Zeros of real analytic

ergodic Laplace eigenfunctions

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Purpose of talk

This is a preliminary report on work in progress on real and complex nodal hypersurfaces of ergodic eigenfunctions.

We consider the eigenvalue problem

$$\Delta\varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

for Laplacians on Riemannian manifolds (M, g) with the properties:

- (M, g) is *real analytic*;
- Its geodesic flow $G^t : S_g^* M \to S_g^* M$ is ergodic.

Problem How are nodal hypersurfaces distributed in the limit $\lambda_j \to \infty$.?

Real versus complex nodal hypersurfaces

We will consider two kinds of nodal hypersurfaces:

- The real nodal hypersurface $Z_{\varphi_j} = \{x \in M : \varphi_j(x) = 0\}$;

- The complex nodal hypersurface $Z_{\varphi_j^{\mathbb{C}}} = \{\zeta \in B^*M : \varphi_j^{\mathbb{C}}(\zeta) = 0\}$, where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of φ_j to the ball bundle B^*M for the natural complex structure adapted to g. (Definitions to come).

Motivating conjecture

We measure distribution of zeros by the probability measure defined by integrating a continuous function over the nodal hypersurfaces

$$\langle [\tilde{Z}_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}, \tag{1}$$

where $d\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional (Haussdorf) surface measure on the nodal hypersurface induced by the Riemannian metric of (M, g).

Conjecture **1** *Let* (M, g) *be a real analytic Riemannian manifold with ergodic geodesic flow, and let* $\{\varphi_j\}$ *be the density one sequence of ergodic eigenfunctions. Then,*

$$\langle [\tilde{Z}_{\varphi_j}], f \rangle \sim \{\frac{1}{Vol(M, g)} \int_M f dVol_g\} \lambda.$$

At this time of writing, even the asymptotics of the area (even in dimension two) has not been proved.

Volumes of nodal hypersurfaces

The best result to date on volumes of nodal hypersurfaces on analytic Riemannian manifolds are the following (note that our λ is the square root of the Δ-eigenvalue.)

Theorem **2** *(Donnelly-Fefferman, Inv. Math. 1988) Suppose that* (M, g) *is real analytic. Then*

$$c_1 \lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda.$$

The conjecture stated above implies an asymptotic formula $\mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \sim C_g \lambda$ in the case of ergodic geodesic flow.

Main result

Theorem **3** *Assume* (M, g) *is real analytic and that the geodesic flow of* (M, g) *is ergodic. Then*

$$\frac{1}{\lambda_j} Z_{\varphi_{\lambda_j}^{\mathbb{C}}} \to \overline{\partial}\partial|\xi|_g, \text{ weakly in } B_g^* M.$$

Here, $\overline{\partial}$ is the Cauchy-Riemann operator for the complex structure on the unit ball bundle with respect to the complex structure adapted to g. Also, $|\xi|_g^2 = \sum_{i,j} g^{ij} \xi_i \xi_j$ is the length-squared of a (co-)vector.

Definition: The adapted complex structure on $B^* M$ *is uniquely characterized by the fact that the maps* $(t, \tau) \in \mathbb{C}^+ \to B^* M$,

$$(t, \tau) \to \tau \dot{\gamma}(t), \; t \in \mathbb{R}, \; \tau \in \mathbb{R}^+$$

are holomorphic curves for any geodesic γ.

Comments

- The Kaehler structure on the cotangent bundle is $\overline{\partial}\partial|\xi|_g^2$. But the limit current is $\overline{\partial}\partial|\xi|_g$. The latter is singular along $M = \{\xi = 0\}$ and the associated volume form is not the symplectic one.

- The reason for the singularity is that the zero set is invariant under the involution $\sigma : T^*M \to T^*M$, $(x, \xi) \to (x, -\xi)$, since the eigenfunction is real valued on M. The fixed point set of σ is M and is also where zeros concentrate. By pushing this further one might be able to prove the conjecture on real zeros.

Bruhat-Whitney complexification

Theorem **4 (Bruhat-Whitney, 1959)** *Let* M *be a real analytic manifold of real dimension* n*. Then there exists a complex manifold* $M_{\mathbb{C}}$ *of complex dimension* n *and a real analytic embedding* $M \to M_{\mathbb{C}}$ *such that* M *is a totally real submanifold of* $M_{\mathbb{C}}$*. The germ of* $M_{\mathbb{C}}$ *is unique.*

Totally real means: Let $J_p : T_{\mathbb{C}}M_{\mathbb{C}} \to T_{\mathbb{C}}M_{\mathbb{C}}$ denote the complex structure on the (complexified) tangent bundle of $M_{\mathbb{C}}$. Then $J_pT_pM \cap T_pM = \{0\}$. I.e. T_pM contains to complex subspaces.

