CONVERGENCE OF BERGMAN GEODESICS ON $\mathbf{CP}^1$ [1]

Jian Song Steve Zelditch

Department of Mathematics
Johns Hopkins University

Dedicated to Yves Colin de Verdière on the occasion of his sixtieth birthday

Abstract

The space $\mathcal{H}$ of Kähler metrics in a fixed Kähler class on a projective Kähler manifold X is an infinite dimensional symmetric space whose geodesics ω_t are solutions of a homogeneous complex Monge-Ampère equation in $X \times A$, where $A \subset \mathbb{C}$ is an annulus. Phong-Sturm have proven that the Monge-Ampère geodesic of Kähler potentials $\varphi(t, z)$ of ω_t may be uniformly approximated by geodesics $\varphi_N(t, z)$ of the finite dimensional symmetric space of Bergman metrics of height N. In this article we prove that $\varphi_N(t, z) \to \varphi(t, z)$ in $C^2([0, 1] \times X)$ in the case of toric Kähler metrics on $X = \mathbf{CP}^1$.

1 Introduction

This article is concerned with geodesics in spaces of Hermitian metrics of positive curvature on an ample line bundle $L \to X$ over a Kähler manifold. Stimulated by a recent article of Phong-Sturm [PS], we study the convergence as $N \to \infty$ of geodesics on the finite dimensional symmetric spaces $\mathcal{H}_N$ of Bergman metrics of 'height N' to Monge-Ampére geodesics on the full infinite dimensional symmetric space $\mathcal{H}$ of C^∞ metrics of positive curvature. Our main result is C^2 convergence of Bergman geodesics to Monge-Ampére geodesics in the case of toric (i.e. S^1-invariant) metrics on $\mathbf{CP}^1$. Although such metrics constitute the simplest case of toric Kähler metrics, the $\mathbf{CP}^1$ case already exhibits much of the complexity of general toric varieties for the approximation problem studied here. The general case will be studied in [SoZ].

The convergence problem raised by Phong-Sturm [PS] (see also Arezzo-Tian [AT] and Donaldson [D2a], Corollary 5) belongs to the intensively studied program initiated by Yau [Y2] of relating the algebro-geometric issue of stability to the analytic issue existence of canonical metrics on holomorphic line bundles. In this program, metrics in $\mathcal{H}_N$ have a simple description in terms of algebraic geometry, while metrics in $\mathcal{H}$ are 'transcendental'. The approximation of transcendental objects in $\mathcal{H}$ by 'rational' objects in $\mathcal{H}_N$ lies at the heart of this program.

The reasons for studying Monge-Ampére geodesics were laid out by Donaldson in [D1] (see also Mabuchi [M] and Semmes [S2]). Formally, $\mathcal{H} = \mathcal{G}_{\mathbb{C}} \backslash \mathcal{G}$ where $\mathcal{G}$ is the group of Hamiltonian symplectic diffeomorphisms of (X, ω); here $\omega \in \mathcal{H}$ is a fixed Kähler form. The geodesics of $\mathcal{H}$ should therefore correspond to orbits of one-parameter subgroups of $\mathcal{G}_{\mathbb{C}}$. Such one parameter subgroups should be important by analogy to finite dimensional settings, where the Hilbert-Mumford criterion relates stability of (X, L) to weights of one-parameter subgroups. Unfortunately, the infinite dimensional group $\mathcal{G}$ does not admit a true complexification. But Monge-Ampère geodesics are well-defined, and they provide a useful replacement for 'one parameter subgroups of $\mathcal{G}_{\mathbb{C}}$'.

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The existence, uniqueness and regularity of such geodesics is connected to existence and uniqueness of metrics of constant scalar curvature. Donaldson asked [D2] if there exist smooth Monge-Ampére geodesics between any pair of metrics $h_0, h_1 \in \mathcal{H}$. The work of Chen [Ch] shows the existence of a unique $C^{1,1}$ geodesic h_t joining h_0 to h_1. The improved regularity of the Monge-Ampère geodesics, due to and Chen-Tian [CT], is sufficient to prove uniqueness of extremal metrics. In the case of toric varieties, the much stronger result is known that the geodesic between any two metrics is C^∞ [G]. In fact, the Monge-Ampére equation can be linearized by the Legendre transform and the symmetric space is flat.

But in general, solutions of the Monge-Ampére equation are difficult to analyze. The remarkable suggestion of Phong-Sturm [PS] and Arrezo-Tian [AT] is to study solutions of the homogeneous Monge-Ampère equation by means of 'algebro-geometric approximations'. It has been proved by Phong-Sturm [PS] (see also [B]) that Bergman geodesics, which are orbits of one-parameter subgroups of $GL(d_N + 1, \mathbb{C})$ between two Bergman metrics, converge uniformly to a given Monge-Ampére geodesic for a general ample line bundle over a Kähler manifold.

The question we take up in this article and in [SoZ] is whether Bergman geodesics converge to Monge-Ampère geodesics in a stronger sense. Convergence in C^2 is especially interesting since it implies that the curvatures and moments maps for the metrics along the Bergman geodesic converge to those along the Monge-Ampère geodesics. In this article and in the subsequent article [SoZ], we study this problem for toric hermitian line bundles over toric Kähler manifolds. In this setting, the Kähler potentials $\varphi_N(t, z)$ of the Bergman metrics along the geodesic have relatively explicit formulae (see 1.8) resembling the free energy of a discrete quantum statistical mechanical model. Convergence in C^0 of the Kähler potential as $k \to \infty$ is analogous to uniform convergence of the free energy in the thermodynamic limit, while convergence of derivatives is related to absence of phase transitions (cf. [E], II.6).

To state our results, we will need some notation. Let $L \to X$ be an ample holomorphic line bundle and denote by $H^0(X, L^N)$ the holomorphic sections of the N^{th} power $L^N \to X$ of L. Given a basis $\mathcal{S}_N = \{S_0, \ldots, S_{d_N}\}$ we define the associated holomorphic embedding

$$\Phi_{\mathcal{S}_N} : X \to \mathbf{CP}^{d_N}, \quad \Phi_{\mathcal{S}}(z) = [S_0(z), \ldots, S_{d_N}(z)]. \tag{1.1}$$

We define the space of Bergman metrics by

$$\mathcal{H}_N = \{\frac{1}{N}\Phi^*_{\mathcal{S}_N}\omega_{FS} \mid \mathcal{S}_N \text{ is a basis of } H^0(X, L^N)\},$$

where ω_{FS} is the Fubini-Study metric on $\mathbf{CP}^{d_N}$. Since $U(d_N + 1)$ is the isometry group of ω_{FS}, $\mathcal{H}_N$ is the symmetric space $GL(d_N + 1, \mathbb{C})/U(d_N + 1, \mathbb{C})$.

Metrics in $\mathcal{H}_N$ are defined by an essentially algebro-geometric construction and are somewhat analogous to rational numbers. A basic fact is that the union

$$\bigcup_{N=1}^{\infty} \mathcal{H}_N \subset \mathcal{H}$$

of Bergman metrics is dense in the C^∞ topology in the space $\mathcal{H}$ of all C^∞ Kähler metrics in a fixed Kähler class $[\omega]$ (see [T, Z]) of positive curvature. Indeed, for each N we have a map

$$\mathcal{S}_N : \mathcal{H} \to \mathcal{H}_N, \ \ h \to \ h_N = (\Phi^*_{\mathcal{S}_N} h_{FS})^{1/N}, \ \ \mathcal{S}_N(h) = \text{an orthonormal basis for } \ h. \tag{1.2}$$

The metric h_N is independent of the choice of orthonormal basis, and $h_N \to h$ in C^∞.

Now let us compare Monge-Ampére geodesics and Bergman geodesics. We let h_0, h_1 be any two hermitian metrics on L in the class $\mathcal{H}$ and write $h_\varphi = e^{-\varphi} h$ relative to a fixed metric h with curvature form $\omega = Ric(h)$. Thus, we have an isomorphism

$$\mathcal{H} \ = \ \{\varphi \in C^\infty(X) \mid \omega_\varphi \ = \ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \ \} \ . \tag{1.3}$$

We may then identify the tangent space $T_\varphi \mathcal{H}$ at $\varphi \in \mathcal{H}$ with $C^\infty(X)$. We define a Riemannian metric on $\mathcal{H}$ as follows: let $\varphi \in \mathcal{H}$ and let $\psi \in T_\varphi \mathcal{H} \simeq C^\infty(M)$ and define

$$||\psi||^2_\varphi \ = \ \int_M |\psi|^2 \ \omega_\varphi^n \ \ . \tag{1.4}$$

With this Riemannian metric, $\mathcal{H}$ is an infinite dimensional negatively curved symmetric space. By [M, S1, S2, D1], the geodesics of $\mathcal{H}$ in this metric are the paths φ_t which satisfy the equation

$$\ddot{\varphi} - |\partial\dot{\varphi}|^2_{\omega_\varphi} = 0. \tag{1.5}$$

This may be interpreted as a Monge-Ampére equation [S1, D1].

