in (39), we can write

$$P_N^c(N\alpha + f) = \frac{e^{-\langle f, \varphi \rangle}}{(2\pi)^m} [k(\tau)e^{-\langle \alpha, \tau \rangle}]^N \int_{\mathbf{T}^m} e^{-iN\langle \alpha, \varphi \rangle} \left[\frac{k(\tau + i\varphi)}{k(\tau)} \right]^N e^{-i\langle f, \varphi \rangle} \, d\varphi, \tag{46}$$

where $\tau \in X$ is arbitrary. (Note that $k(\tau) > 0$ for $\tau \in X$.) To choose a suitable $\tau \in X$, we need to find the point where the function $k(\tau)e^{-\langle \alpha, \tau \rangle}$ attains its minimum. To describe the critical points of this function, we define a map $\mu_P : X \cong \mathbb{R}^m \to P^o$ by

$$\mu_P(\tau) := \partial_\tau \log k(\tau) = \frac{1}{\sum_{\beta \in S} c(\beta) e^{\langle \beta, \tau \rangle}} \sum_{\beta \in S} c(\beta) e^{\langle \beta, \tau \rangle} \beta. \tag{47}$$

The map μ_P defined above is an analogue of the moment map for a Hamiltonian torus action on toric manifolds. Thus we call the map μ_P the *moment map*. Since the set $D(S)$ of differences of vectors in the finite set S spans the whole space X^* (over $\mathbb{R}$), the elements in S are not contained simultaneously in any affine hyperplane in X^*. It is well-known [Fu, p. 83] that the moment map μ_P defines a (real analytic) diffeomorphism between the vector space X and the interior P^o of the polytope P.

We denote the inverse of the moment map μ_P by $\tau_P = \tau_P(x) : P^o \to X$. Then, for every $\alpha \in P^o$, we have $\mu_P(\tau_P(\alpha)) = \alpha \in P^o$.

We note that the center of mass m_S^* is the value of the moment map at the origin: $\mu_P(0) = m_S^*$, $\tau_P(m_S^*) = 0$. The differential of the moment map $\mu_P : X \to P^o$ defines the following linear transform $A(\tau) : X \to X^*$.

$$A(\tau) := \sum_{\beta \in S} \frac{c(\beta) e^{\langle \tau, \beta \rangle}}{k(\tau)} \beta \otimes \beta - \mu_P(\tau) \otimes \mu_P(\tau), \quad \tau \in X, \quad A := A(0).$$

Lemma 1.5. *We set*

$$f_\alpha(\tau) := \log k(\tau) - \langle \alpha, \tau \rangle, \quad \tau \in X, \tag{48}$$

so that $e^{f_\alpha(\tau)} = k(\tau)e^{-\langle \alpha, \tau \rangle}$. Then the Hessian of the function f_α, which is given by $A(\tau)$, is a positive definite for every $\tau \in X$. The vector $\tau_P(\alpha)$ is the unique critical point of the function f_α. In fact, we have

$$f_\alpha(\tau) \geqslant f_\alpha(\tau_P(\alpha)), \quad \tau \in X$$

with equality holds only at $\tau = \tau_P(\alpha)$.

Proof. It is straightforward to show that

$$\partial f(\tau) = \mu_P(\tau) - \alpha, \quad A(\tau) = \partial^2 f(\tau). \tag{49}$$

Although one can prove the positivity of the map $A(\tau)$ for every $\tau \in X$ by exactly the same argument as in [SZ], we give a proof of it for completeness. For each $\beta \in S$, we set $m_\beta(\tau) := c(\beta) e^{\langle \beta, \tau \rangle}/k(\tau)$ so that $\sum_{\beta \in S} m_\beta(\tau) = 1$. We define a probability measure v_S^τ on X^* supported on S, depending on $\tau \in X$, by $dv_S^\tau = \sum_{\beta \in S} m_\beta(\tau)\delta_\beta$, where δ_β denotes the Dirac measure at β. Then, for any vector $x \in X$, we have

$$\langle A(\tau)x, x \rangle = \int_{X^*} g_x(v)^2 \, dv_S^\tau(v) - \left| \int_{X^*} g_x(v) \, dv_S^\tau(v) \right|^2 \geqslant 0,$$

where g_x is a linear function on X^* defined by $g_x(v) = \langle v, x \rangle$, $v \in X^*$, $x \in X \cong \mathbb{R}^m$. The equality in the above holds if and only if g_x is constant on S. In such a case, the function g_x is zero on $D(S)$, since g_x is linear. Thus, by assumption (36), g_x is zero on X^*, and which implies $x = 0$. This shows that $A(\tau)$ is positive definite for any $\tau \in X$.

By (49), the vector $\tau_P(\alpha)$ is the unique critical point of the function f_α, since the map $\mu_P : X \to P^o$ is a diffeomorphism. A Taylor expansion at $\tau = \tau_P(\alpha)$ for the function f_α gives

$$f_\alpha(\tau) = f_\alpha(\tau_P(\alpha)) + \int_0^1 (1-t) \langle A(\tau_P(\alpha) + t(\tau - \tau_P(\alpha)))(\tau - \tau_P(\alpha)), \, \tau - \tau_P(\alpha) \rangle \, dt.$$

Since $A(\tau)$ is positive definite, the last integral is non-negative, and equals zero if and only if $\tau = \tau_P(\alpha)$. This completes the proof. □

It should be noted that the constant $\delta_c(S, \alpha)$ and the matrix $A_c(S, \alpha)$ defined by (28), (29) in Theorem 11 can be written as

$$A_c(S, \alpha) = A(\tau_P(\alpha)), \tag{50}$$

$$\delta_c(S, \alpha) = f_\alpha(\tau_P(\alpha)). \tag{51}$$

Hence the matrix $A_c(S, \alpha)$ is real symmetric and positive definite. It should be noted that the function $\delta_c(S, x)$ on P^o defined in (28) satisfies

$$\delta_c(S, x) = \log(V(S)) - I_S(x), \quad x \in P^o, \tag{52}$$

where the function I_S is the rate function defined by (43).

We choose the vector $\tau \in X$ in (46) as $\tau = \tau_P(\alpha)$. Recall that, by Lemma 1.4, the absolute value of the integrand in (46) equals one precisely on the set $\ker \pi_S \subset \mathbf{T}^m$, where $\pi_S : \mathbf{T}^m \to T(S)$ is a homomorphism. The set $\ker \pi_S$ is a subgroup in $\mathbf{T}^m$ and isomorphic to $\Pi(S) = L(S)/L$, which is a finite group. For each $g \in \ker \pi_S \cong \Pi(S)$, we take a representative $\varphi_g \in X$ so that $g = \exp \varphi_g$. Let $V_g \subset U_g$ be open neighborhoods of the vector $\varphi_g \in X$ such that $U_g \cap \ker \pi_S = \{g\}$ and $\overline{V_g} \subset U_g$, and a branch of the logarithm

$$\log\left(\frac{k(\tau_P(\alpha) + i\varphi)}{k(\tau_P(\alpha))}\right)$$

exists on each of U_g. We choose a constant $c > 0$ so that

$$|k(\tau_P(\alpha) + i\varphi)/k(\tau_P(\alpha))| \leqslant e^{-c} \quad \text{for } \exp \varphi \in \mathbf{T}^m \Big\backslash \bigcup_{g \in \ker \pi_S} V_g.$$

Let χ_g be a smooth function on X supported in the open set U_g and equals one near V_g. Then we can write integral (46) in the following form:

$$P_N^c(N\alpha + f) = \frac{e^{N\delta_c(S,\alpha) - \langle f, \tau_P(\alpha) \rangle}}{(2\pi)^m} \times \left(\sum_{g \in \ker \pi_S} \int_{\mathbf{T}^m} e^{N\Phi_{\alpha,g}(\varphi)} \chi_g(\varphi) e^{-i\langle f, \varphi \rangle} \, d\varphi + O(e^{-Nc}) \right), \tag{53}$$

where the phase function $\Phi_{\alpha,g}(\varphi)$ is given by

$$\Phi_{\alpha,g}(\varphi) = \log\left(\frac{k(\tau_P(\alpha) + i\varphi)}{k(\tau_P(\alpha))}\right) - i\langle \alpha, \varphi \rangle.$$

By definition, the vectors φ_g are in $2\pi L(S)$. This implies that $\langle \beta - \beta', \varphi_g \rangle$ is 2π times an integer for any $\beta, \beta' \in S$. Therefore, the complex number

$$h(g) := e^{i\langle \beta, \varphi_g \rangle} \in U(1), \quad \beta \in S, \quad g \in \ker \pi_S \tag{54}$$

does not depend on the choice of $\beta \in S$ and $\varphi_g \in \exp^{-1}(g) \subset X$. Furthermore, we have

$$k(\tau + i\varphi_g) = h(g)k(\tau), \quad (\partial_\varphi k)(\tau + i\varphi_g) = ih(g)(\partial k)(\tau), \quad \tau \in X. \tag{55}$$

Lemma 1.6. *For each $g \in \ker \pi_S \cong \Pi(S)$, we set*

$$C_g := \{\varphi \in U_g; R\Phi_{\alpha,g}(\varphi) = 0, \partial_\varphi \Phi_{\alpha,g}(\varphi) = 0\}.$$

Then we have $C_g = \{\varphi_g\}$. Furthermore, we have

$$e^{N\Phi_{\alpha,g}(\varphi_g)} = h(g)^N e^{-iN\langle \alpha,\varphi_g\rangle}, \quad \mathrm{Hess}(\Phi_{\alpha,g})(\varphi_g) = -A_c(S,\alpha).$$

Proof. That the real part of the phase function $\Phi_{\alpha,g}$ is less than or equal to zero follows from the Cauchy–Schwarz inequality, since we have the obvious identity

$$R\Phi_{\alpha,g}(\varphi) = \log\left(\frac{|k(\tau_P(\alpha)+i\varphi)|}{k(\tau_P(\alpha))}\right).$$

By the above identity and Lemma 1.4, $R\Phi_{\alpha,g}(\varphi) = 0$ for $\varphi \in U_g$ if and only $\varphi = \varphi_g$. Thus the critical set C_g is empty or consists of the point φ_g. By (55), we have

$$(\partial_\varphi \Phi_{\alpha,g})(\varphi_g) = i\left[\frac{(\partial k)(\tau_P(\alpha)+i\varphi_g)}{k(\tau_P(\alpha))} - \alpha\right] = i[\mu_P(\tau_P(\alpha)) - \alpha] = 0,$$

which shows $C_g = \{\varphi_g\}$. The rest of the assertion can be proved by a similar calculation by using identity (55). □

