Bergman Kernels and Asymptotics of Polynomials I

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Our Topics

These talks are about random polynomials of large degree N in m complex variables. We will equip the space $\mathcal{P}_N^m$ of polynomials of degree N with natural probabality measures. Using asymptotics of Bergman kernels, we will determine:

- The almost sure distribution of (simultaneous) zeros $\{f_1 = \cdots = f_k = 0\}$ of $k \leq m$ polynomials;

- Correlations between zeros.

- The distribution of critical points $\{\nabla f = 0\}$ of f;

- How zeros and critical points are influenced by the Newton polytopes of the polynomials.

Plan of Events

- Talk I: Distribution and correlations between zeros of general polynomials in $\mathbb{CP}^m$ (projective space) and more general Kähler manifolds (M, ω). The theme is:

  Random (Complex) Algebraic Geometry: How are random hypersurfaces in $\mathbb{CP}^m$ or other Kähler manifolds distributed? How do they intersect? Do zeros tend to repel or attract or ignore each other?

- Talk II: Random polynomials with fixed Newton polytope. How does the Newton polytope impact on the distribution of zeros and critical points? As we will see, it creates 'allowed' and 'forbidden' regions.

Polynomials

- Monomials: $\chi_\alpha(z) = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, $\alpha \in \mathbb{N}^m$.

- Polynomial of degree p (complex, holomorphic, not necessarily homogeneous):

$$f(z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq p} c_\alpha \chi_\alpha(z_1, \ldots, z_m).$$

- Homogenize to degree p: introduce new variable z_0 and put: $\hat{\chi}_\alpha(z) = z_0^{p-|\alpha|} z_1^{\alpha_1} \cdots z_m^{\alpha_m}$.

Inner product and Gaussian ensemble

We now define probability measure on $\mathcal{P}^m$. The simplest are the Gaussian measures, which only require an inner product. To define them, we homogenize the polynomials on $(\mathbb{C}^*)^m$ so that homogeneous polynomials F of degree p in $m+1$ complex variables. This identifies:

- $\mathcal{P}^m = H^0(\mathbb{CP}^m, \mathcal{O}(p))$, the space of holomorphic sections of the pth power of the hyperplane line bundle $\mathcal{O}_{\mathbb{CP}^m}(1)$. Equivalently, they are CR functions on S^{2m+1} satisfying $F(e^{i\theta}x) = e^{ip\theta}F(x)$.

  $H^0(\mathbb{CP}^m, \mathcal{O}(p))$ carries the SU($m+1$)-invariant inner product

  $$\langle F_1, \bar{F}_2 \rangle = \int_{S^{2m+1}} F_1 \bar{F}_2 \, d\sigma \ ,$$

  where $d\sigma$ is Haar measure on the $(2m+1)$-sphere S^{2m+1}.

Random $SU(m+1)$ polynomials

An orthonormal basis of $H^0(\mathbb{CP}^m, \mathcal{O}(p))$ is given by $\left\{\|\chi_\alpha\|^{-1}\chi_\alpha\right\}_{|\alpha|\leq p}$, where $\|\cdot\|$ denotes the norm in $H^0(\mathbb{CP}^m, \mathcal{O}(p))$. (Note that $\|\chi_\alpha\|$ depends on p.)

The corresponding SU($m+1$)-invariant Gaussian measure γ_p is defined by

$$(1)\quad d\gamma_p(s) = \frac{1}{\pi^{k_p}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{|\alpha|\leq p} \lambda_\alpha \frac{\chi_\alpha}{\|\chi_\alpha\|},$$

where $k_p = \#\{\alpha : |\alpha| \leq p\} = \binom{m+p}{p}$. Thus, the coefficients λ_α are independent complex Gaussian random variables with mean zero and variance one.