Geodesics in $\mathcal{H}_N$ with respect to the symmetric space metric are given by one-parameter subgroups e^{tA} of $GL(d_N + 1, \mathbb{C})$. That is, let h_0, $h_1 \in \mathcal{H}$ and $\sigma \in GL(d_N + 1, \mathbb{C})$ be the change of basis matrix defined by $\sigma \cdot \underline{\hat{S}}^{(0)} = \underline{\hat{S}}^{(1)}$, where $\underline{\hat{S}}^{(0)} = \mathcal{S}_N(h_0)$ and $\underline{\hat{S}}^{(1)} = \mathcal{S}_N(h_1)$. Without loss of generality, we may assume that σ is diagonal with entries $e^{\lambda_0}, ..., e^{\lambda_{d_N}}$ for some $\lambda_j \in \mathbb{R}$. Let $\underline{\hat{S}}^{(t)} = \sigma^t \cdot \underline{\hat{S}}^{(0)}$ where σ^t is diagonal with entries $e^{\lambda_j t}$. We fix a smooth hermitian metric $h \in \mathcal{H}$ and define

$$h_{\underline{\hat{S}}^{(t)}}(z) = \frac{1}{\left(\left|\underline{\hat{S}}^{(t)}\right|^2\right)^{\frac{1}{N}}},$$

$$h_N(t, z) = h_{\underline{\hat{S}}^{(t)}}(z) = h(z) e^{-\varphi_N(t,z)}.$$

Then $h_N(t, \cdot)$ is the smooth geodesic in $GL(d_N + 1, \mathbb{C})/U(d_N + 1, \mathbb{C})$ joining $h_N(0, \cdot)$ to $h_N(1, \cdot)$. Explicitly, we have

$$\varphi_N(t, z) \ = \ \frac{1}{N} \log \left(\sum_{j=0}^{d_N} e^{2\lambda_j t} |\hat{S}_j^{(0)}|^2_{h^N}(z) \right). \tag{1.6}$$

Thus, the problem is the convergence of $h_N(t, \cdot) \to h(t, \cdot)$ or equivalently of $\varphi_N(t, \cdot) \to \varphi(t, \cdot)$. The following general result is proved in [PS].

Theorem 1.1 *The Bergman geodesics uniformly converge to the Monge-Ampère geodesic in the sense that*

$$\varphi_t(z) = \lim_{k\to\infty} [\sup_{N \geq k} \varphi_N(t, \cdot)]^*(z), \tag{1.7}$$

where, for any bounded function $f : [0, 1] \times X \to \mathbf{R}$*, the upper envelope of* f *is defined by* $f^*(x_0) = \lim_{\epsilon\to 0} \sup_{|x-x_0|<\epsilon} f(x)$.

As mentioned above, our goal here and in [SoZ] is to study the degree of convergence of these geodesics in the case of toric hermitian metrics on a toric line bundle $L \to X$. We define the space $\mathcal{H}_T$ to be the subspace of $\mathcal{H}$ of hermitian metrics for which φ is invariant under the underlying real torus $T = (S^1)^n$ action.

In the case of $\mathbf{CP}^1$, we may assume $L = \mathcal{O}(1)$ and an orthogonal basis $\{S_\alpha^N\}$ of holomorphic sections of $L^N = \mathcal{O}(N)$ is given in an affine chart by the monomials $z^{N\alpha}, N\alpha = 0, \dots, N$. A toric hermitian metric is entirely encoded in the set of L^2 squared norms $\mathcal{Q}_h^N(\alpha) = ||z^{N\alpha}||^2_{h^N}$ of the monomials with respect to powers h^N of the Hermitian metric h (cf. Definition 2.1). Then (1.6) takes the form

$$\varphi_N(t,z) \;=\; \frac{1}{N}\log\left(\sum_{\alpha\in\frac{1}{N}\mathbb{Z}\cap[0,1]} \frac{|z^{N\alpha}|^2_{h_0}}{(\mathcal{Q}^N_{h_0}(\alpha))^{1-t}(\mathcal{Q}^N_{h_1}(\alpha))^t}\right). \tag{1.8}$$

If we write $|z|^2 = e^\rho$, we see the resemblance to the free energy of a quantum statistical model with states parameterized by lattice points in $[0, N]$ [E] (§7). The main result of this article is:

Theorem 1.2 *On $\mathbf{CP}^1$, the Bergman geodesics converge to the toric Monge-Ampère geodesic uniformly*

$$\lim_{N\to\infty} \varphi_N(t,z) = \varphi_t(z), \tag{1.9}$$

uniformly in the C^2 topology on $[0,1]\times\mathbf{CP}^1$.

A natural question is whether the convergence is uniform in higher C^k spaces. We have no reason to doubt this, but our proofs are based on explicit calculation of two derivatives and analysis of the asymptotics of the resulting expressions. The expressions become rather complicated when one takes three or higher derivatives, and it becomes quite messy to check if they converge uniformly. As will be seen in the proof, most of the complications concern the joint asymptotics in the (N, α) parameters of the *norming constants* $\mathcal{Q}_h^N(\alpha)$ near the boundary of the 'moment polytope' $[0, 1]$. The essential simplification in $\mathbf{CP}^1$ over higher dimensional toric varieties is that the approach to the boundary is much simpler for an interval than for the possible convex Delzant polytopes in higher dimensions. Otherwise, the case of $\mathbf{CP}^1$ already exhibits much of the complexity of the general case. In [SoZ], we study the C^2 convergence problem in all dimensions.

Our analysis of the norming constants builds on the work of [STZ1], and may have an independent interest, since the norming constants determine a toric metric. For instance, in [D4] and elsewhere, numerical methods for approximating extremal Kähler metrics on toric varieties are also based on the study of norming constants. It would be interesting to generalize the results on norming constants to higher dimensions. The subsequent article [SoZ] involves quantities which are in a sense dual to norming constants and does not directly provide information on norming constants.

Finally, we thank the referee for some corrections and improvements. As the referee points out, there are interesting connections between the calculations of this article and those of [B]. Our methods can be adapted to the slightly different situation of that article in the toric case, and we hope to present the details elsewhere.

2 Preliminaries

Although we primarily study $\mathbf{CP}^1$ in this article, we set the scene for toric varieties in arbitrary dimensions. Let (X, ω, τ) be a compact toric manifold of complex dimension n and

$$\tau : T^n \to Diff(X, \omega)$$

an effective Hamilton action of the standard real n-torus $T = (S^1)^n$. Let π be the moment map associated to the toric Kähler metric ω

$$\pi : X \to \mathbf{R}^n. \tag{2.1}$$

The image P of π is a Delzant polytope, defined by a set of linear inequalities given by

$$\langle x, v_r \rangle \geq \lambda_r, \quad r = 1, ..., d,$$

where v_r is an inward-pointing normal to the r-th $(n-1)$-dimensional face of P. Define the affine functions $l_r : \mathbf{R}^n \to \mathbf{R}$ by

$$l_r(x) = \langle x, v_r \rangle - \lambda_r.$$

Fix a toric polarization L on X with $[L] = [\omega]$. Let

$$\mathcal{H} = \{h \mid h \text{ is a smooth } T\text{-invariant hermitian metric on } L \text{ such that } Ric(h) > 0\}.$$

Fix $h \in \mathcal{H}$ and let $\omega = Ric(h)$, then

$$\mathcal{H} \cong \{\varphi \in C^\infty(X) \mid \varphi \text{ is } T\text{-invariant and } \ \omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\}.$$

Hence the hermitian metric $h_\varphi \in \mathcal{H}$ and the ω-plurisubharmonic potential $\varphi \in \mathcal{H}$ are related by

$$h_\varphi = h_0 e^{-\varphi}.$$

The L^2-metric on $\mathcal{H}$ is given by

$$||\psi||^2_{\omega_\varphi} = \int_X |\psi|^2 \omega_\varphi^n$$

for any $\psi \in C^\infty(X)$.

For any φ_0 and $\varphi_1 \in \mathcal{H}$, the geodesic φ_t joining φ_0 and φ_1 in $\mathcal{H}$ is defined by

$$\frac{\partial^2 \varphi_t}{\partial t^2} = |\partial \dot{\varphi}_t|^2_{\omega_{\varphi_t}}. \tag{2.2}$$

From a complex geometric viewpoint, the complex torus $(\mathbf{C}^*)^n$ acts on X with an open orbit, and X may be viewed as a compactification of $(\mathbf{C}^*)^n$. On the open orbit, we denote the standard holomorphic coordinates by $(z_1, ..., z_n)$. We also define the real coordinates $\rho_j = \log |z_j|^2$, $j = 1, ..., n$. Then a toric Kähler form has a T-invariant Kähler potential u on the orbit defined by $\omega = \sum_{i,j=1,...,n} \sqrt{-1} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z}_j > 0$. Since u is T-invariant, it can be considered as a function in $\rho = (\rho_1, ..., \rho_n)$ on $\mathbf{R}^n$ and it is convex on $\mathbf{R}^n$. We then define $U(\rho) = u(z)$ on $\mathbf{R}^n$.

The Legendre transform of U defines the symplectic potential G of ω, a convex function on P°. That is,

$$G(x) = \langle x, \rho \rangle - U(\rho)$$

with $x = \nabla U(\rho) \in P \subset \mathbf{R}^n$ given by the moment map. It has the same singularities at the boundary ∂P as the symplectic reference potential

$$G_P(x) = \sum_{r=1}^d l_r(x) \log l_r(x). \tag{2.3}$$

G_P induces a smooth hermitian metric h_P on $L \to X$ (smooth over all of X) with $Ric(h_P) = \sqrt{-1}\partial\overline{\partial} u_P$ on $(\mathbf{C}^*)^n$ and u_P being the Legendre transform of G_P. For background, we refer to [A, D4, Gu].

The following theorem is proved by Guan [G].

Theorem 2.1 *Let h_t be the smooth geodesic joining h_0 and $h_1 \in \mathcal{H}$ for $t \in [0,1]$. The corresponding symplectic potential G_t is given by*

$$G_t(x) = G_P(x) + f_t(x) \tag{2.4}$$

where f_t is a smooth function on $\mathbf{R}^n$ with $\nabla^2 G_t > 0$ on P°. Furthermore,

$$f_t(x) = (1-t)f_0(x) + tf_1(x). \tag{2.5}$$

Hence the geodesic of the symplectic potentials is linear. A very simple proof (cf. [SoZ]) is simply to push forward the energy functional defining the Monge-Ampère geodesics to the polytope and observe that it becomes the Euclidean energy functional there.