Completion of proof of Theorem 11: Let $\alpha \in S$ and $f \in L(S)^*$. We set

$$I_g := \int_{\mathbf{T}^m} e^{N\Phi_{\alpha,g}(\varphi)} \chi_g(\varphi) e^{-i\langle f,\varphi\rangle}\, d\varphi.$$

so that, by (53), the lattice paths counting function $P_N^c(N\alpha+f)$ is written as

$$P_N^c(N\alpha+f) = \frac{e^{N\delta_c(S,\alpha)-\langle f,\tau_P(\alpha)\rangle}}{(2\pi)^m}\left(\sum_{g\in \ker \pi_S} I_g + O(e^{-cN})\right)$$

for some constant $c>0$. To obtain an asymptotic estimate for the integral I_g, we shall use the method of stationary phase with a complex phase function. In fact, by Lemma 1.6 and Theorem 7.7.5 in [Hö], we have

$$I_g = \left(\frac{N}{2\pi}\right)^{-m/2} \frac{e^{N\Phi_{\alpha,g}(\varphi_g)-i\langle f,\varphi_g\rangle}}{\sqrt{\det A_c(S,\alpha)}}(1+O(N^{-1})). \tag{56}$$

Since $f \in L(S)^*$ and $\varphi_g \in 2\pi L(S)$, $\langle f,\varphi_g\rangle$ is 2π times an integer. Furthermore, we have assumed that $\alpha \in S$. Therefore, by Lemma 1.6 and the definition of $h(g) \in U(1)$, we have

$$e^{N\Phi_{\alpha,g}(\varphi_g)-i\langle f,\varphi_g\rangle} = h(g)^N e^{-i\langle N\alpha+f,\varphi_g\rangle} = 1,$$

which shows the asymptotic formula 33. As for the constant $\delta_c(S,\alpha)$, by taking the exponential $e^{\delta_c(S,\alpha)}$, it is easy to prove that $\delta_c(S,\alpha)>0$ if $c(\alpha)\geqslant 1$.

Remarks. The constant $\delta_c(S,\alpha)$ can be negative. To be precise, we set $c = \max_{\beta\in S} c(\beta)$, and $f=0$. Then $P_N^c(N\alpha) \leqslant c^N P_N^1(N\alpha)$, where $P_N^1(N\alpha)$ is the number of (non-weighted) lattice paths

$$P_N(N\alpha) = \sharp\{(\beta_1, \ldots, \beta_N) \in S^N; N\alpha = \beta_1 + \cdots + \beta_N\}.$$

Thus if $c < e^{-\delta_1(S;\alpha)}$, then $P_N^c(N\alpha)$ decays exponentially. This proves that if $c(\beta) < e^{-\delta_1(S;\alpha)}$, then we have $\delta_c(S, \alpha) < 0$.

1.2. Proof of Theorem 10

Next, we shall prove Theorem 10. The same method as in the proof of Theorem 11 will show the following

Proposition 1.7. *Let x be a point in the interior P^o of the polytope P, and let $\gamma_N = Nx + d_N(\gamma_N)$ be a sequence of lattice points in L^* with $d_N(\gamma_N) = o(N)$. Assume that $P_N^c(\gamma_N) \neq 0$ for every sufficiently large N. Then, we have*

$$P_N^c(\gamma_N) = (2\pi N)^{-m/2} \frac{|\Pi(S)| e^{N\delta_c(S,\gamma_N/N)}}{\sqrt{\det A_c(S, \gamma_N/N))}} (1 + O(N^{-1})). \tag{57}$$

In particular, we have

$$\lim_{N\to\infty} \frac{1}{N} \log P_N^c(\gamma_N) = \delta_c(S, x). \tag{58}$$

Proof. The proof is almost the same as the proof of Theorem 11, so we give its proof briefly. In the following, we shall write γ for the sequence γ_N for simplicity of notation. As in (53), we can write

$$P_N^c(\gamma) = \frac{e^{N\delta_c(S,\gamma/N)}}{(2\pi)^m} \left(\sum_{g\in\ker\pi_S} \int_{\mathbf{T}^m} e^{N\Psi_{N,\gamma}(\varphi)} \chi_g(\varphi)\, d\varphi + O(e^{-Nc}) \right)$$

for some constant $c > 0$, where with the phase function $\Psi_{N,\gamma}$ is given by

$$\Psi_{N,\gamma}(\varphi) = \log\left[\frac{k(\tau_P(\gamma/N) + i\varphi)}{k(\tau_P(\gamma/N))}\right] - i\langle \gamma/N, \varphi \rangle.$$

Here, it should be noted that $\delta_c(S;\gamma/N) = \log k(\tau_P(\gamma/N)) - \langle \gamma/N, \tau_P(\gamma/N)\rangle$. The phase function $\Psi_{\gamma,N}$ satisfies $R\Psi_{\gamma,N} \leqslant 1$, and the point φ_g is the only critical point with $R\Psi_{\gamma,N} \leqslant 1$ on the support of χ_g. The Hessian of $\Psi_{\gamma,N}$ at φ_g is $-A(\tau_P(\gamma/N)) = -A_c(S, \gamma/N)$. Although the phase $\Psi_{\gamma,N}$ depends on N, it is directly shown that its C^4-norm on the support of the cut-off function χ_g is bounded in N. Since $d_N(\gamma_N) = o(N)$ and τ_P is continuous on the interior P^o, we have $\gamma/N \to x \in P^o$ as $N \to \infty$ and hence $A(\tau_P(\gamma/N)) \to A(\tau_P(x))$ as $N \to \infty$. This shows that the norm of $A(\tau_P(\gamma/N))$ is bounded from below uniformly in N. We have assumed that $P_N^c(\gamma)\neq 0$ for every sufficiently large N, and hence there exists $\beta_1,\dots,\beta_N\in S$ such that $\gamma=\beta_1+\cdots+\beta_N$. Thus, we have

$$e^{N\Psi_{\gamma,N}(\varphi_g)}=h(g)^N e^{-i\langle\gamma,\varphi_g\rangle}=h(g)^N e^{-i\sum_{j=1}^N\langle\beta_j,\varphi_g\rangle}=1$$

for any $g\in\ker\pi_S$ for every sufficiently large N. Therefore, Eq. (57) follows from Theorem 7.7.5 in [Hö]. Next, we note that $\delta_c(S,\gamma/N)\to\delta_c(S,x)$ and $A_c(S,\gamma_N/N)\to A_c(S,x)$ as $N\to\infty$. Therefore, by taking the logarithm of (57), we obtain (58). □

Completion of proof of Theorem 10: First, note that we have set $A=A(0)$. We use Proposition 1.7 with $x=m_S^*$. We have $\sqrt{\det A(\tau)}=\sqrt{\det A}(1+O(|\tau|))$ near $\tau=0$. Noting $\gamma/N-m_S^*=N^{-1}d_N(\gamma)=O(N^{-(1-s)})$ and $\tau_P(m_S^*)=0$, we have

$$\sqrt{\det A(\tau_P(\gamma/N))}=\sqrt{\det A}(1+O(N^{-(1-s)})).$$

This combined with Proposition 1.7 shows the first assertion in Theorem 10. Next, we consider the exponent $\delta_c(S;\gamma/N)$. Since $A(\tau)=(\partial\mu_P)(\tau)$ is bounded from below and since $\tau_P=\mu_P^{-1}$, we have

$$\tau_P(x)=\tau_P(x)-\tau_P(m_S^*)=A^{-1}(x-m_S^*)+O(|x-m_S^*|^2)$$

near $x=m_S^*$. A Taylor expansion for the function $f_{\gamma/N}(\tau):=\log k(\tau)-\langle\gamma/N,\tau\rangle$ at $\tau=0$ gives

$$f_{\gamma/N}(\tau)=\log(V(S))-N^{-1}\langle d_N(\gamma),\tau\rangle+\langle A\tau,\tau\rangle/2+O(|\tau|^3).$$

These two inequalities with the fact that $\delta_c(S;\gamma/N)=f_{\gamma/N}(\tau_P(\gamma/N))$ show that

$$N\delta_c(S;\gamma/N)=N\log(V(S))-\langle A^{-1}d_N(\gamma),d_N(\gamma)\rangle/(2N)+O(N^{-2}|d_N(\gamma)|^3).$$

From this, it is clear that, if $d_N(\gamma)=o(N^s)$ with $0\leqslant s\leqslant 2/3$, then $O(N^{-2}|d_N(\gamma)|^3)=o(N^{3s-2})$ with $3s-2\leqslant 0$, which completes the proof.

Example. Let us examine Theorems 11 and 10 for the case where $S=p\Sigma\cap\mathbb{Z}^m$ with the standard simplex $\Sigma\subset\mathbb{R}^m$ and a positive integer p. We choose the weight function $c(\beta)=\binom{p}{\beta}$, $\beta\in S$. We take $L=\mathbb{Z}^m\subset X=\mathbb{R}^m$. Then, the finite group $\Pi(S)$ is trivial. For any vector $x=(x_1,\dots,x_m)\in\mathbb{R}^m$ with nonnegative coefficients $x_j\geqslant 0$, we set $|x|=\sum_{j=1}^m x_j$. The weighted lattice paths counting function P_N^c is given by

$$P_N^c(\gamma)=\binom{Np}{\gamma}=\frac{(Np)!}{(Np-|\gamma|)!\gamma_1!\cdots\gamma_m!},\quad \gamma\in Np\Sigma\cap\mathbb{Z}^m.$$

The S-character k, the moment map μ_P and its inverse τ_P are given by

$$k(\tau) = (1 + |e^{\tau}|)^p, \quad \mu_P(\tau) = \frac{pe^{\tau}}{1 + |e^{\tau}|}, \quad \tau_P(x) = \log\left(\frac{x}{p - |x|}\right), \quad x \in P^o, \; \tau \in \mathbb{R}^m,$$

where, for example, we write $\log x = (\log x_1, \ldots, \log x_m)$. It is easy to see that the function $\delta_c(S, x)$ is given by

$$\delta_c(S, x) = \log\left(\frac{p^p}{x^x (p - |x|)^{p-|x|}}\right), \quad x \in p\Sigma^o,$$

where $x^x = x_1^{x_1} \cdots x_m^{x_m}$. Thus, Proposition 1.7 tells us that, for $\gamma = Nx + o(N)$ with $x \in p\Sigma^o$,

$$\lim_{N \to \infty} \frac{1}{N} \log\binom{Np}{\gamma} = \log\left(\frac{p^p}{x^x (p - |x|)^{p-|x|}}\right),$$

which can easily be deduced from Stirling's formula. As for the matrix $A_c(S, x)$, we have the following simple lemma.