General Kähler manifold

L cx line bundle with hermitian metric h

$\downarrow$ curvature of L equals ω

M compact complex/

The Kähler metric/symplectic form ω and metric h gives an inner product on the space $H^0(M, L^N)$ (or $H^0_J(M, L^N)$):

$$\langle s_1, s_2 \rangle = \int_M h_N(s_1, s_2) dV_M$$

where $dV_M = \frac{1}{m!}\omega^m \quad (m = \dim M)$.

Distribution of zeros

We first define 'delta' functions on zero sets of one or several polynomials.

Given $f_1, \ldots, f_k$, $k \leq m$, put $Z_{f_1,\ldots,f_k} = \{z \in (C^*)^m : f_1(z) = \cdots = f_k(z) = 0\}$. $Z_{f_1,\ldots,f_k}$ defines a (k,k) current of integration:

$$\langle \psi, Z_{f_1,\ldots,f_k} \rangle = \int_{Z_f} \psi.$$

By Wirtinger's formula, the integral of a scalar function φ over Z_f can be defined as

$$\int_{Z_{f_1,\ldots,f_k}} \varphi \frac{\omega_{FS}^{n-k}}{(n-k)!}.$$

Expected zero current: Definition

Now consider m independent random polynomials with Newton polytope P, using the conditional probability $d\gamma_{pN}$. We let $\mathbf{E}_N(Z_{f_1,\dots,f_m})$ denote the expected density of their simultaneous zeros. It is the measure on $\mathbb{C}^{*m}$ given by

$$\mathbf{E}_{|P}(Z_{f_1,\dots,f_m})(U) = \int d\gamma_{p|P}(f_1) \cdots \int d\gamma_{p|P}(f_m)$$

$$\times\Big[\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}\Big]\,,$$

for $U \subset \mathbb{C}^{*m}$, where the integrals are over $H^0(\mathbb{CP}^m, \mathcal{O}(pN))$.

Expected distribution of zeros

Theorem 1 We have:

$$\frac{1}{(Np)^m}\mathbf{E}_N(Z_{f_1,\dots,f_m}) \to \omega_{\mathsf{FS}}^m$$

in the sense of weak convergence; i.e., for any open $U \subset \mathbb{C}^{*m}$, we have

$$\frac{1}{(Np)^m}\mathbf{E}_N\Big(\#\{z \in U : f_1(z) = \cdots = f_m(z) = 0\}\Big)$$

$$\to m!\mathrm{Vol}_{\mathbb{CP}^m}(U) \, .$$

Convergence is exponentially fast in the sense that

$$\mathbf{E}_N(Z_{f_1,\dots,f_m}) = (Np)^m\omega_{\mathsf{FS}}^m + O\left(e^{-\lambda N}\right),$$

for some positive continuous function λ.

Almost sure distribution of zeros

In fact, not only is the expected value of $\frac{1}{N}Z_{f_N}$ equal to the Fubini-Study Kähler form ω, but also:

THEOREM. (S–Zelditch, 1998) Consider a *random sequence* $\{f_N\}$ of sections of L^N (or polynomials of degree N), $N = 1, 2, 3, \dots$. Then

$$\frac{1}{N}Z_{f_N} \to \omega \qquad \textit{almost surely}.$$

Expected zeros of $k \leq m$ polynomials

Theorem We have:

$$N^{-1}\mathbf{E}_N(Z_f) \to \omega_{FS}^k$$

Bergman kernels, line bundles and circle bundles

The key ingredient in these results is the Bergman kernel.

The 'Bergman kernel' of a space $\mathcal{S}$ of holomorphic sections of a line bundle $L \to M$ = kernel of the orthogonal projection to $L^2 \to \mathcal{S}$.

For asymptotics, we lift to the S^1 bundle $X = \partial D \to M$ associated to L, i.e. boundary of the associated unit disk bundle D relative to a hermitian metric. Then Π is orthogonal projection from space $\mathcal{L}^2(\partial D)$ of sections to lifts of sections in $\mathcal{S}$. It is of the form $\Pi(x, y) = \sum_j s_j(x)\overline{s_j(y)}$, where $\{s_j\}$ is an orthonormal basis of $\mathcal{S}$.