Definition 2.1 *For any lattice point $N\alpha \in NP \cap \mathbf{Z}^n$, we let $S_\alpha^N \in H^0(X, L^N)$ denote the section which equals the monomial $z^{N\alpha}$ on $(\mathbf{C}^*)^n$ in the standard affine frame. We define the L^2 norm of $S_\alpha^N \in H^0(X, L^N)$ with respect to h_t by*

$$\mathcal{Q}_t^N(\alpha) = \int_X |S_\alpha^N|^2_{h_t^N} \omega_t^n \tag{2.6}$$

where $\omega_t = Ric(h_t)$ and h_t^N the N^{th}-power of h_t. We also define $\mathcal{Q}_P^N(\alpha)$ with respect to h_P by

$$\mathcal{Q}_P^N(\alpha) = \int_X |S_\alpha^N|^2_{h_P^N} \omega_P^n \tag{2.7}$$

where ω_P is the toric Kähler form given by the symplectic potential G_P. The formula for $\mathcal{Q}_t^N(\alpha)$ and $\mathcal{Q}_P^N(\alpha)$ can be extended by real analyticity to all $\alpha \in P$.

Phong and Sturm [PS] introduce the $GL(d_N+1, \mathbf{C})$ geodesics in the space of Bergman metrics to approximate the Monge-Ampere geodesic φ_t.

Definition 2.2 *We define $\mathcal{E}_N(t,z)$ by*

$$\mathcal{E}_N(t,z) = \sum_{N\alpha \in NP \cap \mathbf{Z}^n} \frac{|S_\alpha^N|^2_{h_t^N}}{(\mathcal{Q}_0^N(\alpha))^{1-t}(\mathcal{Q}_1^N(\alpha))^t}, \tag{2.8}$$

and the Szegö kernel Π_N with respect to h_t by

$$\Pi_N(t,z) = \sum_{N\alpha \in NP \cap \mathbf{Z}^n} \frac{|S_\alpha^N|^2_{h_t^N}}{\mathcal{Q}_t^N(\alpha)}. \tag{2.9}$$

Definition 2.3 *We also define for $\alpha \in P$*

1. *the norming constants*

$$Q_t^N(\alpha) = \mathcal{Q}_t^N(\alpha) e^{-NG_t(\alpha)}, \quad Q_P^N(\alpha) = \mathcal{Q}_P^N(\alpha) e^{-NG_P(\alpha)}, \tag{2.10}$$

2. *the norming constants*

$$q_t^N(\alpha) = \frac{Q_t^N(\alpha)}{Q_P^N(\alpha)}, \quad \mathcal{R}_t^N(\alpha) = \frac{q_t^N(\alpha)}{(q_0^N(\alpha))^{1-t}(q_1^N(\alpha))^t} = \frac{Q_t^N(\alpha)}{(Q_0^N(\alpha))^{1-t}(Q_1^N(\alpha))^t}, \tag{2.11}$$

3. the norm squares of the normalized monomial sections

$$\mathcal{P}_\alpha^N(t,z) = \frac{|S_\alpha^N|^2_{h_t^N}(z)}{\mathcal{Q}_t^N(\alpha)}. \tag{2.12}$$

Lemma 2.1

$$\mathcal{E}_N(t,z) = \sum_{N\alpha \in NP\cap \mathbf{Z}^n} \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z). \tag{2.13}$$

Proof Straightforward calculation shows that

$$\begin{aligned}
& \frac{q_t^N(\alpha)}{(q_0^N(\alpha))^{1-t}(q_1^N(\alpha))^t} \\
= \ & \frac{Q_t^N(\alpha)}{(Q_0^N(\alpha))^{1-t}(Q_1^N(\alpha))^t} \\
= \ & e^{N((1-t)G_0(\alpha)+tG_1(\alpha)-G_t(\alpha))}\frac{\mathcal{Q}_t^N(\alpha)}{(\mathcal{Q}_0^N(\alpha))^{1-t}(\mathcal{Q}_1^N(\alpha))^t} \\
= \ & \frac{\mathcal{Q}_t^N(\alpha)}{(\mathcal{Q}_0^N(\alpha))^{1-t}(\mathcal{Q}_1^N(\alpha))^t}.
\end{aligned}$$

The last equality follows from the geodesic equation $G_t(x) = (1-t)G_0(x) + tG_1(x)$.

□

3 Joint (N, α) asymptotics of the norming constants for metrics on $\mathbf{CP}^1$

We first give a useful formula for the norming constants $Q_t^N(\alpha)$ (2.10) which is valid on any toric variety, and then we use it in the case of $\mathbf{CP}^1$ to determine joint (N,α) asymptotics.

Lemma 3.1 *The norming constants $Q_t^N(\alpha)$ and $Q_P^N(\alpha)$ in Definition 2.3 for $\alpha \in P$ are given on any toric variety by*

$$\begin{aligned}
Q_t^N(\alpha) &= (2\pi)^n \int_P e^{-NF_{t,\alpha}(x)}dx \\
Q_P^N(\alpha) &= (2\pi)^n \int_P e^{-NF_{P,\alpha}(x)}dx,
\end{aligned} \tag{3.1}$$

where the phase functions $F_{t,\alpha}(x)$ and $F_{P,\alpha}(x)$ are defined by

$$\left\{ \begin{array}{rcl} F_{P,\alpha}(x) &=& \langle x-\alpha, \nabla G_P(x)\rangle - (G_P(x) - G_P(\alpha)) \\ \\ F_{t,\alpha}(x) &=& \langle x-\alpha, \nabla G_t(x)\rangle - (G_t(x) - G_t(\alpha)). \end{array} \right. \tag{3.2}$$

Proof Let $z = (z_1, ..., z_n) \in (\mathbf{C}^*)^n$ and $\rho = (\rho_1, ..., \rho_n) \in \mathbf{R}^n$ with $\rho_j = \log|z_j|^2$ for $j = 1, ..., n$. We suppose that the Kähler form for g_t is given by $\sum_{i,j=1,...,n}\sqrt{-1}\frac{\partial^2 u_t}{\partial z_i \partial \bar{z}_j}dz_i\wedge d\bar{z}_j$, where $u_t(z)$ is the Kähler potential for the toric Kähler metric g_t on $(\mathbf{C}^*)^n$. Let $U_t(\rho) = u_t(z)$ and $\pi_t = \nabla U_t$:

$\mathbf{R}^n \to P$ be the moment map associated to g_t. Then the symplectic potential G_t on P for g_t is given by the following Legendre transform

$$G_t(x) = \langle x, \rho \rangle - U_t(\rho)$$

with $x = \nabla U_t(\rho) \in P \subset \mathbf{R}^n$. Also $U_t(\rho)$ can be recovered from $G_t(x)$ by the following inverse Legendre transform

$$U_t(\rho) = \langle x, \rho \rangle - G_t(x)$$

with $\rho = \nabla G_t(x)$. Also $\pi_t^*(dx_1...dx_n) = \det\left(\frac{\partial^2 U_t}{\partial \rho_i \partial \rho_j}\right) d\rho_1...d\rho_n$.

$$\begin{aligned}
Q_t^N(\alpha) &= (\sqrt{-1})^n \int_{\mathbf{C}^n} |z|^{N\alpha} e^{-Nu_t(z)-NG_t(\alpha)} \det\left(\frac{\partial^2 u_t}{\partial z_i \partial \bar{z}_j}\right) dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n \\
&= (2\pi)^n \int_{\mathbf{R}^n} e^{N(\langle \alpha, \rho \rangle - U_t(\rho)) - NG_t(\alpha)} \det\left(\frac{\partial^2 U_t}{\partial \rho_i \partial \rho_j}\right) d\rho_1...d\rho_n \\
&= (2\pi)^n \int_P e^{N(\langle \alpha, \nabla G_t(x)\rangle - (\langle x, \nabla G_t(x)\rangle - G_t(x)) - G_t(\alpha))} dx_1...dx_n \\
&= (2\pi)^n \int_P e^{-NF_{t,\alpha}(x)} dx.
\end{aligned}$$

The same argument gives the integral formula for $Q_P^N(\alpha)$.

□

We now specialize to the case of $\mathbf{CP}^1$, where:

- $P = [0,1]$ and the canonical symplectic potential equals $G_P(x) = x \log x + (1-x)\log(1-x)$ (it is the symplectic potential dual to the Fubini-Study Kähler potential);
- For $\alpha \in \frac{1}{N}\mathbb{Z} \cap P$, $\mathcal{Q}_P^N(\alpha) = \binom{N}{N\alpha}^{-1}$, and $Q_P^N(\alpha) = 2\pi \binom{N}{N\alpha}^{-1} e^{-N(\alpha \log \alpha + (1-\alpha)\log(1-\alpha))}$.
- The geodesic of the symplectic potentials $G_t(x)$ is

  $$G_t(x) = G_P(x) + f_t(x)$$

  where $f_t(x) = (1-t)f_0(x) + tf_1(x)$ is a smooth function on $\mathbf{R}$ such that

  $$\frac{d^2}{dx^2} G_t(x) > 0. \tag{3.3}$$

In fact, because $G_t''(x)$ has poles of order 1 at 0 and 1, we have:

Lemma 3.2 *There exists a constant $\Lambda > 0$ such that for any $t \in [0,1]$ and $x \in (0,1)$*

$$x(1-x)G_t''(x) > \Lambda, \qquad x(1-x)G_P''(x) > \Lambda. \tag{3.4}$$

We also evaluate:

$$\begin{cases} F_{P,\alpha}(x) = -\alpha \log x - (1-\alpha)\log(1-x) + \alpha \log \alpha + (1-\alpha)\log(1-\alpha), \\ \\ F_{t,\alpha}(x) = G_P(\alpha) - \alpha \log x - (1-\alpha)\log(1-x) + (x-\alpha)^2 f_{t,\alpha}(x), \end{cases} \tag{3.5}$$

where $f_{t,\alpha}(x) = -\frac{f_t(x)-f_t(\alpha)-f_t'(x)(x-\alpha)}{(x-\alpha)^2}$ with $f_{t,\alpha}(\alpha) = \frac{1}{2}f_t''(\alpha)$. It is easy to check by Taylor expansion that $f_{t,\alpha}(x)$ is smooth in x and t.

We now consider the joint asymptotics in (N, α) of the norming constants. Our main result, Theorem 3.1, is a comparison of the joint asymptotics of a metric norming constant (2.10) to the canonical norming constants $Q_P^N(\alpha)$. The joint asymptotics of the latter can be derived from known (elementary) results on binomial coefficients, and we begin by recalling the relevant background.