Lemma 1.8. *For $x \in p\Sigma^o$, the matrix $A_c(S, x)$, its determinant and its inverse are given by*

$$A_c(S, x) = \left(x_j \delta_{ij} - \frac{1}{p} x_i x_j\right)_{ij}, \quad \det A_c(S, x) = \frac{(p - |x|) x_1 \cdots x_m}{p},$$

$$A_c(S, x)^{-1} = \left(\frac{\delta_{ij}}{x_j} + \frac{1}{p - |x|}\right)_{ij}. \tag{59}$$

Proof. By applying the operators $x_i \partial_{x_i}$ and $x_j \partial_{x_j}$ to the formula $|x|^k = \sum_{|\beta|=k} \frac{k!}{\beta!} x^{\beta}$ with $|x| = \sum x_j$, we have

$$\sum_{|\beta|=k} \frac{k!}{\beta!} x^{\beta} \beta_i \beta_j = k x_i |x|^{k-1} \delta_{ij} + k(k-1)|x|^{k-2} x_i x_j.$$

A direct computation shows that the coefficients $a_{ij}(x)$ of the matrix $A_c(S, x)$ is given by

$$a_{ij}(x) = \sum_{|\beta| \leqslant p} \binom{p}{\beta} \frac{(p - |x|)^{p-|\beta|}}{p^p} x^{\beta} \beta_i \beta_j - x_i x_j.$$

The first part in Eq. (59) follows from these two formulas. We set $D_m(x_1, \ldots, x_m) := \det A_c(S, x)$ with $x = (x_1, \ldots, x_m)$. Then, a simple computation shows that

$$\frac{D_m(x_1,\dots,x_m)}{x_1\cdots x_m}=\frac{D_{m-1}(x_2,\dots,x_m)}{x_2\cdots x_m}-\frac{1}{p}x_1.$$

Thus, the second part in (59) follows from the induction on m. The formula for the inverse matrix is easily verified by a direct computation. □

Thus, by Theorem 11, we have

$$P_N^c(N\alpha)\sim(2\pi N)^{-m/2}\frac{p^{Np+1/2}}{\alpha^{N\alpha+\mathbf{1/2}}(p-|\alpha|)^{N(p-|\alpha|)+1/2}},$$

where $\mathbf{1/2}=(1/2,\dots,1/2)\in\mathbb{R}^m$. By Lemma 1.8, we have $m_S^*=(\frac{p}{1+m},\dots,\frac{p}{1+m})$ and $V(S)=(1+m)^p$. Therefore, by Theorem 10, we obtain

$$P_N^c(\gamma)=\binom{Np}{\gamma}\sim(2\pi Np)^{-m/2}(m+1)^{Np+(m+1)/2}e^{-\frac{m+1}{2Np}(||\gamma-Nm_S^*||^2+|\gamma-Nm_S^*|^2)},$$

where $||x||^2=\sum x_j^2$ for a vector $x\in\mathbb{R}^m$. These formulas can be deduced from Stirling's formula.

2. Application to multiplicities of group representations

In this section, we shall prove Theorems 1 and 5–9 as applications of Theorems 10 and 11. As in the introduction, let G be a compact connected Lie group, and we fix a maximal torus T in G. For any irreducible representation V_λ of G with highest weight λ, the multiplicity of a weight v in the Nth tensor power $V_\lambda^{\otimes N}$ is denoted by $m_N(\lambda;v)$. Similarly, the multiplicity of an irreducible summand V_v in $V_\lambda^{\otimes N}$ with the highest weight v is denoted by $a_N(\lambda;v)$.

2.1. Relation between number of lattice paths and multiplicities

First of all, we shall explain the relations between the weighted number of lattice paths discussed in Section 1 and the multiplicities m_N and a_N in group representations. The main results are Propositions 2.3 and 2.4. In this subsection, we prepare lemmas and propositions.

Let g and t be the Lie algebras of G and T, respectively. We fix an inner product $\langle\cdot,\cdot\rangle$ on g invariant under the adjoint action, which determines an inner product on t invariant under the Weyl group W. In case where G is semisimple, we use the negative Killing form as a fixed inner product. We sometimes identify the spaces g and t with their duals g^* and t^*, respectively, by the fixed inner product. Let $I\subset t$ be the integral lattice, i.e., $I=\exp^{-1}(1)$, and let $I^*\subset t^*$ be its dual lattice, i.e., the lattice of weights. We fix an (open) dual Weyl chamber C in t^*. Let Φ and Φ_+ denote, respectively, the sets of the roots and the positive roots, respectively. Let $B\subset\Phi_+$ be

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the set of the simple roots, so that $f \in C$ if and only if $\langle f, \alpha \rangle > 0$ for all $\alpha \in B$. Let X^* be the linear span of the simple roots in t^*, and let $X = X^{**}$ be its dual space. The vector space X is regarded as a subspace in t by using the fixed inner product. Since the simple roots are linearly independent, they form a basis of the vector space X^*. Thus we have $\dim X^* = \#B =: m$. The subspace $X \subset t$ is spanned by the inverse roots $\alpha^* := 2\kappa^{-1}(\alpha)/\langle \alpha, \alpha \rangle$, where $\kappa : t \to t^*$ is an isomorphism induced by the fixed W-invariant inner product $\langle \cdot, \cdot \rangle$. We also note that all the roots is in X^*.

Each dominant weight $\lambda \in \bar{C} \cap I^*$ corresponds to an irreducible unitary representation V_λ. We define the finite set $M_\lambda \subset I^*$ by the support of the multiplicity function:

$$M_\lambda := \{\mu \in I^*; m_1(\lambda; \mu) \neq 0\},$$

where $m_1(\lambda; \mu)$ denotes the multiplicity of the weight μ in V_λ. Note that the convex hull $Q(\lambda)$ of the W-orbit of λ coincides with the convex hull of M_λ. The dimension of the polytope $Q(\lambda)$ might be less than that of t. However, as we shall see soon, the polytope $Q(\lambda)$ is contained in the affine subspace $X^* + \lambda$ in t^*. Thus, the interior $Q(\lambda)^o$ of $Q(\lambda)$ means, in the following, the interior of $Q(\lambda)$ considered as a polytope in the above affine subspace. If G is semisimple, then clearly $X^* = t^*$, and hence we can use the polytope $Q(\lambda)$ as the polytope P in Section 1. However, in general, the finite set $M_\lambda \subset I^*$ of all the weights in V_λ is not in the subspace X^*. Thus, we have to modify it. Namely, we set

$$S_\lambda = \{\mu - \lambda; \mu \in M_\lambda\}.$$

Lemma 2.1. *We set* $D(S_\lambda) = \{\beta - \beta'; \beta, \beta' \in S_\lambda\}$. *If* $\lambda \in C \cap I^*$, *then we have*

$$\operatorname{span}_{\mathbb{R}} D(S_\lambda) = X^*,$$

where the subspace $X^* \subset t^*$ *is, as above, the linear span of the simple roots.*

Remarks. It should be noted that we denote by C the *open* Weyl chamber. If $\lambda \in \bar{C}$ is contained in a wall, the linear span $\operatorname{span}_{\mathbb{R}} D(S_\lambda)$ will be a proper subspace of X^*. In fact, in the case where $G = U(2)$, the Weyl group is the symmetric group of order $2! = 2$, and the Weyl chamber is a half-plane in a two-dimensional vector space. Thus, if λ is in the wall, which is the unique wall defined by the orthogonal complement of the (unique) positive root, then it is stable under the Weyl group action. Thus, the corresponding set M_λ consists of the single point λ, and the linear span $\operatorname{span}_{\mathbb{R}} D(S_\lambda)$ is the trivial subspace $\{0\}$.

Proof. We first note that the difference $\lambda - v$ between the dominant weight λ and any weight $\mu \in M_\lambda$ is a linear combination of the simple roots with non-negative coefficients (see [BD]). Thus we have $\operatorname{span}_{\mathbb{R}} D(S_\lambda) \subset X^*$. Next, let α be any simple roots. Then, one has $\lambda(\alpha^*) = \lambda - s_\alpha \lambda$, where $s_\alpha \in W$ is reflection with respect to the wall $\ker \alpha \subset t \cong t^*$. Since λ is assumed to lie in the interior of the Weyl chamber, $\lambda(\alpha^*) \neq 0$. Thus, one has $\alpha \in \operatorname{span}_{\mathbb{R}} D(S_\lambda)$ for any simple root α, which implies $X^* = \operatorname{span}_{\mathbb{R}} D(S_\lambda)$. $\square$

We consider the lattice $L^* = X^* \cap I^*$ of weights in X^* as a fixed lattice in X^*, as in Section 1. In Section 1, the lattice $L(S)^*$ spanned by $D(S)$ played a role. In our case, the lattice $L(S_\lambda)^*$ spanned by $D(S_\lambda)$ does not depend on λ for generic λ as follows.

Lemma 2.2. *Let $\Lambda^* \subset X^*$ be the lattice spanned by the roots over $\mathbb{Z}$. Assume that the dominant weight λ is in the open Weyl chamber C. Then we have*

$$L(S_\lambda)^* := \mathrm{span}_{\mathbb{Z}}(D(S_\lambda)) = \Lambda^*.$$

Proof. It is well known that the difference $\mu - \mu'$ of any two weights μ, μ' in M_λ is in the root lattice Λ^*. Thus, we have $L(S_\lambda)^* \subset \Lambda^*$. This holds for arbitrary dominant weight $\lambda \in \bar{C}$. Now, we assume that $\lambda \in C$. This implies that the integer $\lambda(\alpha^*)$ is strictly positive for every simple root α. It is also well-known that the string of weights of the form

$$\lambda, \ \lambda - \alpha, \ \ldots, \ s_\alpha \lambda = \lambda - \lambda(\alpha^*)\alpha$$

is contained in M_λ. In particular, we have $\lambda - \alpha \in M_\lambda$. This shows that $\alpha \in L(S_\lambda)^*$ for every simple root α. Since every root can be expressed as a linear combination of the simple roots with integer coefficients, we have $\Lambda^* \subset L(S_\lambda)^*$, which completes the proof. □

By Lemma 2.1, the finite set S_λ is a subset in L^*. Let $P_\lambda \subset X^*$ be the convex hull of the finite set S_λ. The relation of the polytopes $Q(\lambda)$ and P_λ is

$$P_\lambda = Q(\lambda) - \lambda \subset X^*.$$

The polytope P_λ contains the origin in X^* as a vertex. Finally, we define the weight function c_λ on S_λ by

$$c_\lambda(\beta) := m_1(\lambda; \mu), \quad \beta = \mu - \lambda \in S_\lambda,$$

which is, of course, a strictly positive function on S_λ. Thus, we get the data, X^*, L^*, S_λ, c_λ exactly as in Section 1. Furthermore, we have the following.