In the case $M = \mathbb{CP}^m, L = \mathcal{O}(p)$, $X = S^{2m+1}$ and sections are just homogeneous polynomials restricted to S^{2m+1}. In this case,

$$\Pi_p^{\mathbb{CP}^m}(x, y) = C_m^p \langle x, y \rangle^p.$$

Bergman kernel, expected mass density and expected zero density

The importance of $\Pi_N(z,w)$ is the following:

- $\Pi_N(z,w) = \mathbf{E}(S(z) \cdot \overline{S(w)})$.

- $\overline{\partial}\partial \log \Pi_N(z,z) = \mathbf{E}(Z_s)$.

Using Π_N we define a (metric) Kodaira embedding

$$\Phi_N : X \to H^0(M, L^N), \quad \Phi_N(x) = \Pi_N(\cdot, x).$$

Complex case

Tian-Zelditch Theorem:

$$\frac{1}{N}\Phi_N^*(\omega_{\mathsf{FS}}) \sim \omega + \frac{1}{N}\varphi_1 + \cdots$$

This immediately tells us that

$$\mathbf{E}\left(\frac{1}{N}Z_{s_N}\right) \longrightarrow \omega \ .$$

Correlations between zeros

Although the *expected* distribution of zeros is uniform, the zeros do not behave as if they are thrown down independently.

The zeros are "correlated."

The correlation functions

We let $K^N\left(z^1,\ldots,z^n\right)$ denote the probability density of simultaneous zeros at points $z^1,\ldots,z^n$.

To define the n-point zero correlation measure $K^N_{nk}(z^1, \dots, z^n)$ we form the product measure

(2)

$$|Z_{(s_1,\dots,s_k)}|^n = \Big(\underbrace{|Z_{(s_1,\dots,s_k)}| \times \cdots \times |Z_{(s_1,\dots,s_k)}|}_{n}\Big) \quad \text{on} \quad M$$

To avoid trivial self-correlations, we puncture out the generalized diagonal in M^n to get the punctured product space

(3)

$$M_n = \{(z^1, \dots, z^n) \in M^n : z^p \neq z^q \quad \text{for } p \neq q\}\,.$$

We then restrict $|Z_{(s_1,\dots,s_k)}|^n$ to M_n and define

$$K^N_{nk}(z^1, \dots, z^n) = E(|Z_{(s_1,\dots,s_k)}|^n).$$

The first correlation function K_{1kN} just gives the expected distribution of simultaneous zeros of k sections. We have seen:

$$K_{1k}^{N}(z^0) = c_{mk} N^k + O(N^{k-1})\,,$$

for any positive line bundle.

When $k = m$, the simultaneous zeros almost surely form a discrete set of points and so this case is perhaps the most vivid. Roughly speaking, $K_{nk}^{N}(z^1, \ldots, z^n)$ gives the probability density of finding simultaneous zeros at $(z^1, \ldots, z^n)$.

Are zeros a gas?

Problems

- Do zeros ignore each other? This would mean $K^N_{nk}(z^1, \ldots, z^n) \equiv 1$.

- Do they Repel? Attract ?

- What is the correlation length? I.e. the length over which zeros interact?

Small length scales

The density of zeros increases linearly with N. If we scale by a factor $\sqrt{N}$, the expected density of zeros stays constant. This is the correlation length.