The joint asymptotics of binomial coefficients $\binom{N}{m}$ in (N,m) and the closely related canonical norming constants $Q_P^N(\alpha)$ have several regimes accordingly as α belongs to an 'interior region' or a 'boundary region'. First let us consider the 'interior,' where $\alpha \in [\frac{1}{N^{3/4}}, 1-\frac{1}{N^{3/4}}]$. The standard Sterling asymptotics for factorial and binomials applies in the region and gives

$$\binom{N}{N\alpha} \sim \frac{1}{\sqrt{2\pi N\alpha(1-\alpha)}}\, e^{-N(\alpha\log\alpha+(1-\alpha)\log(1-\alpha)}, \tag{3.6}$$

and more precisely the asymptotics

$$Q_P^N(\alpha) = 2\pi\binom{N}{N\alpha}^{-1} e^{-NG_P(\alpha)} = 2\pi\sqrt{(2\pi)N\alpha(1-\alpha)}\exp\left(O\left(\frac{1}{N\alpha}+\frac{1}{N-N\alpha}\right)\right). \tag{3.7}$$

We observe that the asymptotics are highly non-uniform as $\alpha \to 0$ or $\alpha \to 1$.

In the left 'boundary region' $\alpha \in [0, \frac{1}{N^{3/4}}]$, we cannot use Stirling's formula up the boundary and rather use that

$$\binom{N}{m} = A(N,m)\frac{N^m}{m!}, \text{ with } A = \Pi_{j=1}^{m-1}(1-\frac{j}{N}).$$

Using that $\ln A = \sum_{j=1}^{m-1}\ln(1-\frac{j}{N})$, and $\ln(1-x) \sim -x$ one has

$$\sum_{j=1}^{N\alpha-1}\ln(1-\frac{j}{k}) \sim \sum_{j=1}^{N\alpha-1} -\frac{j}{N} \sim \frac{(N\alpha)^2}{2N} = o(1)$$

if $N\alpha = o(\sqrt{N})$. It follows that if $(N\alpha) = o(\sqrt{N})$, then $\binom{N}{N\alpha} \sim \frac{N^{N\alpha}}{(N\alpha)!}$, and further that

$$\begin{aligned}
2\pi\left(Q_P^N(\alpha)\right)^{-1} &= \tbinom{N}{N\alpha}(\tfrac{N\alpha}{N})^{N\alpha}(1-\tfrac{N\alpha}{N})^{N-N\alpha} \\
&= \tfrac{(N\alpha)^{N\alpha}}{(N\alpha)!}(1-\tfrac{1}{N})(1-\tfrac{2}{N})\cdots(1-\tfrac{N\alpha}{N})(1-\tfrac{N\alpha}{N})^{N-N\alpha} \\
&= (1-\tfrac{(N\alpha)}{N})^N\tfrac{(N\alpha)^{(N\alpha)}}{(N\alpha)!}(1-\tfrac{1}{N})(1-\tfrac{2}{N})\cdots(1-\tfrac{N\alpha}{N})(1-\tfrac{N\alpha}{N})^{-N\alpha}.
\end{aligned} \tag{3.8}$$

We record the following:

Lemma 3.3 *There exists a constant $C > 0$ such that for all $\alpha \in [0,1] \cap \frac{1}{N}\mathbf{Z}$*

$$Q_P^N(\alpha) \geq C. \tag{3.9}$$

Proof In the interior region, (3.7) implies the lower bound $Q_P^N(\alpha) \geq CN^{1/8}$. In the boundary region, we can continue to use Stirling's formula as long as $N\alpha \to \infty$ to obtain

$$\binom{N}{N\alpha} \sim \left(\frac{Ne}{N\alpha}\right)^{N\alpha}(2\pi N\alpha)^{-1/2} \implies \binom{N}{N\alpha}\left(\frac{N\alpha}{N}\right)^{N\alpha}(1-\frac{N\alpha}{N})^{N-N\alpha} \sim (2\pi N\alpha)^{-1/2},$$

so $Q_P^N(\alpha) \to \infty$ there as well. If $N\alpha \leq K$ then the exact formula (3.8) gives positive upper bound independent of N. We note that it equals 1 when $\alpha = 0$.

We now turn to general metrics. The following comparison inequality is the principal technical tool in the proof of C^2 convergence of the geodesics (see Definition 2.3).

Theorem 3.1 *There exists a constant $C > 0$ such that for all integer $N > 0$, $\alpha \in P$ and $t \in [0,1]$*

$$\frac{1}{C} \leq q_t^N(\alpha) \leq C. \tag{3.10}$$

Furthermore, if we let π_t and π_P be the moment maps associated to the toric Kähler metrics g_t and g_P and define

$$\Omega_t(\alpha) = \left(\frac{\det \nabla^2 G_P(\alpha)}{\det \nabla^2 G_t(\alpha)}\right)^{\frac{1}{2}},$$

then $\Omega_t(\alpha)$ extends to a continuous function on P and

$$\lim_{N\to\infty} q_t^N(\alpha) = \Omega_t(\alpha) \tag{3.11}$$

uniformly for $\alpha \in P$.

Indeed, $\Omega_t(\alpha) = \left(\frac{\det \nabla^2 U_t(\pi_t^{-1}(\alpha))}{\det \nabla^2 U_P(\pi_P^{-1}(\alpha))}\right)^{\frac{1}{2}}$ is the ratio of the volume forms of Kähler metrics g_t and g_P on $(\mathbf{C}^*)^n$, although $\pi_t^{-1}(\alpha)$ and $\pi_P^{-1}(\alpha)$ do not necessarily coincide.

The following corollaries play an important role in the proof of the main result.

Corollary 3.1 *There exists a constant $C > 0$ such that for all integer $N > 0$, $\alpha \in [0,1]$ and $t \in [0,1]$, the ratios $\mathcal{R}$ of Defintion 2.3 satisfy*

$$\frac{1}{C} \leq \mathcal{R}_t^N(\alpha) \leq C. \tag{3.12}$$

Furthermore,

$$\lim_{N\to\infty} \mathcal{R}_t^N(\alpha) = \frac{\Omega_t(\alpha)}{(\Omega_0(\alpha))^{1-t}(\Omega_1(\alpha)^t}, \tag{3.13}$$

uniformly for $\alpha \in [0,1]$.

The next Corollary follows immediately from Theorem 3.1 and Lemma 3.3.

Corollary 3.2 *There exist $C > 0$ such that for all $\alpha \in [0,1] \cap \frac{1}{N}\mathbf{Z}$*

$$Q_t^N(\alpha) \geq C. \tag{3.14}$$

We divide the proof of Theorem 3.1 into an analysis of norming constants in an interior region of $[0,1]$ and in a boundary region.

3.1 Interior estimates

We begin by studying $Q_t^N(\alpha)$ where α lies in the (left) 'interior interval' $\alpha \in [\frac{1}{N^{3/4}}, \frac{2}{3}]$. It is then possible to obtain joint (N, α) asymptotics by a complex stationary phase method. The discussion is essentially the same for the right interior interval $[\frac{1}{3}, 1 - \frac{1}{N^{3/4}}]$ and is omitted.

Proposition 3.1 *Let $\alpha \in [\frac{1}{N^{3/4}}, \frac{2}{3}]$ and $M = N\alpha$. Then there exist uniformly bounded functions $A_{t,k}(\alpha)$ on the interior region, such that*

$$Q_t^N(\alpha) \sim \frac{2\pi^{\frac{3}{2}}\alpha}{(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha))^{\frac{1}{2}}(M)^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{A_{t,k}(\alpha)}{M^k} = \frac{2\pi^{\frac{3}{2}}}{(G_t''(\alpha))^{\frac{1}{2}}(N)^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{A_{t,k}(\alpha)}{M^k}, \tag{3.15}$$

in the sense that for any $R \in \mathbf{Z}^+$ there exists $C_R > 0$ such that

$$\left| Q_t^N(\alpha) - \frac{2\pi^{\frac{3}{2}}\alpha}{(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha))^{\frac{1}{2}}(M)^{\frac{1}{2}}} \sum_{k=0}^{R} \frac{A_{t,k}(\alpha)}{M^k} \right| \leq \frac{C_R\alpha}{(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha))^{\frac{1}{2}}(M)^{\frac{1}{2}}} M^{-(R+1)}. \tag{3.16}$$

In particular, $A_{t,0} = 1$.

Corollary 3.3 *Let $\alpha \in [\frac{1}{N^{3/4}}, \frac{2}{3}]$ and $M = N\alpha$. There is a complete asymptotic expansion for large M*

$$q_t^N(\alpha) \sim \frac{1}{(M(1 + \alpha(1-\alpha) f_t''(\alpha)))^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{B_{t,k}(\alpha)}{M^k} = \left(\frac{G_P''(\alpha)}{G_t''(\alpha)}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{B_{t,k}(\alpha)}{M^k} \tag{3.17}$$

in the sense that for any $R \in \mathbf{Z}^+$ there exists $C_R > 0$ such that

$$\left| q_t^N(\alpha) - \frac{1}{(M(1 + \alpha(1-\alpha) f_t''(\alpha)))^{\frac{1}{2}}} \sum_{k=0}^{R} \frac{B_{t,k}(\alpha)}{M^k} \right| \leq C_R M^{-(R+1)}. \tag{3.18}$$

In particular, $B_{t,0} = 1$ and there exists $C > 0$ such that

$$0 < \frac{1}{C} \leq q_t^N(\alpha) \leq C. \tag{3.19}$$

The proof of Proposition 3.1 proceeds by a sequence of Lemmas. The first concerns the phase $F_{t,\alpha}$ (3.5).

Lemma 3.4 *α is the only critical point of $F_{t,\alpha}(x)$ and we have*

$$F_{t,\alpha}''(\alpha) = G_t''(\alpha) > 0, \qquad (x - \alpha)F_{t,\alpha}'(x) \geq 0. \tag{3.20}$$

Proof Differentiating (3.2) shows that $F_{t,\alpha}'(x) = (x - \alpha)G_t''(x)$. The second derivative is readily obtained and it is positive by Lemma 3.2.