Proposition 2.3. *Let $P_N^{c_\lambda}(\gamma)$, $\gamma \in L^*$ be the lattice paths counting function in L^* with the weight function c_λ and the set of the allowed steps S_λ. Then we have*

$$m_N(\lambda; \mu) = P_N^{c_\lambda}(\mu - N\lambda)$$

for every $\mu \in NQ(\lambda)$.

Proof. Let χ_λ be the character of V_λ, which is considered as a function on t. The character χ_λ is given explicitly by

$$\chi_\lambda(\varphi) = \sum_{\mu \in M_\lambda} m_1(\lambda; \mu) e^{2\pi i \langle \mu, \varphi \rangle}, \quad \varphi \in t. \tag{60}$$

The character of the tensor power $V_\lambda^{\otimes N}$ is the Nth power χ_λ^N of the character χ_λ. Since the multiplicity $m_N(\lambda;\mu)$ is the coefficients of $e^{2\pi i\langle\mu,\varphi\rangle}$ in χ_λ^N, we have

$$m_N(\lambda;\mu) = \sum_{\mu_1,\dots,\mu_N\in M_\lambda,\ \mu=\mu_1+\cdots+\mu_N} m_1(\lambda,\mu_1)\cdots m_1(\lambda,\mu_N).$$

This shows that $m_N(\lambda;\mu)=0$ if $\mu\notin NQ(\lambda)$. On the other hand, consider, as in Section 1, the weighted polytope character:

$$k(w) = \sum_{\beta\in S_\lambda} c_\lambda(\beta)e^{\langle\beta,w\rangle}, \quad w\in X^{\mathbb{C}}. \tag{61}$$

Then, the lattice paths counting function $P_N^{c_\lambda}(\gamma)$ for $\gamma\in L^*$ is the coefficient of $e^{\langle\gamma,w\rangle}$ in $k(w)^N$. By the definition of the finite set S_λ, we can rewrite the function $k(i\varphi)$ for $\varphi\in X$ as

$$k(i\varphi) = e^{-i\langle\lambda,\varphi\rangle}\chi_\lambda(\varphi/2\pi), \quad \varphi\in X(\subset t). \tag{62}$$

Thus, the coefficient $P_N^{c_\lambda}(\mu-N\lambda)$ of $e^{i\langle\mu-N\lambda,\varphi\rangle}$ in $k(i\varphi)^N$ coincides with $m_N(\lambda;\mu)$, concluding the assertion. □

Next, we discuss the multiplicities of irreducible subrepresentations in the tensor power $V_\lambda^{\otimes N}$. Our strategy to prove Theorem 9 is based on the following alternating sum formula.

Proposition 2.4. *We fix a dominant weight $\lambda\in\bar{C}\cap I^*$. Let ρ be half the sum of the positive roots: $\rho=\frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$. Then we have*

$$\begin{aligned} a_N(\lambda;\mu) &= \sum_{w\in W}\operatorname{sgn}(w)m_N(\lambda;\mu+\rho-w\rho) \\ &= \sum_{w\in W}\operatorname{sgn}(w)P_N^{c_\lambda}(\mu-N\lambda+\rho-w\rho), \quad \mu\in\bar{C}\cap I^*, \end{aligned}$$

where the weighted lattice paths counting function $P_N^{c_\lambda}$ with the weight function c_λ and the set of the allowed steps S_λ in L^*.

Proof. The second equality follows from Proposition 2.3. Although the first equality is a special case of expression (8) in [GM], we give a proof for completeness. Consider the character χ_λ^N of $V_\lambda^{\otimes N}$, which has the following expression:

$$\chi_\lambda^N = \sum_{\mu\in\bar{C}\cap I^*} a_N(\lambda;\mu)\chi_\mu, \tag{63}$$

where χ_μ is the character of an irreducible representation with the highest weight μ. By the Weyl character formula, we have

$$\Delta\chi_\mu = \sum_{w\in W} \mathrm{sgn}(w) e^{2\pi i w(\mu+\rho)},$$

where Δ is the Weyl denominator $\Delta = \sum_{w\in W} \mathrm{sgn}(w) e^{2\pi i w\rho}$. Multiplying (63) by the Weyl denominator Δ, we have

$$\Delta\chi_\lambda^N = \sum_{\mu\in\bar{C}\cap I^*,\, w\in W} \mathrm{sgn}(w) a_N(\lambda;\mu) e^{2\pi i w(\mu+\rho)}, \tag{64}$$

which tells us that the multiplicity $a_N(\lambda;\mu)$ for $\mu\in\bar{C}\cap I^*$ is the coefficient of $e^{2\pi i(\mu+\rho)}$ in $\Delta\chi_\lambda^N$. But, the character χ_λ^N has the decomposition into the weights for T. Therefore we also have

$$\Delta\chi_\lambda^N = \sum_{\gamma\in I^*,\, w\in W} \mathrm{sgn}(w) m_N(\lambda;\gamma) e^{2\pi i(\gamma+w\rho)}. \tag{65}$$

In (65), the term $e^{2\pi i(\mu+\rho)}$ appears for $\gamma\in I^*$ with $\gamma = \mu+\rho-w\rho$ for every $w\in W$. (Note that $\rho - w\rho$ is a weight for every $w\in W$.) Therefore, the coefficient of $e^{2\pi i(\mu+\rho)}$ in (65) is given by

$$\sum_{w\in W} \mathrm{sgn}(w) m_N(\lambda;\mu+\rho-w\rho),$$

which proves the assertion. □

Next, we assume that G is semisimple. In this case, we simply use the set M_λ for the finite set S as in Section 1. Furthermore, we have the following

Lemma 2.5. *Assume that G is semisimple. Then, for any dominant weight λ in the open Weyl chamber C, the center of mass $Q^*(\lambda)\in Q(\lambda)$ of the polytope $Q(\lambda)$ defined by* (3) *is the origin.*

Proof. Clearly, the center of mass $Q^*(\lambda)$ is invariant under the action of the Weyl group W. Thus, for any simple root α and any element w in W, we have $\langle Q^*(\lambda), w\alpha-\alpha\rangle = 0$. By taking $w = s_\alpha$, one see that $Q^*(\lambda)$ is orthogonal to any roots. Let $x_\lambda = \kappa^{-1}Q^*(\lambda)\in t$. Then, x_λ is in $\ker(\alpha)$ for any root α, which implies that $t_\lambda := \exp(x_\lambda)\in T$ is in the kernel determined by each root α. This implies that t_λ is in the center of G (see [BD]). But, the Lie group G is assumed to be semisimple, and hence the center is finite. Therefore, we have $x_\lambda = 0$, and hence $Q^*(\lambda) = 0$. □

2.2. Proof of Theorems 1

First of all, we shall prove Theorem 1. By using Proposition 2.3, we have

$$m_{\lambda,N} = \frac{1}{V(S_\lambda)^N} \sum_{\nu - N\lambda \in NP_\lambda} P_N^{c_\lambda}(\nu - N\lambda)\delta_{\nu/N} = \frac{1}{V(S_\lambda)^N} \sum_{\gamma \in NP_\lambda} P_N^{c_\lambda}(\gamma)\delta_{\gamma/N+\lambda}, \qquad (66)$$

where the weighted volume of the finite set S_λ is given by

$$V(S_\lambda) = \dim V_\lambda = \sum_{\nu - \lambda \in S_\lambda} c_\lambda(\nu - \lambda), \quad c_\lambda(\nu - \lambda) = m_1(\lambda; \nu).$$

The probability measure $m_{S_\lambda,N}$ on X^*, discussed in Section 1, is given by

$$m_{S_\lambda,N} = \frac{1}{V(S_\lambda)^N} \sum_{\gamma \in NP_\lambda} P_N^{c_\lambda}(\gamma)\delta_{\gamma/N}, \qquad (67)$$

which is different from $m_{\lambda,N}$ in the term $\delta_{\gamma/N+\lambda}$ and $\delta_{\gamma/N}$. Thus, for any compact supported continuous function f on t^*, let f_λ be the function obtained by translating f by λ: $f_\lambda(x) = f(x+\lambda)$. Then, we have

$$\int_{X^*} f_\lambda(x)\, dm_{S_\lambda,N} = \int_{t^*} f(x)\, dm_{\lambda,N}. \qquad (68)$$

The point m_{S_λ} is equal to $Q^*(\lambda) - \lambda$, where, as in Introduction, the point $Q^*(\lambda)$ is given in (3), and hence, by Proposition 1.1, we have $m_{\lambda,N} \to \delta_{Q^*(\lambda)}$ weakly as $N \to \infty$. □

2.3. Proof of Theorems 2, 3 and Corollary 4

Next, we shall prove Theorem 3. By Proposition 1.3 and (62), the measures $\{m_{S_\lambda,N}\}$ satisfies the large deviation principle with the rate function

$$I_{S_\lambda}(x) = \sup_{\tau \in X}\{\langle x+\lambda, \tau\rangle - \log(\chi_\lambda(\tau/2\pi i)/(\dim V_\lambda))\}.$$

As in (68), we have $dm_{\lambda,N} = (\phi_\lambda)_* dm_{S_\lambda,N}$ with $\phi_\lambda(x) = x + \lambda$, namely $m_{\lambda,N}(B) = m_{S_\lambda,N}(B - \lambda)$. Thus, the measure $m_{\lambda,N}$ satisfies the large deviation principle with the rate function $I_{S_\lambda}(x-\lambda) = I_\lambda(x)$, where the function $I_\lambda(x)$ is given in (8), which proves Theorem 3. Theorem 2 follows from its lattice path version (Proposition 1.2). (See also the proof of Theorems 6 and 7 below for the description of the matrix A_λ.)

To prove Corollary 4, we need the following lemmas.

Lemma 2.6. *Let $C_N(\lambda) \subset \bar{C}$ be a set of dominant weights defined by*

$$C_N(\lambda) = \{\mu \in \bar{C} \cap I^*; a_N(\lambda;\mu) \neq 0\}.$$

Then, for a weight $\nu \in I^*$, the alternating sum

$$\sum_{\sigma \in W} \operatorname{sgn}(\sigma) m_N(\lambda; \nu + \rho - \sigma\rho) = 0 \tag{69}$$

if and only if $\nu + \rho \notin W(\mu + \rho)$ for every $\mu \in C_N(\lambda)$.