We rescale to study correlations on this scale:

$$K^N\left(\frac{z^1}{\sqrt{N}}, \ldots, \frac{z^n}{\sqrt{N}}\right) \longrightarrow K^\infty_{nkm}(z^1, \ldots, z^n)$$

We then define n-*point scaling limit zero correlation function*

(4)

$$\widetilde{K}^\infty_{nkm}(z^1, \ldots, z^n)$$

$$= \lim_{N\to\infty} \left(c_{mk} N^k\right)^{-n} K^N_{nk}\left(\frac{z^1}{\sqrt{N}}, \ldots, \frac{z^n}{\sqrt{N}}\right).$$

Universal scaling limit

The n-point scaling limit zero correlation function $\widetilde{K}^{\infty}_{nkm}(z^1, \dots, z^n)$ is given by a universal rational function, homogeneous of degree 0, in the values of the function $e^{i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2}$ and its first and second derivatives at the points $(z, w) = (z^p, z^{p'})$, $1 \leq p, p' \leq n$. Alternately it is a rational function in $z^p_q, \bar{z}^p_q, e^{z^p\cdot\bar{z}^{p'}}$

The function $e^{i\Im(z\cdot\bar{w})-\frac{1}{2}|z-w|^2}$ which appears in the universal scaling limit is (up to a constant factor) the Bergman kernel $\Pi^{\mathbf{H}}_1(z, w)$ of level one for the reduced Heisenberg group $\mathbf{H}^n_{\text{red}}$ (cf. §**??**).

Correlation length revisted

We have

$$\kappa_{km}(r) = 1 + O(r^4 e^{-r^2}), \quad r \to +\infty.$$

On the scale $\frac{r}{\sqrt{N}}$, correlations are short range in that they differ from the case of independent random points by an exponentially decaying term.

Explicit universal scaling limit

We have explicit formulas for $\widetilde{K}^{\infty}_{nkm}$ in all dimensions and codimensions.

The most vivid case is when $k = m$, where the simultaneous zeros of k-tuples of sections almost surely form a set of discrete points. Here is the small distance asymptotics of κ_{mm} in all dimensions.

Theorem **1** *For small values of r, we have*

$$\kappa_{mm}(r) = \begin{cases} \frac{m+1}{4} r^{4-2m} + O(r^{8-2m}), & \textit{as} \quad r \to 0, \\ \\ 1 + O(r^4 e^{-r^2}), & r \to +\infty. \end{cases}$$

Dimensional dependence

- When $m = 1, \kappa_{mm}(r) \to 0$ as $r \to 0$ and one has "zero repulsion."

- When $m = 2$, $\kappa_{mm}(r) \to 3/4$ as $r \to 0$ and one has a kind of neutrality.

- With $m \geq 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and there is some kind of attraction between zeros. One is more likely to find a zero at a small distance r from another zero than at a small distance r from a given point; i.e., zeros tend to clump together in high dimensions.

Ideas of Proof

A key object is the Bergman-Szegö projector

$$\Pi_{|P}(x,y) = \sum_{\alpha \in P} \frac{\widehat{\chi}_\alpha(x)\overline{\widehat{\chi}_\alpha(y)}}{||m_\alpha||^2}$$

of $H^0(\mathbb{CP}^m, \mathcal{O}(p), P)$. Its importance stems from:

- $\mathbf{E}_N\left(|f(z)|^2\right) = \Pi_N(z,z)$;

- $\mathbf{E}_N(Z_f) = \bar{\partial}\partial \log \Pi_N(z,z)$;

- $\mathbf{E}_N(Z_{f_1,\ldots,f_k}) = [\bar{\partial}\partial \log \Pi_N(z,z)]^{\wedge k}$;

Here, we use the Poincare-Lelong formula:

$$\begin{aligned} Z_s &= \frac{\sqrt{-1}}{2\pi}\left(\partial\bar{\partial}\log\|s\|_h^2 + \partial\bar{\partial}\log h\right) \\ &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|s\|_h^2 + c_1(L,h) \end{aligned}$$

$$\mathbf{E}(Z_s) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log \Pi_N(z,z) + c_1(L,h)$$

