Statistics of supersymmetric vacua in string/M theory

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Supergravity theory

An *effective supergravity theory* $\mathcal{T}$ is defined by data $(\mathcal{C}, K, W)$:

- a configuration space $\mathcal{C}$, a complex manifold with Kähler metric ω, with Kähler potential $K(z, \bar{z})$, $\omega = \partial\bar{\partial}K$.

- The associated holomorphic line bundle $\mathcal{L}$ over $\mathcal{C}$, such that $c_1(\mathcal{L}) = -\omega$.

- A hermitian metric $||W||$ and a natural holomorphic metric connection ∇ with curvature ω.

- A superpotential W, which is a (locally) holomorphic section of $\mathcal{L}$.

The potential

Physicists write a section locally as a holomorhic function W. Then the Hermitian metric is written

$$||W||^2 = e^K |W|^2;$$

and the covariant derivative of a section W is $\nabla_i W \equiv \partial_i W + (\partial_i K)W$.

The scalar potential V is the following function on $\mathcal{C}$,

$$V = e^K \left(g^{i\bar{j}} (D_i W)(\bar{D}_{\bar{j}} W^*) - 3|W|^2 \right) \tag{1}$$

where $W^*(\bar{z})$ is the complex conjugate section and

$$\bar{D}_{\bar{j}} W^* = \bar{p}_{\bar{j}} W^* + (\bar{p}_{\bar{j}} K) W^*.$$

The choice of V is dicated by supersymmetry (see Wess-Bagger).

Vacua

A *vacuum* is then a critical point $p \in \mathcal{C}$ of V. The vacua are further distinguished as follows:

A *supersymmetric* vacuum is one in which the covariant gradient $D_i W = 0$. This can be seen to imply $V' = 0$, but the converse is not true.

A *non-supersymmetric* vacuum is one in which $D_i W \neq 0$. The norm of the gradient,

(2) $$M_{susy}^4 \equiv e^K g^{i\bar{j}} D_i W D_{\bar{j}} W^*,$$

is then referred to as the *scale of supersymmetry breaking*.

Cosmological constant

The value of V at a critical point is the *cosmological constant* Λ of that vacuum. These are divided into $\Lambda = 0$, the Minkowski vacua, $\Lambda > 0$, the *de Sitter* (or dS) vacua, and $\Lambda < 0$, the *Anti-de Sitter* (or AdS) vacua. It is easy to see that supersymmetric vacua can only be Minkowski or AdS. The Minkowski vacua are simultaneous solutions of $D_i W = W = 0$; in this case $D_i W = \partial_i W$ and the existence of such vacua is independent of the Kähler potential. On the other hand, this is an overdetermined set of equations, so supersymmetric Minkowski vacua are not generic.

Stable vacua

A *stable* vacuum is one in which small fluctuations of the fields will not grow exponentially. One might think that the condition for this is for the Hessian V'' to be non-negative definite. This is correct in a Minkowski vacuum, but in curved space-time the linearized equations of motion also get a contribution from the background curvature, and in AdS the spatial boundary conditions are treated differently, changing the discussion.

For AdS_4, the correct stability condition is [**?**]

$$\partial_i \partial_j V \geq -\frac{3}{4} \Lambda g_{ij}.$$

It can be shown that this is always satisfied by supersymmetric vacua.

Physical questions about vacua

- Find the expected number of stable supersymmetric vacua, or the ratio between the expected numbers of supersymmetric and stable non-supersymmetric vacua. More generally, one wants to know which parameters control this ratio.

- Having found stable non-supersymmetric vacua, the next most basic question is to understand the cosmological constant Λ, the value of V at the minimum. Our universe at present is clearly very similar to Minkowski space-time, but according to recent observations there is a non-zero "dark energy" which can be fit by taking $\Lambda > 0$ and small. If this is actually a cosmological constant, in the distant future the universe will asymptotically approach the de Sitter space-time.

- If there are many candidate vacua, the next general question which emerges is their distribution in the configuration space. Although this is complicated in its details, one would like to know whether or not the vacua which pass the previous tests are roughly uniformly distributed with respect to the natural measure (the volume form obtained from the Kähler form).