Proof. First, note that in (64), the terms $w(\mu+\rho)$ with $w \in W$ and $\mu \in \bar{C}$ are all distinct since $\mu + \rho \in C$ for every $\mu \in \bar{C}$. Thus, in (64), the coefficient of $e^{2\pi i(\nu+\rho)}$ vanish if and only if $\nu + \rho \notin W(\mu+\rho)$ for every $\mu \in C_N(\lambda)$. Then, comparing (64) with (65), the coefficient of $e^{2\pi i(\nu+\rho)}$ in (65) is given by the alternating sum in (69), proving the lemma. □

Lemma 2.7. *Let ρ be half the sum of the positive roots. For each $w \in W$, we define a map $\psi_{w,N} : t^* \to t^*$ by $\psi_{w,N}(x) = x - (\rho - w\rho)/N$. Then we have*

$$\sum_{w \in W} \operatorname{sgn}(w)(\psi_{w,N})_* dm_{\lambda,N}|_{\bar{C}} = \frac{B_N(\lambda)}{(\dim V_\lambda)^N} dM_{\lambda,N},$$

where $|_{\bar{C}}$ denotes the restriction to the closed Weyl chamber $\bar{C}$, and $B_N(\lambda)$ is defined in (9).

Proof. A direct computation with Lemma 2.6 shows that

$$\begin{aligned}\sum_{w \in W} \operatorname{sgn}(w)(\psi_{w,N})_* m_{\lambda,N} &= \frac{1}{(\dim V_\lambda)^N} \sum_{\nu \in I^*; \nu+\rho \in W(C_N(\lambda)+\rho)} \sum_{w \in W} \operatorname{sgn}(w) \\ &\quad \times m_N(\lambda; \nu + \rho - w\rho)\delta_{\nu/N}. \\ &= \frac{1}{(\dim V_\lambda)^N} \sum_{\mu \in C_N(\lambda)} \sum_{\sigma, w \in W} \operatorname{sgn}(w) \\ &\quad \times m_N(\lambda; \mu + \rho - \sigma^{-1} w\rho)\delta_{\frac{\sigma(\mu+\rho)-\rho}{N}},\end{aligned}$$

where, for the second line, the invariance of the multiplicity $m_N(\lambda; \cdot)$ under the Weyl group has been used. Now, we restrict the above functional on the closed Weyl chamber $\bar{C}$. The point $\frac{\sigma(\mu+\rho)-\rho}{N}$ is in $\bar{C}$ if and only if $\sigma(\mu+\rho) \in \bar{C} + \rho$ since $\bar{C}$ is a cone. But, in the sum above, μ is a dominant weight. Thus, only $\sigma = 1$ term is in $\bar{C}$. Thus, the assertion follows from Proposition 2.4. □

Completion of proof of Corollary 4: First of all, we shall prove upper bound in the large deviation principle. Note that any $\mu \in C_N(\lambda)$ is of order $O(N)$ uniformly, since it is in the convex polytope $NQ(\lambda)$. By the Weyl dimension formula, we have

$$\dim V_\mu = O(N^d), \quad \mu \in C_N(\lambda),$$

with d the number of the positive roots. Then, again the Weyl dimension formula shows

$$(\dim V_\lambda)^N = \sum_{\mu \in C_N(\lambda)} a_N(\lambda;\mu)(\dim V_\mu) = B_N(\lambda)O(N^d).$$

Let $F \subset \bar{C}$ be a closed set. Then, by Lemma 2.7,

$$\frac{1}{N}\log(M_{\lambda,N}(F)) = \frac{1}{N}\log\left(\sum_w \operatorname{sgn}(w)m_{\lambda,N}(F + (\rho - w\rho)/N)\right) + O(N^{-1}\log N).$$

For any positive integer $n > 0$, we set

$$F_n := \left\{x \in \bar{C}; \inf_{y \in F} |x - y| \leqslant 1/n\right\},$$

which is of course a closed set in $\bar{C}$. We choose a constant $a > 0$ so that $|\rho - w\rho| \leqslant a$ for every $w \in W$. Then, clearly $F + (\rho - w\rho)/N \subset F_t$ for $a/N \leqslant 1/n$. Hence, for every n, we have

$$\frac{1}{N}\log M_{\lambda,N}(F) \leqslant \frac{1}{N}\log m_{\lambda,N}(F_n) + O(N^{-1}\log N).$$

Since the measures $m_{\lambda,N}$ satisfies the large deviation principle, we obtain

$$\limsup_{N\to\infty} \frac{1}{N}\log M_{\lambda,N}(F) \leqslant -\inf_{x \in F_n} I_\lambda(x),$$

where the rate function $I_\lambda(x)$ is given by (8). Now, we claim that

$$\lim_{n\to\infty} a_n = \inf_{x \in F} I_\lambda(x), \quad a_n := \inf_{x \in F_n} I_\lambda(x), \tag{70}$$

which will completes the proof, where the existence of the limit in the left-hand side is shown as follows. The set F_n is decreasing: $F_n \supset F_{n+1}$, and the sequence $\{a_n\}$ is non-decreasing. This sequence is bounded from above by $a := \inf_{x\in F} I_\lambda(x)$ because $F = \cap_{n \leqslant 1} F_n$. Thus, $a_\infty := \lim_{n\to\infty} a_n$ exists. In particular $a \geqslant a_\infty$. The rate function $I_\lambda(x)$ is lower-semicontinuous, and is *good* in the sense that its sublevel set $I_\lambda^{-1}[0,\alpha]$ is compact for every $\alpha > 0$ (see [DZ]). Thus, the function I_λ attains its minimum on each closed set. Let $x_n \in F_n$ be a point such that $I_\lambda(x_n) = a_n$. Note that x_n is in the compact set $I_\lambda^{-1}[0,a]$, and hence it has a convergent subsequence. We also denote it by x_n. Since F is closed, there exists a point $y_n \in F$ such that $\inf_{y\in F} |y - x_n| = |y_n - x_n| \leqslant 1/n$, and, as a result, $\{y_n\}$ contains a convergent sequence. Therefore, the limit $x := \lim x_n$ is in F. By the lower-semicontinuity, we have

$$a_\infty = \lim_{n\to\infty} I_\lambda(x_n) \geqslant I_\lambda(x) \geqslant a = \inf_{x\in F} I_\lambda(x) \geqslant a_\infty,$$

which establishes (70). □

2.4. Proof of Theorems 6 and 7

By Theorem 11 and Proposition 2.3, we have an asymptotic estimate of the multiplicity $m_N(\lambda; Nv+f)$ if $v_0 \in Q(\lambda)^o$ and $f \in \Lambda^*$. To compute the exponent $\delta_{c_\lambda}(S_\lambda, v_0-\lambda)$ and the linear transform $A_{c_\lambda}(S_\lambda, v_0-\lambda)$ from X to X^* in Theorem 11, we note that the moment map (47) for $S = S_\lambda$ is given by

$$\mu_{P_\lambda} : X \ni x \to \mu_\lambda(x) - \lambda \in P_\lambda,$$

where μ_λ is defined in (14). Thus, we have $\tau_\lambda(v_0) = \tau_{P_\lambda}(v_0-\lambda)$. From this, we have $\delta_{c_\lambda}(S_\lambda, v_0-\lambda) = \delta_\lambda(v_0)$. The positivity of the linear transform $A_{c_\lambda}(S_\lambda, v_0-\lambda)$ from X to X^* is proved in Section 1. A direct computation by using definition (29) shows that

$$A_{c_\lambda}(S_\lambda, v_0-\lambda) = \sum_{\mu\in M_\lambda} k_\mu(v_0)(\mu-\lambda)\otimes(\mu-\lambda) - (v_0-\lambda)\otimes(v_0-\lambda),$$

$$k_\mu(v_0) := \frac{m_1(\lambda;\mu)e^{\langle \mu, \tau_\lambda(v_0)\rangle}}{\sum_{\mu'\in M_\lambda} m_1(\lambda;\mu')e^{\langle \mu', \tau_\lambda(v_0)\rangle}}$$

where, for any $f \in X^*$, $f\otimes f : X \to X^*$ is defined by $(f\otimes f)x = \langle x, f\rangle f$, $x \in X$. By definition ((14)), we have $\sum_\mu k_\mu(v_0)\mu = \mu_\lambda(\tau_\lambda(v_0)) = v_0$. From this, it is easy to see that $A_{c_\lambda}(S_\lambda, v_0-\lambda)$ coincides with the linear transform $A^0_\lambda(v_0)$ on X. This shows that $A_\lambda(v_0)$ is positive definite as a linear transform from X to X^*, and it is equal to $A_{c_\lambda}(S_\lambda, v_0-\lambda)$. The positivity of the exponent $\delta_\lambda(v_0)$ follows from the assumption that the weight v_0 occurs in V_λ. This completes the proof of Theorem 6. Similarly, Theorem 7 is proved by using Proposition 1.7. □

2.5. Proof of Theorem 5

Before proving Theorem 5, we shall state more general result, which corresponds to Theorem 10.

Theorem 2.8. *Let $0 \leqslant s \leqslant 2/3$. Let $v \in NQ(\lambda)$ be a weight of the form*

$$v = NQ^*(\lambda) + d_N(v), \quad |d_N(v)| = o(N^s).$$

Assume that $m_N(\lambda; v) \neq 0$ for every sufficiently large N. Then, we have

$$m_N(\lambda; v) = (2\pi N)^{-m/2}|\Pi(G)|(\dim V_\lambda)^N \frac{e^{-\langle A_\lambda^{-1} d_N(v), d_N(v)\rangle/(2N)}}{\sqrt{\det A_\lambda}}(1+\varepsilon_N),$$

where

$$\varepsilon_N = \begin{cases} O(N^{-(1-s)}) & \text{for } 0 \leqslant s \leqslant 1/2, \\ o(N^{3s-2}) & \text{for } 1/2 < s \leqslant 2/3, \end{cases}$$

and the positive definite linear transformation $A_\lambda : X \to X^*$ is given by

$$A_\lambda = A_\lambda(Q^*(\lambda)) = \frac{1}{\dim V_\lambda} \sum_{\mu \in M_\lambda} m_1(\lambda; \mu)\mu \otimes \mu - Q^*(\lambda) \otimes Q^*(\lambda).$$

Proof. This follows from Theorem 10 and Proposition 2.3, and the computations for the exponent and the matrix by the same method as in the proof of Theorem 6. □

Completion of Proof of Theorem 5: Assume that G is semisimple. Then, by Lemma 2.5, $Q^*(\lambda) = 0$. Thus, $d_N(\lambda)$ is γ itself. Hence, Theorem 5 is a direct consequence of Theorem 2.8.