Co-area formula for correlations

Generalized Kac-Rice formula

Consider random sections $s \in H^0(M, L^N)$. We let

$$D_N(x^1, \ldots, x^n; \xi^1, \ldots, \xi^n; z^1, \ldots, z^n)$$

denote the joint probability distrbution of

$$x^j = s(z^j), \ \xi^j = \nabla s(z^j), \quad j = 1, \ldots, n.$$

The zero correlations are given by $K^N(z^1, \ldots, z^n)$

$$= \int D_N(0, \xi, z) \prod_{j=1}^{n} \left(\|\xi^j\|^2 d\xi^j \right)$$

The JPD

$$D_N(\bullet, \bullet, z^1, \ldots, z^n) = \mathcal{J}^1_{z*} \nu_N$$

is also a Gaussian. It may be expressed in terms of the Bergman kernel.

JPD

Define random variables $x_j^p,\ \xi_{jq}^p$

(5)

$$s(z^p) = \sum_{j=1}^{k} x_j^p e_j^p, \quad \nabla s(z^p) = \sum_{j=1}^{k} \sum_{q=1}^{m} \xi_{jq}^p dz_q^p \otimes e_j^p, \qquad p =$$

We have:

(6)

$$D_n(x, \xi; z) = \frac{\exp\langle -\Delta_n^{-1} v, v\rangle}{\pi^{kn(1+m)} \det \Delta_n}\,, \qquad v = (x \xi)\,,$$

where

(7) $$\Delta_n = \begin{pmatrix} A_n & B_n \\ B_n^* & C_n \end{pmatrix}$$

$$A_n = \left(A_{j'p'}^{jp}\right) = \left(\mathbf{E} x_j^p \bar{x}_{j'}^{p'}\right), \quad B_n = \left(B_{j'p'q'}^{jp}\right) = \left(\mathbf{E} x_j^p \bar{\xi}_{j'q'}^{p'}\right)$$

$$j, j' = 1, \ldots, k; \quad p, p' = 1, \ldots, n; \quad q, q' = 1, \ldots, m.$$

(We note that $A_n,\ B_n,\ C_n$ are $kn \times kn,\ kn \times knm,\ knm \times knm$ matrices, respectively; j, p, q index the rows, and j', p', q' index the columns.)

The function $D_n(0,\xi;z)$ is a Gaussian function, but it is not normalized as a probability density. It can be represented as

(8) $$D_n(0,\xi;z) = Z_n(z) D_{\Lambda_n}(\xi;z),$$

where

(9) $$D_{\Lambda_n}(\xi;z) = \frac{1}{\pi^{knm} \det \Lambda_n} \exp\left(-\langle \Lambda_n^{-1}\xi,\xi\rangle\right)$$

is the Gaussian density with covariance matrix

(10)
$$\Lambda_n = C_n - B_n^* A_n^{-1} B_n = \left(C^{jpq}_{j'p'q'} - \sum_{j_1,p_1,j_2,p_2} \bar{B}^{j_1 p_1}_{jpq} \Gamma^{j_1 p_1}_{j_2 p_2} B^{j_2 \ldots}_{j' p \ldots}\right.$$

and

(11) $$Z_n(z) = \frac{\det \Lambda_n}{\pi^{kn} \det \Delta_n} = \frac{1}{\pi^{kn} \det A_n}.$$

This reduces formula (**??**) to

(12)
$$K_n(z) = \frac{1}{\pi^{kn} \det A_n} \left\langle \prod_{p=1}^{n} \det\left(\xi^{p*}\gamma^p \xi^p\right) \right\rangle_{\Lambda_n}$$

Heisenberg model and Bergman kernel scaling

Near diagonal asymptotics of the Szegö kernel on a symplectic manifold:

Choose normal coordinates $\{z_j\}$ centered at a point $P_0 \in M$ and a 'preferred' local frame for L. We then have:

$$N^{-m}\Pi_N(P_0 + \frac{u}{\sqrt{N}}, \frac{\theta}{N}; P_0 + \frac{v}{\sqrt{N}}, \frac{\varphi}{N})$$

$$\sim \frac{1}{\pi^m} e^{i(\theta-\varphi)+u\cdot\bar{v}-\frac{1}{2}(|u|^2+|v|^2)} \cdot \left[1 + \sum_{r=1}^{K} N^{-\frac{r}{2}} b_r(P_0, u, v) + \cdots\right],$$

where b_r is an even/odd polynomial of degree $\leq 5r$.