□

Now we make a substitution of variables. Let $y = \frac{x-\alpha}{\alpha}$, $M = N\alpha$. We then have

$$Q_t^N(\alpha) = 2\pi\alpha \int_{-1}^{\frac{1}{\alpha}-1} e^{-M\mathcal{F}_{t,\alpha}(y)} dy \tag{3.21}$$

with new phase function

$$\begin{aligned} \mathcal{F}_{t,\alpha}(y) &= \tfrac{1}{\alpha} F_{t,\alpha}(\alpha(1+y)), \quad \mathcal{F}_{P,\alpha}(y) = \tfrac{1}{\alpha} F_{P,\alpha}(\alpha(1+y)) \\ \mathcal{F}_{t,\alpha}(y) &= -\left(\log(1+y) + \tfrac{1-\alpha}{\alpha}\log\tfrac{1-\alpha-\alpha y}{1-\alpha} + \alpha y^2 f_{t,\alpha}(\alpha(1+y))\right). \end{aligned} \tag{3.22}$$

Lemma 3.5 *The phase has the following properties:*

1. $\mathcal{F}_{t,\alpha}(y)$ *is strictly decreasing on* $(-1,0)$ *and strictly increasing on* $(0, \frac{1}{\alpha}-1)$ *with a unique critical (minimum) point at* $y=0$ *with* $\mathcal{F}_{t,\alpha}(0) = 0$.
2. *If* $y_0 > 0$, *then* $\inf_{y\geq y_0} \mathcal{F}'_{t,\alpha}(y) \geq C(y_0) > 0$ *where* $C(y_0)$ *is independent of* α, t.
3. *If* $y_0 < 0$, *then* $\inf_{y\in[-1,y_0]} |\mathcal{F}'_{t,\alpha}(y)| \geq C(y_0) > 0$ *where* $C(y_0)$ *is independent of* α, t.
4. *The Hessian of* $\mathcal{F}_{t,\alpha}$ *of* $y = 0$ *is non-degenerate and*
$$\mathcal{F}''_{t,\alpha}(0) = \alpha G''_t(\alpha) = \frac{1}{1-\alpha} + \alpha f''_t(\alpha) > 0.$$
5. $\mathcal{F}_{t,\alpha}(y)$ *and all of its derivatives are uniformly bounded for* $\alpha \in [0, \frac{2}{3}]$ *and for* y *in any compact set of* $(-1, \frac{1}{\alpha}-1)$.

Proof Comparing with (3.5) and Lemma 3.4 shows that

$$\begin{aligned} \mathcal{F}'_{t,\alpha}(y) &= \frac{1}{\alpha}\frac{dF_{t,\alpha}(x)}{dx}\frac{dx}{dy} = F'_{t,\alpha}(x) \\ &\quad = (x-\alpha)G''_t(x) = \alpha y G''_t(\alpha(1+y)) = \frac{x-\alpha}{x}(xG''_t(x)) = \frac{y}{1+y}(xG''_t(x)) \\ &= -\frac{1}{1+y} + \frac{1-\alpha}{1-\alpha-\alpha y} + 2\alpha y f_{t,\alpha}(\alpha(1+y)) + \alpha^2 y^2 f'_{t,\alpha}(\alpha(1+y)) \\ \mathcal{F}''_{t,\alpha}(y) &= \alpha F''_{t,\alpha}(x) = \alpha G''_t(x) + \alpha(x-\alpha)G'''_t(x) = \alpha G''_t(\alpha(1+y)) + \alpha^2 y G'''_t(\alpha(1+y)) \\ &= \frac{1}{(1+y)^2} + \frac{\alpha(1-\alpha)}{(1-\alpha-\alpha y)^2} \\ &\quad +2\alpha f_{t,\alpha}(\alpha(1+y)) + (2\alpha y + 2\alpha^2 y) f'_{t,\alpha}(\alpha(1+y)) + \alpha^2 y^2 f_{t,\alpha}(\alpha(1+y)). \end{aligned}$$

By Lemma 3.2, $xG''_t(x)$ has a uniform positive lower bound, hence by the formula $\mathcal{F}'_{t,\alpha}(y) = \frac{y}{1+y}(xG''_t(x))$, $\mathcal{F}'_{t,\alpha}(y) = 0$ if and only if $y = 0$. Also $\mathcal{F}'_{t,\alpha}(y) < 0$ on $(-1,0)$ and $\mathcal{F}'_{t,\alpha}(y) > 0$ on $(0, \frac{1}{\alpha}-1)$. The same formula implies (2)-(3) since the factor $|\frac{y}{1+y}|$ then has a uniform lower bound.

Again by Lemma 3.2, $G''_t(\alpha)$ has poles at 0 and 1, hence $\alpha G''_t(\alpha)$ is uniformly bounded below from 0 for $\alpha \in [0, \frac{2}{3}]$. In particular, at the critical point, we have (cf. Lemma 3.4),

$$\mathcal{F}''_{t,\alpha}(0) = \alpha F''_{t,\alpha}(\alpha) = \frac{1}{1-\alpha} + \alpha f''_t(\alpha) = \alpha G''_t(\alpha) > 0.$$

□

Lemma 3.6 *There exist δ and $C > 0$ such that*

$$\left| 1 - \frac{2\pi\alpha \int_{-\frac{1}{2}}^{1} e^{-M\mathcal{F}_{t,\alpha}(y)} dy}{Q_t^N(\alpha)} \right| \le \frac{Ce^{-\delta M}}{M}. \tag{3.23}$$

Proof By Lemma 3.5 (2), there exists $\Lambda > 0$ independent of (t,α) such that

$$\mathcal{F}_{t,\alpha}(y) \ge \mathcal{F}_{t,\alpha}(1) + \frac{\Lambda}{2}(y-1), \quad \text{for } y \ge 1. \tag{3.24}$$

Using also that $\mathcal{F}_{t,\alpha}$ increases on $(0, \frac{1}{2})$, we have

$$\begin{aligned}
\int_1^{\frac{1}{\alpha}-1} e^{-M\mathcal{F}_{t,\alpha}(y)} dy &\le \int_1^{\frac{1}{\alpha}-1} e^{-\frac{\Lambda}{2}M(y-1)-M\mathcal{F}_{t,\alpha}(1)} dy \\
&\le \frac{2e^{-M\mathcal{F}_{t,\alpha}(1)}}{\Lambda M} \\
&\le \frac{4e^{-M(\mathcal{F}_{t,\alpha}(1)-\mathcal{F}_{t,\alpha}(\frac{1}{2}))}}{\Lambda M} \int_0^{\frac{1}{2}} e^{-M\mathcal{F}_{t,\alpha}(y)} dy \\
&\le \frac{Ce^{-\delta M}}{2M\alpha} Q_t^N(\alpha), \quad \text{where } \delta := 2 \inf_{y \in [\frac{1}{2},1]} \mathcal{F}'_{t,\alpha}(y).
\end{aligned}$$

In the last line we again used Lemma 3.5 (2).

By the same argument, there exists $\delta > 0$ (independent of (t,α) so that

$$\int_{-1}^{-\frac{1}{2}} e^{-M\mathcal{F}_{t,\alpha}(y)} dy \le \frac{Ce^{-\delta M}}{2M\alpha} Q_t^N(\alpha).$$

Indeed, by Lemma 3.2 (3), $\mathcal{F}_{t,\alpha}$ is decreasing on $(-1,0)$ and there exists $-\Lambda < 0$ independent of (t,α) so that

$$\mathcal{F}_{t,\alpha}(y) \ge \mathcal{F}_{t,\alpha}(-\frac{1}{2}) - \frac{\Lambda}{2}(y + \frac{1}{2}). \tag{3.25}$$

As above,

$$\begin{aligned}
\int_{-1}^{-\frac{1}{2}} e^{-M\mathcal{F}_{t,\alpha}(y)} dy &\le \frac{2e^{-M\mathcal{F}_{t,\alpha}(-\frac{1}{2})}}{\Lambda M} \\
&\le \frac{8e^{-M(\mathcal{F}_{t,\alpha}(-\frac{1}{2})-\mathcal{F}_{t,\alpha}(-\frac{1}{4}))}}{\Lambda M} \int_{-\frac{1}{2}}^{-\frac{1}{4}} e^{-M\mathcal{F}_{t,\alpha}(y)} dy \\
&\le \frac{Ce^{-\delta M}}{2M\alpha} Q_t^N(\alpha), \quad \text{where } \delta := 2 \inf_{y \in [-\frac{1}{2},-\frac{1}{4}]} |\mathcal{F}'_{t,\alpha}(y)|.
\end{aligned}$$

The lemma is proved by combining the above inequalities and

$$\left| 1 - \frac{2\pi\alpha \int_{-\frac{1}{2}}^{1} e^{-M\mathcal{F}_{t,\alpha}(y)} dy}{Q_t^N(\alpha)} \right| = \left| \frac{2\pi\alpha \int_{-1}^{-\frac{1}{2}} e^{-M\mathcal{F}_{t,\alpha}(y)} dy + 2\pi\alpha \int_1^{\frac{1}{\alpha}-1} e^{-M\mathcal{F}_{t,\alpha}(y)} dy}{Q_t^N(\alpha)} \right| \le \frac{Ce^{-\delta M}}{M}.$$

$\square$

Proof of Proposition 3.1

We introduce a smooth cut-off function η such that $\eta = 1$ on $[-\frac{1}{2}+\epsilon, 1-\epsilon]$ for some fixed $\epsilon > 0$ (independent of (t, N, α)) and with $\eta = 0$ outside $(-\frac{1}{2}, 1)$ and write

$$Q_t^N(\alpha) = I_t^N(\alpha) + II_t^N(\alpha), \text{ with } I_t^N(\alpha) = 2\pi\alpha \int_{-\frac{1}{2}}^{1} e^{-M\mathcal{F}_{t,\alpha}(y)}\eta(y)dy.$$