- Finally, one would like to consider more global definitions of stability. In particular, a vacuum with $\Lambda = \Lambda_1 > 0$ can tunnel or decay to another vacuum (refs) with $\Lambda_1 > \Lambda = \Lambda_2 \geq 0$, at a rate roughly given by the exponential of the action

$$\exp - \int dz \sqrt{V(z) - g_{i\bar{j}}(z)\dot{z}^i \dot{\bar{z}}^{\bar{j}}}.$$

- Finally, there are questions which are simpler than literally counting critical points, but still relevant. Our basic example is the following. The problem of counting critical points of a section is not a holomorphic problem, because of the presence of the nonholomorphic connection ∂K. Thus, critical points come with a non-zero Morse sign sgn $\det D_i D_j W$ (here i,j are real coordinates).

  The simplest definition of "counting critical points" is to count with this sign,

$$I = \sum_{DW(p)=0} \text{sgn } \det_{i,j} D_i D_j W|_p.$$

  This was called the "supergravity index" in [**?**], and is closely related to the Morse index of the norm of the section, $e^K|W|$, but is not literally the same if W has zeroes. It is invariant under appropriate continuous deformations of the section and metric.

Since this quantity is easier to compute than the literal number of critical points, it is of interest to know in what circumstances and how well it estimates the number of critical points, of the two types we discussed.

- Similar comments might be relevant for counting critical points of V. Of course, at a minimum $V'' > 0$ this sign is positive, but as we mentioned the AdS stability condition allows for some negative eigenvalues.

- To summarize, we would like to know the expected distribution of critical points of W (supersymmetric) and of V (nonsupersymmetric if $DW \neq 0$), the expected distribution of $||W||^2$ at the critical points in the supersymmetric case, and the expected joint distribution of M_{susy}^4 and V in the nonsupersymmetric case.

Connection

We start with a choice of a d complex dimensional Calabi-Yau manifold M, *i.e.* a complex Kähler manifold with $c_1(M) = 0$. The topological data of M of primary relevance for us is its middle cohomology $H^d(M, \mathbb{Z})$. This carries an intersection form

$$\eta^{ij} = (\Sigma^i, \Sigma^j),$$

symmetric (antisymmetric) for d even (odd). This form is preserved by a linear action of $SO(b_+(M), b_-(M); \mathbb{Z})$ for d even, or $Sp(b_d(M); \mathbb{Z})$ for d odd. We denote this group as $G(\eta)$.

By Yau's theorem, such a manifold admits Ricci-flat Kähler metrics, which come in a family parameterized locally by a choice of complex structure on $\mathcal{M}$ and a choice of Kähler class. One can show that the moduli space of complex structures $\mathcal{M}_c(M)$ is an (open) manifold of dimension $b_{d-1,1}(M)$.

One can also think of the complex structure as locally determined by a choice of nonvanishing holomorphic d-form Ω, or by its cohomology class $[\Omega] \in H^d(M, \mathbb{C})$. In some cases, this has been extended to a global Torelli theorem (ref Voisin). Conversely, Ω is uniquely determined by the complex structure up to its overall scale.

Let $\mathcal{M}_c(M)$ be the moduli space of complex structures; the association we just described gives a local embedding

$$\mathcal{M}_c(M) \to \mathbb{CP}(H^d(M, \mathbb{C})) \equiv \mathbb{CP}(V),$$

the "period map." Pairing this with a basis for $H_d(M, \mathbb{Z})$ gives a basis for the periods, which we can denote Π_i.

The moduli space of complex structures carries a natural Kähler metric, given by the Kähler potential

$$e^{-K} = \int_M \bar{\Omega} \wedge \Omega.$$

Then, the periods are locally sections of the line bundle $\mathcal{L}$ associated to this metric. They are not global sections over $\mathcal{M}_c(M)$, because of monodromy: $\mathcal{M}_c(M)$ is singular in codimension one (for a hypersurface $f = 0$, this is on the discriminant locus) and a closed loop encircling this singularity induces a Picard-Lefschetz monodromy on $H_d(M, \mathbb{Z})$. We can regard the periods either as sections of $V \otimes \mathcal{L}$ where V is a flat $G(\eta)$ vector bundle, or simply as sections of $\mathcal{L}$ over an appropriate covering space of $\mathcal{M}_c(M)$.