2.6. Proof of Theorems 9 and 8

For any $w \in W$, the weight $\rho - w\rho$ is in the root lattice Λ^*. Therefore, we can apply Theorem 6 for $f = \rho - w\rho$ and $v_0 = v$. Now, Theorem 9 follows from Proposition 2.4

2.6.1. Proof of Theorem 8

As mentioned in the Introduction, our approach to the irreducible multiplicities based on Proposition 2.4 does not seem to be the most efficient for the central limit region. Our steepest descent method easily gives the principal term, but the remainder estimate becomes tricky since one needs to use cancellations occurring in the alternating sum over the Weyl group. Hence, we use the method of Biane [B] in this region. Although it is not new, we include it for the sake of completeness. We also add some details not in [B].

We begin with:

Lemma 2.9. *Assume that G is semisimple. For any fixed dominant weight λ and the positive integer $N > 0$, we set*

$$NM_\lambda = \{\mu = v_1 + \cdots + v_N; v_j \in M_\lambda, j = 1, \ldots, N\}.$$

Let μ be a dominant weight such that $\mu \notin NM_\lambda$. Then $a_N(\lambda; \mu) = 0$.

Proof. First of all, we note that if V and W are two representations of G, the weights in $V \otimes W$ are of the form $\mu + v$ where μ is a weight in V and v is that in W. If the dominant weight μ is not in NM_λ, we have $m_N(\lambda; \mu) = 0$ and hence $a_N(\lambda; \mu) = 0$. □

Proof of Theorem 8. Since G is assumed to be semisimple, we may use the polytope $Q(\lambda)$ as P in Section 1 and M_λ as the finite set S. Thus, the torus $\mathbf{T}^m$ essentially coincides with the maximal torus T. The finite group $\Pi(G)$ is isomorphic to the kernel of the surjective homomorphism $\pi_\lambda : \mathbf{T}^m \to T(G) := X/(2\pi\Lambda)$. We also note that $\Lambda^* \subset I_\lambda^* \subset L^*$, where $L^* = I^*$ is the full weight lattice, where I_λ^* is the lattice spanned by M_λ over $\mathbb{Z}$.

By the Weyl integration formula (or by using Propositions 2.3, 2.4 and the integral formula (39)), we have

$$a_N(\lambda;\mu) = \frac{(\dim V_\lambda)^N}{(2\pi)^m} \int_{\mathbf{T}^m} e^{-i\langle \mu+\rho,\varphi\rangle} K(\varphi)^N J(\varphi)\, d\varphi, \tag{71}$$

where we set $K(\varphi) = \chi_\lambda(\varphi/2\pi)/\dim V_\lambda$ and $J(\varphi) = \Delta(\varphi/2\pi)$ being χ_λ the character of V_λ and Δ the Weyl denominator $\Delta(H) = \sum_{w\in W} \operatorname{sgn}(w) e^{2\pi i\langle w\rho, H\rangle}$. As in the proof of Theorem 11 (Section 1), we use the cut-off function χ around the origin so that a branch of the logarithm $\log K$ exists on $\operatorname{Supp}\chi$. We also use the function $\chi_g = \chi(\varphi - \varphi_g)$, where $\varphi_g \in 2\pi\Lambda$ is a (fixed) representative of $g \in \ker \pi_\lambda \cong \Pi(G)$, i.e., $g = \exp \varphi_g \in \mathbf{T}^m$ $\pi_\lambda(\exp\varphi_g) = 1$. Then, by Lemma 1.4, we have

$$a_N(\lambda;\mu) = \frac{(\dim V_\lambda)^N}{(2\pi)^m} \left(\sum_{g\in\ker \pi_\lambda} \int e^{-i\langle \mu+\rho,\varphi\rangle} K(\varphi)^N J(\varphi)\chi_g(\varphi)\, d\varphi + O(e^{-cN}) \right)$$

for some constant $c>0$. Now, we make a change of variable $\varphi \mapsto \varphi + \varphi_g$ for each integral in the above. Then, we will have the term

$$e^{-i\langle \mu+\rho,\varphi_g\rangle} h(g)^N J(\varphi+\varphi_g) = \sum_{w\in W} \operatorname{sgn}(w)[e^{-i\langle \mu+\rho,\varphi_g\rangle} h(g)^N e^{i\langle w\rho,\varphi_g\rangle}] e^{i\langle w\rho,\varphi\rangle} \tag{72}$$

in the integrand, where $h(g) = e^{i\langle \nu,\varphi_g\rangle}$ for $g \in \ker\pi_\lambda \cong \Pi(G)$ which does not depends on the choice of $\nu \in M_\lambda$. Note $\rho - w\rho \in \Lambda^*$ for every $w \in W$. Thus $\langle \rho - w\rho, \varphi_g\rangle$ is 2π times an integer. We assume that $\mu \in NM_\lambda$. Then, clearly we have $h(g)^N = e^{i\langle \mu,\varphi_g\rangle}$. Therefore, expression (72) is equal to $J(\varphi)$, and hence we have

$$a_N(\lambda;\mu) = \frac{(\dim V_\lambda)^N |\Pi(G)|}{(2\pi)^m} \int e^{-i\langle \mu+\rho,\varphi\rangle} K(\varphi)^N J(\varphi)\chi(\varphi)\, d\varphi + O(e^{-cN}). \tag{73}$$

By changing the variable $\varphi \mapsto \varphi/N^{1/2}$, we have

$$a_N(\lambda;\mu) = \frac{|\Pi(G)|(\dim V_\lambda)^N}{(2\pi)^m N^{m/2}} I(N),$$

$$I(N) := \int e^{-i\langle \mu+\rho,\varphi\rangle/N^{1/2}} K(\varphi/N^{1/2})^N J(\varphi/N^{1/2}) \chi(\varphi/N^{1/2})\, d\varphi$$

modulo $O(e^{-cN})$. As in [B], we set $\kappa(\varphi) = \prod_{\alpha>0} \langle \alpha, \varphi\rangle$, which is a polynomial of degree $d = \#\Phi_+$, the number of the positive roots. Then, it is easy to show that $J(\varphi/N^{1/2}) = (\frac{i}{N^{1/2}})^d \kappa(\varphi)(1 + |\varphi|^{2d} O(N^{-1}))$. Since $|K(\varphi)|^2$ is real, and since the first derivative of K at the origin is zero (Lemma 2.5), we can choose $r > 0$ such that

$$|K(\varphi)|^2 \leqslant 1 - c\langle A_\lambda \varphi, \varphi \rangle \leqslant e^{-c\langle A_\lambda \varphi, \varphi \rangle}, \quad |\varphi| < r. \tag{74}$$

Replacing χ by a cut-off function whose support is small enough, we have

$$\int |K(\varphi/N^{1/2})|^N |\kappa(\varphi)||\varphi|^{2d} \chi(\varphi/N^{1/2})\, d\varphi = O(1),$$

and hence

$$I(N) = (i/N^{1/2})^d I_1(N)(1 + O(1/N)),$$

$$I_1(N) = \int e^{-i\langle \mu+\rho, \varphi \rangle / N^{1/2}} K(\varphi/N^{1/2}) \kappa(\varphi) \chi(\varphi/N^{1/2}).$$

For simplicity, we set $A_N(\varphi) = e^{-i\langle \mu+\rho, \varphi\rangle/N^{1/2}} \kappa(\varphi)$. A Taylor expansion of $\log K$ at the origin gives

$$K(\varphi/N^{1/2})^N = e^{-\langle A_\lambda \varphi, \varphi \rangle/2 - iT(\varphi)/N^{1/2}} e^{NR_4(\varphi/N^{1/2})}, \tag{75}$$

where $R_4(\varphi) = O(|\varphi|^4)$ locally uniformly. Concerning this expansion, we write

$$I_1(N) = \int A(\varphi) e^{-\langle A_\lambda \varphi, \varphi \rangle/2 - iT(\varphi)/N^{1/2}}\, d\varphi + \sum_{j=1}^{3} \mathbb{I}_j(N), \tag{76}$$

where we set

$$\mathbb{I}_1(N) = \int A(\varphi)(K(\varphi/N^{1/2}) - e^{-\langle A_\lambda \varphi, \varphi \rangle/2 - iT(\varphi)/N^{1/2}}) \chi(\varphi/N^{1/4})\, d\varphi,$$

$$\mathbb{I}_2(N) = \int A(\varphi) K(\varphi/N^{1/2})^N (1 - \chi(\varphi/N^{1/4})) \chi(\varphi/N^{1/2})\, d\varphi,$$

$$\mathbb{I}_3(N) = -\int A(\varphi) e^{-\langle A_\lambda \varphi, \varphi \rangle/2 - iT(\varphi)/N^{1/2}} (1 - \chi(\varphi/N^{1/4}))\, d\varphi.$$

Here we note that $\chi(\varphi/N^{1/4})\chi(\varphi/N^{1/2}) = \chi(\varphi/N^{1/4})$ for sufficiently large N. For the integral $\mathbb{I}_1(N)$, the integrand vanish for $|\varphi| > cN^{1/4}$ for some c. Thus, by (75), we have $|e^{NR_4(\varphi/N^{1/2})}| = O(1)$, and $NR_4(\varphi/N^{1/2}) = |\varphi|^4 O(1/N)$. Therefore we have $|\mathbb{I}_1(N)| = O(1/N)$. For the integral $\mathbb{I}_2(N)$, $\varphi/N^{1/2}$ is bounded. Thus, by (74), we have

$$|\mathbb{I}_2(N)| \leqslant \int_{|\varphi| \geqslant N^{1/4}} e^{-c\langle A_\lambda \varphi, \varphi \rangle /2} |\kappa(\varphi)| \, d\varphi = O(N^{(d+m-1)/4} e^{-cN^{1/2}}).$$

Similarly, it is easy to see that $\mathbb{I}_3(N) = O(N^{(m-2)/4} e^{-cN^{1/2}})$. Finally, we consider the first integral in (76), which can be written in the form

$$\int A(\varphi) e^{-\langle A_\lambda \varphi, \varphi \rangle /2 - iT(\varphi)/N^{1/2}} \, d\varphi = \int e^{-i\langle \mu+\rho, \varphi \rangle /N^{1/2}} \kappa(\varphi) e^{-\langle A_\lambda \varphi, \varphi \rangle /2} \, d\varphi (1 + O(1/N^{1/2})).$$