By Lemma 3.6, $II_t^N \leq \frac{Ce^{-\delta M}}{2M\alpha} Q_t^N(\alpha)$, hence $Q_t^N(\alpha)\left(1 + O(\frac{e^{-\delta M}}{2M\alpha})\right) = I_t^N(\alpha)$, and therefore

$$Q_t^N(\alpha) = I_t^N(\alpha)\left(1 + O(\frac{e^{-\delta M}}{2M\alpha})\right). \tag{3.26}$$

We now evaluate $I_t^N(\alpha)$ asymptotically with respect to the parameter M by the method of complex stationary phase with non-degenerate complex phase functions (Theorem 7.7.5 in [H]), with α, t as parameters. Using the evaluation of the Hessian in Lemma 3.5, we obtain

$$\left| I_t^N(\alpha) - \frac{2\pi^{\frac{3}{2}}\alpha}{(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha))^{\frac{1}{2}}(M)^{\frac{1}{2}}} \sum_{k=0}^{R} \frac{A_{t,k}(\alpha)}{M^k} \right| \leq \frac{C_R\alpha}{(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha))^{\frac{1}{2}}(M)^{\frac{1}{2}}} M^{-(R+1)}, \tag{3.27}$$

where the $A_{t,k}(\alpha)$ are obtained by applying powers of the inverse Hessian operator

$$\frac{1}{\left(\frac{1}{(1-\alpha)} + \alpha f_t''(\alpha)\right)} \frac{d^2}{dy^2}$$

to $e^{-MR_3(y;t,\alpha)}$ and R_3 is the cubic remainder in the Taylor expansion of $\mathcal{F}_{t,\alpha}(y)$ at $y = 0$. The inverse Hessian is uniformly bounded above in the interior region, so R_3 is uniformly bounded with uniformly bounded derivatives when $\alpha \in [0, \frac{2}{3}]$ and $y \in [-\frac{1}{2}+\epsilon, 1-\epsilon]$ (cf. Lemma 3.5 (5). Therefore the stationary phase coefficients and remainder are uniformly bounded in the interior region.

The Proposition follows by combining the complex stationary phase asymptotics with (3.26).

□

3.2 Boundary estimates

We now give joint asymptotic estimates of norming constants in the boundary zone where $0 < \alpha \leq \frac{1}{N^{3/4}}$. The exclusion of $\alpha = 0$ is not important since the norming constants also equal 1 there. The main result of this section is:

Proposition 3.2 *Assume $0 < \alpha \leq \frac{1}{N^{3/4}}$. Then we have*

$$q_t^N(\alpha) = 1 + O(N^{-\frac{1}{3}}). \tag{3.28}$$

The proof of Proposition 3.2 consists of a number of Lemmas.

First we will localize the integral Q_t^N and Q_P^N.

Lemma 3.7 *Suppose* $0 < \alpha \le \frac{1}{N^{3/4}}$. *Then there exists constants* $\delta, C > 0$ *such that*

$$\begin{cases} \dfrac{\alpha \int_{\frac{1}{\alpha N^{2/3}}}^{\frac{1}{\alpha}-1} e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy}{Q_t^N(\alpha)} & \le \; Ce^{-\delta N^{1/3}}, \\ \dfrac{\alpha \int_{\frac{1}{\alpha N^{2/3}}}^{\frac{1}{\alpha}-1} e^{-\alpha N \mathcal{F}_{P,\alpha}(y)} dy}{Q_P^N(\alpha)} & \le \; Ce^{-\delta N^{1/3}}. \end{cases} \tag{3.29}$$

Proof We now localize the integral (3.21) into the subinterval $[-1, \frac{1}{\alpha N^{2/3}}]$ by showing that the integral over $[\frac{1}{\alpha N^{2/3}}, \frac{1}{\alpha} - 1]$ is relatively negligeable. In the boundary region, $\frac{1}{\alpha N^{2/3}} \ge N^{1/12}$.

As in Lemma 3.6, there exists a uniform positive constant $\Lambda > 0$ such that $\mathcal{F}_{t,\alpha}(y) \ge \mathcal{F}_{t,\alpha}(\frac{1}{\alpha N^{2/3}}) + \Lambda(y - \frac{1}{\alpha N^{2/3}})$ on $[\frac{1}{\alpha N^{2/3}}, \frac{1}{\alpha} - 1]$ and we obtain

$$\begin{aligned} \int_{\frac{1}{\alpha N^{2/3}}}^{\frac{1}{\alpha}-1} e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy &\le \frac{C}{\alpha N} e^{-\alpha N \mathcal{F}_{t,\alpha}(\frac{1}{\alpha N^{2/3}})} \\ &\le \frac{C}{\alpha N} e^{-\epsilon N^{1/3}}, \end{aligned}$$

using the fact (cf. Lemma 3.5) that $\mathcal{F}_{t,\alpha}(y) \ge \epsilon(y-1) + \mathcal{F}_{t,\alpha}(1)$ as $y \to \infty$; here, $\epsilon > 0$ is a positive and uniform constant.

To prove that the integral over $[\frac{1}{\alpha N^{2/3}}, \frac{1}{\alpha} - 1]$ is relatively negligeable, we give a lower bound for the integral on the whole interval $[-1, \frac{1}{\alpha} - 1]$. In fact a very crude lower bound suffices, so we choose a convenient subinterval. By Lemma 3.5, $\mathcal{F}'_{t,\alpha}(y)$ is uniformly bounded on any compact subset of $[0, \frac{1}{\alpha} - 1]$ when α is the boundary region. Using that $\mathcal{F}_{t,\alpha}(y) \le \mathcal{F}_{t,\alpha}(1) + C_0(y-1)$ on $[1, 2]$ for some $C_0 > 0$, we have

$$\begin{aligned} \int_{-1}^{\frac{1}{\alpha}-1} e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy &\ge \int_1^2 e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy \\ &\ge e^{-\alpha N \mathcal{F}_{t,\alpha}(1)} \int_1^2 e^{-C_0 \alpha N (y-1)} dy \\ &\ge \frac{C}{\alpha N} e^{-\alpha N \mathcal{F}_{t,\alpha}(1) - C_0 \alpha N} \\ &\ge \frac{C}{\alpha N} e^{-C_0 N^{1/4}}. \end{aligned}$$

Therefore

$$\frac{\alpha \int_{\frac{1}{\alpha N^{2/3}}}^{\frac{1}{\alpha}-1} e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy}{Q_t^N(\alpha)} \le Ce^{-\epsilon N^{1/3} + C_0 N^{1/4}} \le Ce^{-\delta N^{1/3}}$$

for some $\delta > 0$.

□

3.3 Proof of Proposition 3.2

Proof By definition and the previous lemma, we have

$$
\begin{aligned}
q_t^N(\alpha) &= \frac{Q_t^N(\alpha)}{Q_P^N(\alpha)} \\
&= \frac{\int_{-1}^{\frac{1}{\alpha N^{2/3}}} e^{-\alpha N \mathcal{F}_{t,\alpha}(y)} dy}{\int_{-1}^{\frac{1}{\alpha N^{2/3}}} e^{-\alpha N \mathcal{F}_{P,\alpha}(y)} dy}(1+O(e^{N^{-\frac{1}{3}}})) \\
&= \frac{\int_{-1}^{\frac{1}{\alpha N^{2/3}}} e^{-\alpha N \mathcal{F}_{P,\alpha}(y)+\alpha^2 N y^2 f_{t,\alpha}(\alpha(1+y))} dy}{\int_{-1}^{\frac{1}{\alpha N^{2/3}}} e^{-\alpha N \mathcal{F}_{P,\alpha}(y)} dy}(1+O(e^{-N^{-\frac{1}{3}}})) \\
&= (1+O(N^{-\frac{1}{3}})),
\end{aligned}
$$

where in the last line, we Taylor expand the exponential $e^{-\alpha^2 N y^2 f_{t,\alpha}(\alpha(1+y))}$ and recall that in the boundary layer, $\alpha^2 N y^2 f_{t,\alpha}(\alpha(1+y)) = O(N^{-\frac{1}{3}})$.

This completes the proof of Proposition 3.2.

When $\alpha = 0$,

$$
q_t^N(0) = \frac{\int_0^1 e^{N\log(1-x) - N x^2 f_{t,\alpha}(x)} dx}{\int_0^1 e^{N\log(1-x)} dx}.
$$

Notice that the phase is strictly decreasing on $[0,1]$, one can apply the similar argument as before with the substitution $y = N^{2/3}x$. We leave it as an excercise to show $q_t^N(0) \sim 1$ as $N \to \infty$.

□

3.4 Completion of proof of Theorem 3.1

It is easy to see in the boundary layer $0 < \alpha \le \frac{1}{N^{3/4}}$,

$$
\frac{\det \nabla^2 G_t(\alpha)}{\det \nabla^2 G_P(\alpha)} = \frac{\frac{1}{\alpha(1-\alpha)} + f_t''(\alpha)}{\frac{1}{\alpha(1-\alpha)}} = 1 + O(\alpha).
$$

Therefore $\frac{Q_t^N(\alpha)}{Q_P^N(\alpha)}$ continuously extends to P.

Consider the interior of P with $\frac{1}{N^{3/4}} \le \alpha \le \frac{2}{3}$. By Corollary 3.3, we have

$$
q_t^N(\alpha) = \frac{Q_t^N(\alpha)}{Q_P^N(\alpha)} = \left(\frac{1}{1+f_t''(\alpha)}\right)^{\frac{1}{2}} + O(\frac{1}{\alpha N}) = \frac{\det \nabla^2 G_t(\alpha)}{\det \nabla^2 G_P(\alpha)} + O(\frac{1}{\alpha N}) = \frac{\det \nabla^2 G_t(\alpha)}{\det \nabla^2 G_P(\alpha)} + O(\frac{1}{N^{1/4}}).
$$

Consider the boundary layer of P with $0 \le \alpha \le \frac{1}{N^{3/4}}$. By Lemma 3.2, we have

$$
q_t^N(\alpha) = \frac{Q_t^N(\alpha)}{Q_P^N(\alpha)} = 1 + O(\frac{1}{N^{1/3}}) = \frac{\det \nabla^2 G_t(\alpha)}{\det \nabla^2 G_P(\alpha)} + O(\frac{1}{N^{1/3}}).
$$

Therefore for any $\alpha \in P$, we have

$$
q_t^N(\alpha) = \frac{\det \nabla^2 G_t(\alpha)}{\det \nabla^2 G_P(\alpha)} + O(\frac{1}{N^{1/4}}). \tag{3.30}
$$

This proves Theorem 3.1.