We now have the ingredients required to define the effective supergravity theories (in the flux sector) for the string compactifications of interest. They each have $\mathcal{C} = \mathcal{M}_c(M)$ and $K = K_M$. Finally, for each choice of the "flux" $N \in H^d(M, \mathbb{Z})$, the theory $\mathcal{T}(N)$ has

$$W = (\Omega, N). \tag{3}$$

The simplest example of the structure above is to take M to be a complex torus $\mathbb{C}^d/L$ with L a $2d$-dimensional lattice. We then have $\mathcal{M}_c(M)$ the Siegel upper half plane, with coordinate τ a $d \times d$ complex matrix with positive definite imaginary part. The Kähler potential is $K = -\log\det \Im\tau$, and a $\mathbb{Z}$-basis for the periods can be taken to be the minors of τ of all ranks: $1, \tau_{ij}, \ldots, (\text{cof } \tau)^{ij}, \det\tau$.

The case $d = 3$ arises in heterotic string compactification on T^6.

F theory compactification on $T^8/\mathbb{Z}_2$ leads to a problem which is can be obtained from the case $d = 4$ of what we discussed, by restriction to the subspace of $\mathcal{M}_c$ in which τ is a direct sum of a rank 1 and a rank 3 matrix (in other words, the torus must admit an elliptic fibration).

The physical problem of interest is then more or less the one we discussed, in which the ensemble of sections is the set of periods of $H_d(M, \mathbb{Z})$. In the case of even d, one takes the sum with fixed self-intersection (N, N).

The same problems can be discussed for more general Calabi-Yau manifolds by using the theory of deformation of Hodge structure. The periods satisfy Picard-Fuchs equations, which are generalized hypergeometric equations. Their explicit solutions are known in many cases.

Quite a lot is known about the possible singular behaviors of these periods, which are expected to have an important effect on the distributions we are discussing. One common case is for a pair of conjugate periods (under the intersection form) to behave as z and $z \log z$. Critical points of sections

$$W = Mz + Nz \log z$$

then can have exponentially small values for Λ and M^4_{susy}, giving a mechanism to realize the small numbers hoped for in the previous discussion.

In physics, the $d = 3$ case is extensively studied, and leads to a structure called "special geometry". This essentially states that the image of the period map is a complex Lagrangian submanifold of $H^3(M, \mathbb{C})$, and thus is determined by a single holomorphic function (the "prepotential"). This is extremely useful in the applications and leads to a fairly developed theory for finding these critical points.

Mathematical discussion

We begin with a precise definition of critical points of a holomorhic section $s \in H^0(M, L)$. It depends on a connection ∇ on $L =$ the Chern connection $\nabla = \nabla_h$ which preserves the metric h and satisfies $\nabla'' s = 0$ for any holomorphic section s. Here, $\nabla = \nabla' + \nabla''$ is the splitting of the connection into its $L \otimes T^{*1,0}$ resp. $L \otimes T^{*0,1}$ parts. We denote by Θ_h the curvature of h.

Definition: *Let $(L, h) \to M$ be a Hermitian holomorphic line bundle over a complex manifold M, and let $\nabla = \nabla_h$ be its Chern connection. A critical point of a holomorphic section $s \in H^0(M, L)$ is defined to be a point $z \in M$ where $\nabla s(z) = 0$, or equivalently, $\nabla' s(z) = 0$. We denote the set of critical points of s with respect to the Chern connection ∇ of a Hermitian metric h by $Crit(s, h)$.*

Critical points depend on the metric

The set of critical points $Crit(s, h)$ of s, and even its number $\#Crit(s, h)$, depends on ∇_h or equivalently on the metric h.

Recall that the connection 1-form α of ∇_h is given in a local frame e by

$$\nabla e = \alpha \otimes e, \quad \alpha = \partial \log h.$$

In the local frame, $s = fe$ where f is a local holomorphic function and then $\nabla s = (\partial f + \alpha f) \otimes e$. Thus the critical point equation in local coordinates reads:

$$(4) \qquad \partial f = -f\alpha \iff -\partial \log f = \partial \log h.$$

This is a real C^∞ equation, not a holomorphic one.