By using the homogeneity of the polynomial κ of degree d, it is easy to see that

$$\int e^{-i\langle \mu+\rho, \varphi \rangle /N^{1/2}} \kappa(\varphi) e^{-\langle A_\lambda \varphi, \varphi \rangle /2} \, d\varphi = \frac{i^d (2\pi)^{m/2}}{\sqrt{\det A_\lambda}} \kappa(\partial)(e^{-\langle A_\lambda^{-1} \varphi, \varphi \rangle /2})((\mu+\rho)/N^{1/2}).$$

As in [B], by using the fact that the polynomial κ is alternating with respect to the W-action, it is not hard to see that

$$\kappa(\partial)(e^{-\langle A_\lambda^{-1} \varphi, \varphi \rangle /2}) = (-1)^d \kappa(A_\lambda^{-1} \varphi) e^{-\langle A_\lambda^{-1} \varphi, \varphi \rangle /2}.$$

Therefore, we have

$$a_N(\lambda; \mu) = \frac{|\Pi(G)| (\dim V_\lambda)^N \kappa(A_\lambda^{-1}(\mu+\rho))}{(2\pi)^{m/2} N^{d+m/2} \sqrt{\det A_\lambda}} e^{-\langle A_\lambda^{-1}(\mu+\rho), (\mu+\rho) \rangle /2N} (1 + O(1/N^{1/2})).$$

Note that the inner product $\langle A_\lambda^{-1} x, y \rangle$ is invariant under the action of the Weyl group. Therefore, by the Weyl dimension formula, we have

$$\kappa(A_\lambda^{-1}(\mu+\rho)) = (\dim V_\mu) \prod_{\alpha > 0} \langle A_\lambda^{-1} \rho, \alpha \rangle,$$

which concludes the assertion. □

3. Example: $G = U(2)$

In the previous sections, we have obtained the asymptotics of the multiplicities of weights and irreducibles in high tensor power V_λ^N of a fixed irreducible representation V_λ.

The leading term of our asymptotic formula are described by the constant $\delta_\lambda(v)$ and the determinant $\det A_\lambda(v)$ of the matrix $A_\lambda(v)$. In general, it seems somewhat difficult to calculate them explicitly. The most subtle point is the inverse of the "moment map" $\tau_\lambda(v) \in X$. Furthermore, in Theorem 9, the term of the Weyl denominator might vanish. The aim of this section is to discuss them for the group $G = U(2)$.

Roughly speaking, for $G = U(2)$, the corresponding lattice paths model is *Example* in Section 1 with the weight function $c \equiv 1$. (But for general $G = U(m+1)$, it is not identically 1.)

To begin with, we recall some of facts about representation theory for $G = U(m+1)$. Let $T \subset U(m+1)$ $(m \geqslant 1)$ be the maximal torus of all diagonal matrices in the unitary group $U(m+1)$. The Lie algebra t of T consists of all diagonal matrices with pure imaginary entries. We identify t with $\mathbb{R}^{m+1}$ by $(x_1, \ldots, x_{m+1}) \mapsto 2\pi i \mathrm{diag}(x_1, \ldots, x_{m+1})$. Let e_j $(j = 1, \ldots, m+1)$ be the standard basis for $\mathbb{R}^{m+1}$, and let e_j^* be the dual basis. The Weyl group W is the symmetric group S_{m+1} of order $(m+1)!$. We use the usual Euclidean inner product to identify $t \cong \mathbb{R}^{m+1}$ with its dual. The integer lattice and the lattice of weights are identified with $\mathbb{Z}^{m+1}$. We choose the positive open Weyl chamber C given by

$$C = \{\gamma = (\gamma_1, \ldots, \gamma_{m+1}); \gamma_1 > \cdots > \gamma_{m+1}\}.$$

The roots of (G, T) are $\alpha_{i,j} := e_i^* - e_j^*$, $i \neq j$; the positive roots; $\alpha_{i,j}$, $i < j$, and the simple roots; $\alpha_j := \alpha_{j,j+1}$, $j = 1, \ldots, m$. The subspace $X^* \subset t^* \cong \mathbb{R}^{m+1}$ spanned by the simple roots is identified with

$$X \cong X^* = \left\{(x_1, \ldots, x_{m+1}) \subset \mathbb{R}^{m+1}; \sum x_j = 0\right\},$$

which is identified with the Lie algebra of $T \cap SU(m+1)$. Half the sum of the positive roots ρ is given by

$$\rho := \frac{1}{2} \sum_{1 \leqslant i < j \leqslant m+1} \alpha_{i,j} = \frac{1}{2} \sum_{j=1}^{m} (m + 2 - 2j) e_j^*. \tag{77}$$

The alternating sum $A(\gamma)$ for the functional $\gamma \in t^*$ is a function on t given by

$$A(\gamma)(\varphi) := \sum_{w \in S_{m+1}} \mathrm{sgn}(w) e^{2\pi i \langle w\gamma, \varphi \rangle}, \quad \varphi \in t \cong \mathbb{R}^{m+1}.$$

Then the Weyl character formula states that, for a dominant weight $\lambda \in \bar{C} \cap \mathbb{Z}^{m+1}$, the character χ_λ for the irreducible representation V_λ corresponding to λ is given by

$$\chi_\lambda(\varphi) = \frac{A(\lambda + \rho)(\varphi)}{\Delta(\varphi)}, \quad \varphi \in t,$$

where Δ is the Weyl denominator $\Delta = A(\rho)$. In the case where $G = U(m+1)$, one can compute the alternating sum $A(\gamma)$ from the definition, and, as a result, the character χ_λ is given by the *Schur polynomial* s_{ζ_λ} for the partition $\zeta_\lambda = (\lambda_1 - \lambda_{m+1}, \ldots, \lambda_m - \lambda_{m+1}, 0)$:

$$\chi_\lambda(\varphi) = (\xi_1 \cdots \xi_{m+1})^{\lambda_{m+1}} s_{\zeta_\lambda}(\xi_1(\varphi), \ldots, \xi_{m+1}(\varphi)),$$

$$s_{\zeta_\lambda} := \frac{\det(\xi_i(\varphi)^{(\lambda_j-\lambda_{m+1})+m+1-j})}{\det(\xi_i(\varphi)^{m+1-j})}, \quad \xi_j := e^{2\pi i e_j^*},$$

where the denominator in the above is Vandermond's determinant (difference product):

$$D(\xi_1, \ldots, \xi_m) := \prod_{1 \leqslant i < j \leqslant m+1} (\xi_i - \xi_j).$$

If $\lambda_{m+1} \geqslant 0$, then the above is just the Schur polynomial s_λ with the partition λ.

Now we fix a dominant weight $\lambda \in C \cap \mathbb{Z}^{m+1}$. For simplicity, we assume that $\lambda_{m+1} \geqslant 0$ so that the character χ_λ is precisely the Schur polynomial s_λ.

It is well-known (see [FH]) that the multiplicity $m_1(\lambda; \mu)$ of a partition μ (which is equivalent to say that μ is a dominant weight with non-negative entries) is given by the *Kostka number* $K_{\lambda\mu}$ which is the coefficients in the Schur polynomial s_λ of the symmetric sum of the monomials corresponding to μ. It is also well-known [FH] that $K_{\lambda\mu} \neq 0$ if and only if the partition μ satisfies

$$\sum_{j=1}^{i} \mu_j \leqslant \sum_{j=1}^{i} \lambda_j, \quad i = 1, \ldots, m, \tag{78}$$

and $\sum_{j=1}^{m+1} \mu_j = \sum_{j=1}^{m+1} \lambda_j$. (The last condition is necessary, since the weights in V_λ is in the convex hull of the W-orbit of λ.)

We note that the relation between our weighted character function k and the character χ_λ is expressed as

$$k(\tau) = e^{-\langle \lambda, \tau \rangle} \chi_\lambda(\tau/2\pi i) = e^{-\langle \lambda, \tau \rangle} s_\lambda(e^{\tau_1}, \ldots, e^{\tau_m}), \quad \tau = (\tau_1, \ldots, \tau_m) \in X(\subset t). \tag{79}$$

Note that, in the above, the character χ_λ is extended to the complexification $t^{\mathbb{C}}$. In particular, we have

$$\log k(\tau) - \langle v - \lambda, \tau \rangle = \log s_\lambda(e^\tau) - \langle v, \tau \rangle, \quad \tau \in X. \tag{80}$$

Therefore, as in (51), (48), the constant $\delta_\lambda(v)$ is given by

$$\delta_\lambda(v) = \log s_\lambda(e^{\tau_\lambda(v)}) - \langle v, \tau_\lambda(v) \rangle. \tag{81}$$

Now, consider the case where $m = 1$. We take $\lambda = (\lambda_1, \lambda_2) \in C \cap \mathbb{Z}^2$, $\lambda_1 > \lambda_2 \geqslant 0$. We set $n_\lambda = \lambda_1 - \lambda_2 > 0$. Then, the Schur polynomial $s_\lambda(\xi_1, \xi_2)$ in two variables corresponding to the partition λ is given by

$$s_\lambda(\xi_1, \xi_2) = \frac{\xi_1^{\lambda_1+1}\xi_2^{\lambda_2} - \xi_1^{\lambda_2}\xi_2^{\lambda_1+1}}{\xi_1 - \xi_2} = \sum_{j=0}^{n_\lambda} \xi_1^{\lambda_1-j}\xi_2^{\lambda_2+j}. \tag{82}$$

Therefore, the weights in the irreducible representation V_λ are of the form

$$v_j := \lambda - j\alpha, \quad j = 0, \dots, n_\lambda, \tag{83}$$

where α is the unique positive (simple) root $\alpha = (1, -1)$. All these weights have multiplicity one: $m_1(\lambda; v_j) = 1$. Therefore, the multiplicity for the high tensor power $V_\lambda^{\otimes N}$ is given by (see Proposition 2.3)

$$m_N(\lambda; \mu) = \#\{(j_1, \dots, j_N); 0 \leqslant j_k \leqslant n_\lambda, \mu = N\lambda - (j_1 + \cdots j_N)\alpha\}.$$

The polytope P_λ is given by

$$P_\lambda = \{\tau\alpha \in X^*; -n_\lambda \leqslant \tau \leqslant 0\}.$$

Thus, we have the following

Lemma 3.1. *For every $j = 0, \dots, n_\lambda$, v_j is a weight in the interior of $Q(\lambda) = P_\lambda + \lambda$ if and only if $1 \leqslant j \leqslant n_\lambda - 1$. Furthermore, v_j is a dominant weight in the interior of $Q(\lambda)$ if and only if $1 \leqslant j \leqslant \frac{n_\lambda}{2}$.*

Next, we shall calculate the moment map $\mu_{P_\lambda} : X \to P_\lambda$ defined in (47).