Complex oscillatory integrals

This is obtained from the Boutet de Monvel - Sjostrand parametrix for the full Bergman kernel: $\Pi(x,y) = \sum_{N=1}^{\infty} \Pi_N(x,y)$:

$$\Pi(x,y) \sim \int_0^{\infty} e^{\psi(x,y)} s(x,y,\lambda) d\lambda,$$

where

- $\psi(x,y) = (1 - \lambda\bar{\mu}\frac{\sqrt{a(x,y)}}{\sqrt{a(x,x)a(y,y)}}$.
- $s(x,y,\lambda) \sim \lambda^m \sum_{j=0}^{\infty} a_j(x) \lambda^{-j}$.

Asymptotics of Π_N

Π_N is a Fourier coefficient of Π:

$$\Pi_N(x,y) = \int_{S^1} \Pi(x, e^{i\theta} y) e^{iN\theta} d\theta.$$

Hence, we get

$$\Pi_N(z,z) = \int_0^\infty \int_{S^1} e^{N\Psi(z,\theta,\lambda)} a_N(z,\theta,\lambda,$$

with

- $\Psi(x,\theta) = (1 - \psi(x, e^{i\theta} x)).$
- $a_N(x,\theta) = s(x, e^{i\theta} x, \lambda).$

Applying stationary phase: (The critical points occur only at $\varphi = 0$.)

$$\Pi_N(z,z) \sim N^m \sum_{j=0}^{\infty} a_j(z) N^{-j}.$$

Supersymmetric formula

The limit n-point correlation functions are given by $K^{\infty}_{nkm}(z^1, \dots, z^n) =$

$$\frac{C}{\det A(z)^k} \int \frac{1}{\det[I + \Lambda(z)\Omega]} d\eta \,.$$

$$\begin{aligned} A(z) &= \left(\Pi_1^{\mathbf{H}}(z^p, z^q)\right) \\ &= \left(\pi^{-m} e^{i\Im(z^p \cdot \bar{z}^q) - \frac{1}{2}|z^p - z^q|^2}\right) \end{aligned}$$

Λ is constructed from A and its first and second covariant derivatives.

$$\Omega = \left(\Omega^{pjq}_{p'j'q'}\right) = \left(p^p_{p'} p^q_{q'} \eta^{p'}_{j'} \bar{\eta}^p_j\right)$$

$(1 \le p, p' \le n, 1 \le j, j' \le k, 1 \le q, q' \le m)$, where the $\eta_j, \bar{\eta}_j$ are anti-commuting (fermionic) variables, and $d\eta = \prod_{j,p} d\bar{\eta}^p_j d\eta^p_j$. The above integral is a *Berezin integral.*

Further results

Variations of these methods give:

- The distribution and correlation of critical points.
- Mass concentration and L^p norms.
- Analysis of random holomorphic maps $x \to [S_0(x), \cdots, S_n(x)]$ to projective space and other varieties.

Further directions

We have mainly studied complex geometry. But real geometry is also rich and is more probabalistic: certain events (such as the Euler characteristic of a zero set) become random in the real case.

Moreover, our results to date mostly involve local behaviour of sections, varieties, maps. Scaling asymptotics occur on small balls. One would like to 'globalize'. Global problems in the real case include:

- Expected number of components of $f^{-1}(0)$ for a random real polynomial.
- Expected betti numbers of components.
- Expected number of nodal domains of random spherical harmonics.

In the complex case, one would like to have probabalistic existence proofs of quantitatively transversal sections, maps, Lefschetz pencils,...