□

4 Proof of the main theorem 1.2

4.1 The C^0-convergence

Proposition 4.1 *There exists a constant $C > 0$ such that for any $z \in \mathbf{CP}^1$*

$$\left| \frac{1}{N} \log \mathcal{E}_N(t,z) \right| \leq \frac{C \log N}{N}. \tag{4.1}$$

Proof By Corollary 3.1, there exists a constant $C > 0$ such that $\frac{1}{C} \leq \mathcal{R}_t^N(\alpha) \leq C$ for any $\alpha \in P$. Then

$$\frac{1}{C} \leq \frac{\mathcal{E}_N(t,z)}{\Pi_N(t,z)} \leq C.$$

The proposition is proved by applying the Tian-Yau-Zelditch expansion [Z], which asserts that there exists a C^∞ asymptotic expansion,

$$\Pi_N(t,z) = N \left(1 + \frac{a_1(t,z)}{N} + \cdots \right), \tag{4.2}$$

which may be differentiated any number of times. It obviously implies that

$$\left| \frac{1}{N} \log \mathcal{E}_N(t,z) \right| \leq \frac{C \log N}{N}.$$

□

We note that the convergence rate is best possible. The C^0 convergence with this rate was proved for general Kähler manifolds in [B]; a simple proof along the above lines for general toric Kähler manifolds will appear in [SoZ].

4.2 A localization lemma

To obtain C^2 convergence, we have to estimate weighted sums of $\mathcal{P}_\alpha^N(t,z)$ for $\alpha \in P \cap \frac{1}{N}\mathbf{Z}$. The following localization lemma enables to replace the sum of $\mathcal{P}_\alpha^N(t,z)$ by its partial sum for α in small neighborhoods of $\pi_t(\rho)$, where $\rho = \log |z|^2$. Fix $z = e^{\rho/2+i\theta} \in X$ with $x = \pi_t(\rho)$.

Lemma 4.1 *For any $\delta > 0$, there exist $0 < \delta_1 < \delta$, $0 < \delta_2 < \delta$, $\epsilon > 0$ and $C > 0$ such that for any α and $\beta \in [0,1] \cap \frac{1}{N}\mathbf{Z}$ with $|\alpha - x| < \delta_1$ and $|\beta - x| \geq 2\delta_2$, we have*

$$\frac{P_\alpha^N(t,z)}{P_\beta^N(t,z)} \leq Ce^{-\epsilon N}. \tag{4.3}$$

Proof First let's assume $x \in (0,1)$.

$$\begin{aligned}
\frac{P_\beta^N(t,z)}{P_\alpha^N(t,z)} &= \frac{e^{-N((x-\beta)G_t'(x)-(G_t(x)-G_t(\beta)))}}{e^{-N((x-\alpha)G_t'(x)-(G_t(x)-G_t(\alpha)))}} \frac{Q_t^N(\alpha)}{Q_t^N(\beta)} \\
&= e^{-N(G_t(\beta)-G_t(\alpha)-G_t'(x)(\beta-\alpha))} \frac{Q_t^N(\alpha)}{Q_t^N(\beta)} \\
&= e^{-N(G_t(\beta)-G_t(\alpha)-G_t'(\alpha)(\beta-\alpha))+N(G_t'(x)-G_t'(\alpha))(\beta-\alpha)} \frac{Q_t^N(\alpha)}{Q_t^N(\beta)} \\
&= e^{-NG_t''(\gamma)(\beta-\alpha)^2+N(G_t'(x)-G_t'(\alpha))(\beta-\alpha)} \frac{Q_t^N(\alpha)}{Q_t^N(\beta)}
\end{aligned}$$

for some γ between α and β.

Notice that G_t'' is uniformly bounded below from 0 and G_t' is equicontinuous on [0,1]. Therefore we can choose $\delta_1 << \delta_2$ so that there exits $\epsilon > 0$ and

$$e^{-NG_t''(\gamma)(\beta-\alpha)^2+N(G_t'(x)-G_t'(\alpha))(\beta-\alpha)} \leq e^{-2\epsilon N}.$$

Also

$$\frac{P_\alpha^N(t,z)}{P_\beta^N(t,z)} \leq Ce^{-\epsilon N}.$$

When $x = 0$ or 1, the same estimate can be proved by similar argument as above making use of the monotonicity of G_t.

□

4.3 The C^2-convergence

We now prove the main results giving bounds on two space-time derivatives of $\varphi_N(t,z)$. The main ingredients are the bounds of $\mathcal{R}_t^N$ in Corollary 3.1 and a comparison to derivatives of the Szegö kernel $\Pi_N(t,z)$ for the metric ϕ_t deriving from Lemma 2.1. By (4.2) it is straightforward to determine the derivatives of $\Pi_N(t,z)$.

The following lemma is the consequence of the family version of the Tian-Yau-Zelditch expansion.

Lemma 4.2 *We have the following uniform convergence in the C^∞ topology on $[0,1]\times\mathbf{CP}^1$*

$$\lim_{N\to\infty}\frac{1}{N}\log\Pi_N(t,z) = 0. \tag{4.4}$$

Corollary 4.1 *All derivatives of $\frac{1}{N}\log\Pi_N(t,z)+u_t(z)$, of order great than zero, are uniformly bounded on X.*

Proof Although $u_t(z)$ is not a well-defined function on X, $e^{-u_t(z)}$ extends to a hermitian metric on the line bundle so that, by applying global vector fields, any derivatives of $u_t(z)$ are well defined functions on X and are uniformly bounded.

□

Proposition 4.2

$$\lim_{N\to\infty}\left|\left|\frac{1}{N}\log\mathcal{E}_N(t,z)\right|\right|_{C^2([0,1]\times X)} = 0. \tag{4.5}$$

Proof Fix $z\in\mathbf{CP}^1$, and put $x=\pi_t(z)$. To prove the C^2 convergence of $\frac{1}{N}\log\mathcal{E}_N(t,z)$ it suffices by (4.2) to prove C^2 convergence for $\frac{1}{N}(\log\mathcal{E}_N(t,z)-\log\Pi_N(t,z))$. We use Lemma 2.1 to simplify the formula for $\mathcal{E}_N(t,z)$.

Second order convergence in pure space derivatives

We first consider pure space derivatives. By $\sum_\alpha$ and $\sum_{\alpha,\beta}$, we mean $\sum_{\alpha\in P\cap\frac{1}{N}\mathbf{Z}}$ and $\sum_{\alpha,\beta\in P\cap\frac{1}{N}\mathbf{Z}}$. If x is in the 'interior region' of $[0,1]$, we may use the coordinates $z=e^{\rho/2+i\theta}$, and

$$\begin{aligned}
&\frac{1}{N}\left|\frac{\partial^2}{\partial\rho^2}\log\mathcal{E}_N(t,z)-\frac{\partial^2}{\partial\rho^2}\log\Pi_N(t,z)\right|\\
=\;& N\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{2\left(\sum_\alpha\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}-\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{2\left(\sum_\alpha\mathcal{P}_\alpha^N(t,z))\right)^2}\right|\\
=\;& N\left(\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{2\left(\sum_\alpha\mathcal{P}_\alpha^N(t,z)\right)^2}\right)\\
&\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{\left(\sum_\alpha\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}\frac{\left(\sum_\alpha\mathcal{P}_\alpha^N(t,z)\right)^2}{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}-1\right|\\
\leq\;& \left|\frac{1}{N}\frac{\partial^2}{\partial\rho^2}\log\Pi_N(t,z)+\frac{\partial^2}{\partial\rho^2}u_t(z)\right|\\
&\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{\left(\sum_\alpha\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}\frac{\left(\sum_\alpha\mathcal{P}_\alpha^N(t,z)\right)^2}{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}-1\right|\\
\leq\;& C\left|\frac{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{\sum_{\alpha,\beta}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}\frac{\left(\sum_\alpha\mathcal{P}_t^N(\alpha,z)\right)^2}{\left(\sum_\alpha\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}-1\right|\\
\leq\;& C\left|\frac{\sum_{\alpha,\beta\in B_x(\delta)}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}{\sum_{\alpha,\beta\in B_x(\delta)}(\alpha-\beta)^2\mathcal{P}_\alpha^N(t,z)\mathcal{P}_\beta^N(t,z)}\frac{\left(\sum_{\alpha\in B_x(\delta)}\mathcal{P}_t^N(\alpha,z)\right)^2}{\left(\sum_{\alpha\in B_x(\delta)}\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}-1+O\left(e^{-\epsilon N}\right)\right|\\
\leq\;& C\left(\sup_{\alpha\in B_x(\delta)}\mathcal{R}_t^N(\alpha)-\inf_{\alpha\in B_x(\delta)}\mathcal{R}_t^N(\alpha)\right)+O\left(e^{-\epsilon N}\right)
\end{aligned}$$

for some fixed $\epsilon>0$, where $B_x(\delta)=\{\alpha\in[0,1]\cap\frac{1}{N}\mathbf{Z}\mid|\alpha-x|<\delta\}$.