Examples: Spheres and tori

BH complexifications of spheres and tori can be identified with their full cotangent spaces:

1. S^n It is defined by $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$. Its BH complexification is the complex quadric

$S^2_{\mathbb{C}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 1\}$.

If we write $z_j = x_j + i\xi_j$, the equations become $|x|^2 - |\xi|^1 = 1, \langle x, \xi \rangle = 0$.

2. $T^n = \mathbb{R}^n/\mathbb{Z}^n$ The BH complexification is $\mathbb{C}^n/\mathbb{Z}^n = T^n \times \mathbb{R}^n \equiv T^*M$.

The complexified exponential map is:

$$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$

Analytic continuation of the wave kernel

Theorem **5** *Let* $E(t,x,y)$ *denote the kernel of* $E(t) := e^{it\sqrt{\Delta}}$*. Then for* ϵ *sufficiently small,*

- $E(t,x,y)$ *can be analytically continued to a holomorphic function in the strip* $0 \leq \Im t \leq \epsilon$*;*

- *For fixed* (x,ϵ)*,* $E(i\epsilon,x,y)$ *can be analytically continued in* x *to a holomorphic function* $E(i\epsilon,x,z)$ *with* $z \in M_\epsilon$*.*

Analytic continuation of the wave kernel

A more precise description:

Theorem **6** $E(i\epsilon, z, y) : L^2(M) \to H^2(\partial M_\epsilon)$ *is a complex Fourier integral operator of order* $-\frac{m-1}{4}$ *associated to the canonical relation*

$$\Gamma = \{(y, \eta, \exp_y(i\epsilon)\eta/|\eta|)\} \subset T^*M \times \Sigma_\epsilon.$$

Moreover,

$$E(i\epsilon) : H^{-\frac{m-1}{4}}(M) \to H^2(\partial B^*_\epsilon M)$$

is an isomorphism.

Analytic continuation of eigenfunctions

The holomorphic extension of φ_λ is obtained by applying a complex Fourier integral operator:

$$E(i\tau)\varphi_\lambda = e^{-\tau\lambda}\varphi_\lambda^{\mathbb{C}}. \tag{2}$$

This implies connections between the geodesic flow and the growth rate and zeros of $\varphi_\lambda^{\mathbb{C}}$.

Corollary **7** *Each eigenfunction φ_λ has a unique holomorphic extensions to M_ϵ satisfying*

$$\sup_{m\in M_\epsilon} |\varphi_\lambda^{\mathbb{C}}(m)| \leq C_\epsilon \lambda^{m+1} e^{\epsilon\lambda}.$$

In particular, eigenfunctions extend holomorphically to the maximal Grauert tube in the adapted complex structure.

Outline of proof of distribution of complex zeros

Theorem **8** *Assume the geodesic flow of* (M, g) *is ergodic. Then*

$$|U_\lambda|^2 = \frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial B^*_\epsilon M)}} \to 1, \text{ *weakly in* } L^1(B^*_\epsilon M).$$

(Ergodicity on the hypersurfaces implies ergodicity in the tube.)

Strong limit of logarithm

Quantum ergodicity $\log |U_j|^2$
plus pluri-subhamonicity implies:

$$\frac{1}{\lambda_j} \partial\bar{\partial} \log |U_j|^2 \to 0, \text{ weakly in } \mathcal{D}'(M_1).$$

By Poincare- Lelong:

$$[\tilde{Z}_j] = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2.$$

Since

$$\partial\bar{\partial} \log |U_j|^2 = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2 - \partial\bar{\partial} \log ||\tilde{\varphi}_j^{\mathbb{C}}||^2_{\partial M_\epsilon},$$

we find asymptotics of $[\tilde{Z}_j]$ from asymptotics of $\log |\tilde{\varphi}_j^{\mathbb{C}}|^2$.

Norm asymptotics

The final step is to prove:

Lemma **9**

$$\frac{1}{\lambda}\log||\varphi_\lambda^{\mathbb{C}\sqrt{\rho}}||_{L^2(\partial M_{\sqrt{\rho}})} \sim |\xi|_g.$$