Lemma 3.2. *We identify X^* with $\mathbb{R}$ through the identification $\mathbb{R} \ni \tau \mapsto \tau\alpha \in X^*$. We set $h(\tau) = k(\tau\alpha)$. Then the moment map μ_{P_λ} is given by*

$$\mu_{P_\lambda}(\tau\alpha) = f(\tau)\alpha, \quad f(\tau) = \frac{h'(\tau)}{2h(\tau)}. \tag{84}$$

The functions $h(\tau)$ and $f(\tau)$ are given explicitly by

$$h(\tau) = e^{-n_\lambda \tau} \frac{\sinh(n_\lambda + 1)\tau}{\sinh \tau} = \sum_{k=0}^{n_\lambda} x^k, \quad x = e^{-2\tau},$$

$$f(\tau) = \frac{(n_\lambda + 1)\sinh(\tau)\cosh((n_\lambda + 1)\tau) - \cosh(\tau)\sinh((n_\lambda + 1)\tau)}{2\sinh(\tau)\sinh((n_\lambda + 1)\tau)} - \frac{n_\lambda}{2}.$$

Furthermore, for $0 \leqslant \tau$ if and only if $-\frac{n_\lambda}{2} \leqslant f(\tau) < 0$, and $f(0) = -\frac{n_\lambda}{2}$.

Proof. Since we have $h'(\tau) = \langle (\partial k)(\tau\alpha), \alpha \rangle$ and $\langle \alpha, \alpha \rangle = 2$, the differential $(\partial k)(\tau\alpha)$ is given by $(\partial k)(\tau\alpha) = \frac{1}{2}h'(\tau)\alpha$. The equation (84) follows from this and the definition of the moment map. The explicit expression for the function $h(\tau)$ follows from (79) and (82), and that for $f(\tau)$ is shown by a direct computation. Next, it is easy to show that, by using the expression for $h(\tau)$ in terms of a polynomial in $x = e^{-2\tau}$, $f(0) = n_\lambda/2$. Also, we have $\lim_{\tau \to +\infty} f(\tau) = 0$ and $\lim_{\tau \to -\infty} f(\tau) = n_\lambda$. From this the rest of the assertion follows. □

Finally, we shall examine that the term of the Weyl denominator in Theorem 9 does not vanish for generic dominant weight in the case where $G = U(2)$.

Proposition 3.3. *Let* v_j $(1 \leqslant j \leqslant n_\lambda/2)$ *be a dominant weight defined in* (83). *We set* $\tau_j := \tau_\lambda(v_j)$: τ_j *is the unique non-negative number satisfying* $f(\tau_j) = -j$, *where* $f(\tau)$ *is defined by* (84). *Then the multiplicity* $a_N(\lambda; Nv_j)$ *of* V_{Nv_j} *in* $V_\lambda^{\otimes N}$ *has the following asymptotic formula*:

$$a_N(\lambda; Nv) = (2\pi N)^{-1/2} e^{-N(n_\lambda - 2j)} \left(\frac{\sinh(n_\lambda + 1)\tau_j}{\sinh \tau_j} \right)^N (a_\lambda(j) + O(N^{-1})),$$

where the positive constant $a_\lambda(j)$ *is given by*

$$a_\lambda(j) = 2e^{-\tau_j} \sqrt{\frac{2\sinh^4 \tau_j \sinh^2(n_\lambda + 1)\tau_j}{\sinh^2(n_\lambda + 1)\tau_j - (n_\lambda + 1)^2 \sinh^2 \tau_j}}.$$

The leading term a_j *vanishes if and only if* n_λ *is even and* $j = n_\lambda/2$. *In this case, the dominant weight* v_j $(j = n_\lambda/2)$ *is in the unique wall of the Weyl chamber* C.

Proof. The non-negativity of the number τ_j follows form Lemma 3.2 and that v_j is a dominant weight, i.e., $1 \leqslant j \leqslant n_\lambda/2$. The lattice $L^* = X^* \cap I^* = X^* \cap \mathbb{Z}^2$ is spanned by the simple root α. Thus we have $\Lambda = L$, and hence the finite group $\Pi(U(2))$ is trivial. Note that the Weyl denominator $\Delta(\tau\alpha/2\pi i)$ is given by

$$\Delta(\tau\alpha/2\pi i) = 2\sinh\tau,$$

which is non-negative for $\tau = \tau_j$ and zero if and only if $\tau = 0 = \tau_{n_\lambda/2}$. By (81), the positive constant $\delta_\lambda(v_j)$ is given by

$$e^{\delta_\lambda(v_j)} = h(\tau_j)^N e^{2j\tau_j} = e^{-(n_\lambda - 2j)\tau_j} \left(\frac{\sinh(n_\lambda + 1)\tau_j}{\sinh \tau_j} \right).$$

Note that half the sum of the positive roots is given by $\rho = \alpha/2$, and hence $\langle \rho, \tau_j \alpha \rangle = \tau_j$. Recall that the matrix $A_\lambda(v_j)$ is equal to $A(\tau_\lambda(v))$ where $A(\tau)$ $(\tau \in X)$ is the derivative of the moment map $\mu_P(\tau)$. In our case, $A(\tau)$ is a positive real number given by

$$A(\tau) = \frac{h(\tau)h''(\tau) - h'(\tau)^2}{2h(\tau)^2} = \frac{\sinh^2(n_\lambda + 1)\tau - (n_\lambda + 1)^2 \sinh^2 \tau}{2\sinh^2 \tau \sinh^2(n_\lambda + 1)\tau}.$$

(Note that, since $\langle \alpha, \alpha \rangle = 2$, $\alpha \otimes \alpha$ is identified with the multiplication by 2.) Therefore, the assertion follows from Theorem 9. $\square$

4. Final comments

We close with some remarks on lattice paths and also on the symplectic interpretation of our problems and results.

4.1. Further relations between multiplicities of irreducibles and lattice paths

A number of relations are known between lattice path counting problems to that of determining multiplicities of weights in tensor powers $V_\lambda^{\otimes N}$. We used formulae (22) and (23) in terms of weighted multiplicities of lattice paths. There are other formulae which express multiplicities in terms of unweighted but constrained sums.

One is given by Theorem 2 of the paper of Grabiner–Magyar [GM]: *Let C be the Weyl chamber of a reductive complex Lie algebra, V be a finite dimensional representation, S be the set of weights of V and L be a lattice containing S and* ρ. *Then the number* $b_{\rho,\rho+\mu,N}$ *of walks of N steps from* ρ *to* $\rho+\mu$ *which stay strictly within C equals the multiplicity of the irreducible with highest weight* μ *in* $V^{\otimes N}$. To use this formula, one needs to count lattice paths satisfying the constraint, for which the only known tool seems to be the Gessel–Zeilberger formula [GZ]. The resulting formula then the right-hand side of the identity in Proposition 2.4, which we have analyzed in this paper. Many further (and much more general) relations between characters and multiplicities to sums over special lattice paths are discussed in [Lit].

4.2. Symplectic model

The reader may note a resemblance between the problems studied in this paper and the well-known problem of finding asymptotics of weight multiplicities in $V_{N\lambda}$, where $V_{N\lambda}$ is the irreducible with highest weight $N\lambda$ (see e.g. [H,GS]). In both cases, the possible weights lie in $Q(N\lambda)$ and one may define analogous distribution of weights of $V_{N\lambda}$. However, the relation is not very close, since our problem is about the thermodynamic limit rather than the semiclassical limit. We add a few remarks to clarify the relations.

We recall the symplectic interpretation of the latter multiplicity problem: the maximal torus $\mathbf{T}$ acts by conjugation on the co-adjoint orbit O_λ associated to V_λ in a Hamiltonian fashion, with moment map given by the orthogonal projection $\mu_\lambda : O_\lambda \to \mathbf{t}^*$ to the Cartan dual subalgebra. The image is given by $\mu_\lambda(O_\lambda) = Q(\lambda)$. As proved by G. Heckman, multiplicities of weights in $V_{N\lambda}$ become asymptotically distributed according to the (Duistermaat-Heckman) measure, namely the push-forward $\mu_{\lambda *} dVol_\lambda$ the symplectic volume measure of O_λ under the orthogonal projection to $\mathbf{t}^*$ [H,GS].

The limit formula in Theorem 1 also has a symplectic interpretation: To $V_\lambda^{\otimes N}$ corresponds the symplectic manifold

$$O_\lambda^N := O_\lambda \times \cdots \times O_\lambda \quad (N \text{ times}).$$

Then **T** acts on O_λ^N with moment map

$$\mu_\lambda^N : O_\lambda^N \to \mathbf{t}^*, \mu_\lambda^N(x_1, \ldots, x_N) = \mu_\lambda(x_1) + \cdots + \mu_\lambda(x_N). \tag{85}$$

The image of the moment map is the convex polytope $Q(N\lambda) = N\mu(O_\lambda)$, and one may define the Duistermaat-Heckman type measure on $Q(\lambda)$ by:

$$dm_\lambda^N := D_N^{-1}(\mu_\lambda^N)_*(dVol_\lambda \times \cdots \times dVol_\lambda) \quad (N \text{ times}) \tag{86}$$

on $Q(\lambda)$, where $D_N x = Nx$ is the dilation operator. Equivalently, this latter measure is defined by

$$\int_{Q(\lambda)} f(x) dm_\lambda^N(x) = \int_{O_\lambda \times \cdots \times O_\lambda} f\left(\frac{\mu_\lambda(x_1) + \cdots + \mu_\lambda(x_N)}{N}\right) \times dVol_\lambda(x_1) \times \cdots \times dVol_\lambda(x_N). \tag{87}$$

Thus, dm_λ^N is the distribution of the sum of the (vector valued) independent random variables $\mu_\lambda(x_j)$, the law of large numbers implies that the limit equals the mean value of the random variables:

$$dm_\lambda^N \to \delta_{Q^*(\lambda)}, \quad \text{weakly as } N \to \infty. \tag{88}$$

This measure represents the thermodynamic limit of the classical spin chain with phase space O_λ at each site, while our problem involves the thermodynamic limit of the quantum spin chain. The two problems are quite distinct until one lets the weight $\lambda \to \infty$ along a ray, i.e. considers the joint asymptotics of weights in $V_{M\lambda}^{\otimes N}$. The Heckman theorem says that if N is fixed and $M \to \infty$ then the quantum problem converges to the classical one. It would be interesting to investigate the joint asymptotics as both parameters become large.

5. Uncited references

[GW,La,P,Sp]

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