$R_t^\infty(\alpha)=\lim_{N\to\infty}R_t^N(\alpha)$ is continuous on $[0,1]$ and the convergence is uniform. Therefore for any $\epsilon'>0$, there exists $\delta>0$ and sufficiently large N' such that for all $N\geq N'$

$$\sup_{\alpha\in B_x(\delta)}\mathcal{R}_t^N(\alpha)-\inf_{\alpha\in B_x(\delta)}\mathcal{R}_t^N(\alpha)\leq\epsilon'.$$

In other words, for any $\epsilon'>0$, there exists a sufficiently large N' such that for all $N\geq N'$,

$$\frac{1}{N}\left|\frac{\partial^2}{\partial\rho^2}\log\mathcal{E}_N(t,z)-\frac{\partial^2}{\partial\rho^2}\log\Pi_N(t,z)\right|<\epsilon'.$$

If x is close to the boundary of $[0,1]$, without loss of generality we fix a holomorphic coordinate system $\{z\}$ for $\mathbf{CP}^1$ near the north pole z_N such that $z=0$ at z_N. Let $r=|z|$. Then

$$
\begin{aligned}
&\frac{1}{N}\left|\frac{\partial^2}{\partial r^2}\log\mathcal{E}_N(t,z)-\frac{\partial^2}{\partial r^2}\log\Pi_N(t,z)\right| \\
= \;& N\left|\frac{\sum_{\alpha,\beta>0}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}^N_{\alpha-\frac{1}{N}}(t,z)\mathcal{P}^N_{\beta-\frac{1}{N}}(t,z)}{\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}-\frac{\sum_{\alpha,\beta>0}(\alpha-\beta)^2\mathcal{P}^N_{\alpha-\frac{1}{N}}(t,z)\mathcal{P}^N_{\beta-\frac{1}{N}}(t,z)}{\left(\sum_\alpha \mathcal{P}_\alpha^N(t,z)\right)^2}\right| \\
\leq \;& C\left|\frac{\left(\sum_{\alpha,\beta>0}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}^N_{\alpha-\frac{1}{N}}(t,z)\mathcal{P}^N_{\beta-\frac{1}{N}}(t,z)\right)\left(\sum_\alpha \mathcal{P}_\alpha^N(t,z)\right)^2}{\left(\sum_{\alpha,\beta>0}(\alpha-\beta)^2\mathcal{P}^N_{\alpha-\frac{1}{N}}(t,z)\mathcal{P}^N_{\beta-\frac{1}{N}}(t,z)\right)\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2}-1\right|.
\end{aligned}
$$

By localizing the summand and similar argument for the interior, we can show that for any $\epsilon'>0$, there exists a sufficiently large N' such that for all $N\geq N'$,

$$
\frac{1}{N}\left|\frac{\partial^2}{\partial r^2}\log\mathcal{E}_N(t,z)-\frac{\partial^2}{\partial r^2}\log\Pi_N(t,z)\right|<\epsilon'.
$$

Second order convergence in pure time derivatives

We now consider time derivatives. Let $G_i(x)=G_P(x)+f_i(x)$, for $i=0,1$. Let $f_t(x)=(1-t)f_0(x)+tf_1(x)$ and $\frac{\partial}{\partial t}f_t(x)=v(x)=f_1(x)-f_0(x)$. Also $U_t(\rho)=u_t(z)$. By Legendre transform, $U_t(\rho)=x\rho-G_t(x)$ with $\rho=G_t'(x)$ and $x=U_t'(\rho)$. Calculate

$$
\frac{\partial}{\partial t}U_t(\rho)=\dot{x}\rho-\frac{\partial}{\partial t}G_t(x)-G_t'(x)\dot{x}=-\frac{\partial}{\partial t}G_t(x)=-v(x)
$$

and

$$
\frac{\partial^2}{\partial t^2}U_t(\rho)=-v'(x)\dot{x}=(v'(x))^2U_t''(\rho)=(v'(x))^2\frac{\partial^2}{\partial \rho^2}u_t(z).
$$

Straightforward calculation shows that

$$
\begin{aligned}
\frac{1}{N}\frac{\partial}{\partial t}\log\mathcal{E}_N(t,z)+\frac{\partial}{\partial t}u_t(z) &= \frac{1}{N}\frac{\sum_\alpha \log\frac{Q_0^N(\alpha)}{Q_1^N(\alpha)}\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)}{\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)} \\
&= \frac{1}{N}\frac{\sum_\alpha\left(-Nv(\alpha)+\log\frac{q_0^N(\alpha)}{q_1^N(\alpha)}\right)\mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)}{\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{N}\frac{\partial^2}{\partial t^2}\log\mathcal{E}_N(t,z)+\frac{\partial^2}{\partial t^2}u_t(z) \\
= \;& N\frac{\sum_{\alpha,\beta}\left((v(\alpha)-v(\beta))+\frac{1}{N}\log\frac{q_0^N(\beta)q_1^N(\alpha)}{q_0^N(\alpha)q_1^N(\beta)}\right)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{2\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
= \;& I_1^N(t,z)+I_2^N(t,z)+I_3^N(t,z).
\end{aligned}
$$

$$
\begin{aligned}
& I_1^N(t,z) \\
= \;& N\frac{\sum_{\alpha,\beta}(v(\alpha)-v(\beta))^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{2\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
\sim \;& N\frac{\sum_{\alpha,\beta\in B_x(\delta)}(\alpha-\beta)^2\left(\frac{v(\alpha)-v(\beta)}{\alpha-\beta}\right)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{2\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
\sim \;& N(v'(x))^2\frac{\sum_{\alpha,\beta\in B_x(\delta)}(\alpha-\beta)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{2\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
& +N\frac{\sum_{\alpha,\beta\in B_x(\delta)}(\alpha-\beta)^2\left(\left(\frac{v(\alpha)-v(\beta)}{\alpha-\beta}\right)^2-(v'(x))^2\right)\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{2\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
\sim \;& \left((v'(x))^2+\sup_{\alpha\neq\beta\in B_x(\delta)}\left(\left(\frac{v(\alpha)-v(\beta)}{\alpha-\beta}\right)^2-(v'(x))^2\right)\right)\left(\frac{1}{N}\frac{\partial^2}{\partial\rho^2}\log\mathcal{E}_N(t,z)+\frac{\partial^2}{\partial\rho^2}u_t(z)\right) \\
\sim \;& (v'(x))^2\frac{\partial^2}{\partial\rho^2}u_t(z)=\frac{\partial^2}{\partial t^2}u_t(z)
\end{aligned}
$$

as $\delta\to 0$.

Therefore $\lim_{N\to\infty} I_1^N(t,z)=\frac{\partial^2}{\partial t^2}u_t(z)$.

$$
\begin{aligned}
I_2^N(t,z) \;&=\; \frac{\sum_{\alpha,\beta}(v(\alpha)-v(\beta))\log\frac{q_0^N(\beta)q_1^N(\alpha)}{q_0^N(\alpha)q_1^N(\beta)}\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
&\sim\; \frac{\sum_{\alpha,\beta\in B_x(\delta)}(v(\alpha)-v(\beta))\log\frac{q_0^N(\beta)q_1^N(\alpha)}{q_0^N(\alpha)q_1^N(\beta)}\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
&\sim\; 0
\end{aligned}
$$

as $\delta\to 0$.

Therefore $\lim_{N\to\infty} I_2^N(t,z)=0$.

$$
\begin{aligned}
I_3^N(t,z) \;&=\; \frac{1}{N}\frac{\sum_{\alpha,\beta}\left(\log\frac{q_0^N(\beta)q_1^N(\alpha)}{q_0^N(\alpha)q_1^N(\beta)}\right)^2\mathcal{R}_t^N(\alpha)\mathcal{R}_t^N(\beta)\mathcal{P}_\alpha^N(t,z)\mathcal{P}_t^N(\beta,z)}{\left(\sum_\alpha \mathcal{R}_t^N(\alpha)\mathcal{P}_\alpha^N(t,z)\right)^2} \\
&\sim\; 0.
\end{aligned}
$$

Therefore $\lim_{N\to\infty} I_3^N(t,z)=0$.

We conclude from the above calculation that

$$
\lim_{N\to\infty}\frac{\partial^2}{\partial t^2}\log\mathcal{E}_N(t,z)=0.
$$

By a similar argument, which we leave to the reader, the mixed space-time derivatives of $\log\mathcal{E}_N(t,z)$ also uniformly converges to 0. Therefore $\frac{1}{N}\log\mathcal{E}_N(t,z)$ has bounded second derivatives and $\frac{1}{N}\log\mathcal{E}_N(t,z)$ converges in C^2 to 0. $\square$

We now conclude the proof of the main result:

Proof of Theorem 1.2

Notice that $\varphi_N(t,\cdot) - \varphi_t(\cdot) = \frac{1}{N}\log \mathcal{E}_N(t,\cdot)$. Therefore Theorem 1.2 is proved. $\square$

4.4 Final remarks and questions

We conclude with some questions:

- Limits on the degree of convergence of $\varphi_N(t,z) \to \varphi(t,z)$ are related to the distribution of complex zeros of the holomorphic extension of $\mathcal{E}_t^N$. In the toric case, in the coordinates $x = e^\rho$, $\mathcal{E}_t^N$ is a positive real polynomial of a real variable. As observed by Lee-Yang in the context of partition functions of statistical mechanical models, the degree of convergence of $\frac{1}{N}\log \mathcal{E}_t^N$ to its limit is related to the limit distribution of the complex zeros of $\mathcal{E}_t^N$ along the real domain. For a modern study of complex zeros of partition functions with references to the literature, see [BBCKK]. It would be interesting to study the complex zeros in the case of toric varieties.

- The formula for $\mathcal{E}_t^N(z)$ in Lemma 2.1 exhibits this function as the value on the diagonal of a Toeplitz type operator with multiplier $\mathcal{R}_t^N(\alpha)$. More precisely, it is the Berezin lower symbol of the Toeplitz type operator. For background we refer to [STZ2]. The question whether it is a Toeplitz operator in any standard sense is essentially the same question as to the existence of asymptotics of $\mathcal{E}_t^N(z)$ and joint asymptotics of $\mathcal{R}_t^N(\alpha)$. When this multiplier is a symbol, the sum has the general form of a Bernstein polynomial in the sense of [Z2] and admits a complete asymptotic expansion. It would be very helpful if there exists a more 'abstract' approach to this Toeplitz operator by constructing its Toeplitz symbol instead of its Berezin symbol. The leading order Toeplitz symbol is calculated in Corollary 3.1.

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