Another definition of critical point

An essentially equivalent definition which only makes use of real functions is to define a critical point as a point w where

$$d|s(w)|_h^2 = 0. \tag{5}$$

Since

$$d|s(w)|_h^2 = 0 \iff 0 = \partial|s(w)|_h^2 = h_w(\nabla' s(w), s(w))$$

it follows that $\nabla' s(w) = 0$ as long as $s(w) \neq 0$. So this notion of critical point is the union of the zeros and critical points. Another essentially equivalent critical point equation which puts the zero set of s at $-\infty$ is

$$d \log |s(w)|_h^2 = 0. \tag{6}$$

This is the equation studied by Bott in his Morse theoretic proof of the Lefschetz hyperplane theorem, which is based on the observation that the Morse index of any such critical point is at least m.

Statistics of sections

Our purpose is to study the statistics of critical points of random sections with respect to complex Gaussian measures γ on the space $H^0(M, L)$ of holomorphic sections, or more generally on subspaces $\mathcal{S} \subset H^0(M, L)$. The Gaussian measure $\gamma = \gamma_{\mathcal{S},h,V}$ is induced by the inner product

$$(7) \qquad \langle s_1, s_2 \rangle = \int_M h(s_1(z), s_2(z)) dV(z)$$

on $H^0(M, L)$, where dV is a fixed volume form on M. When the metric and volume form and subspace are fixed we denote the Gaussian measure more simply by γ. By definition,

$$(8) \qquad d\gamma(s) = \frac{1}{\pi^d} e^{-\|c\|^2} dc \,, \qquad s = \sum_{j=1}^{d} c_j e_j,$$

where dc is Lebesgue measure and $\{e_j\}$ is an orthonormal basis basis for $\mathcal{S}$ relative to h, dV. We also denote the expected value of a random variable X on with respect to γ by $\mathbf{E}_\gamma$ or simply by $\mathbf{E}$ if the Gaussian measure γ is understood.

Density of critical points

The critical point density of a fixed section s with respect to h (or ∇_h) is the measure

(9) $$C_s^h := \sum_{z \in Crit(s,h)} \delta_z,$$

where δ_z is the Dirac point mass at z.

Definition: The expected density of critical points of $s \in \mathcal{S} \subset H^0(M, L)$ with respect to h and V is defined by

(10) $$K^{\text{crit}}_{\mathcal{S},h,V}(z)\, dV(z) = \mathbf{E}_{\gamma_{\mathcal{S},h,V}} C_s^h\,,$$

i.e.,

(11)

$$\int_M \varphi(z) K^{\text{crit}}_{\mathcal{S},h,V}(z)\, dV(z) = \int_{\mathcal{S}} \left[\sum_{z: \nabla_h s(z)=0} \varphi(z) \right] d\gamma_{\mathcal{S},h,V}($$

Expected number of critical points

Definition:

We further define the expected number of critical points by

$$\mathcal{N}^{crit}(\mathcal{S}, h, V) = \int_M K^{\text{crit}}_{\mathcal{S},h,V}(z)\, dV(z). \tag{12}$$

To state the result, it is most convenient to introduce a local frame (non-vanishing holomorphic section) e_L for L over an open set $U \subset M$ containing z, and local coordinates $(z_1, \ldots, z_m)$ on U in which the local connection form has zero pure holomorphic and anti-holomorphic derivatives. We denote the curvature $(1,1)$-form of ∇ in these coordinates by $\Theta = \sum_{i,\bar{j}=1}^m \Theta_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and refer to the $m \times m$

matrix $[\Theta_{i,\bar{j}}]$ as the curvature matrix of ∇. We also denote by $\mathrm{Sym}(m,\mathbb{C})$ the space of complex $m \times m$ symmetric matrices.

Theorem **1** *Let $(\mathcal{S}, \nabla, \gamma)$ denote as above a finite-dimensional subspace $\mathcal{S} \subset H^0(M, L)$ of holomorphic sections of a holomorphic line bundle $(L, \nabla) \to M$ with connection on an m-dimensional complex manifold, together with a Gaussian measure γ on $\mathcal{S}$. Assume that $\mathcal{S}$ satisfies the 2-jet spanning property. Given a local frame e_L and adapted coordinates $z = (z_1, \ldots, z_m)$, there exist positive-definite Hermitian matrices*

$$A(z) : \mathbb{C}^m \to \mathbb{C}^m \,, \quad \Lambda(z) : \mathrm{Sym}(m,\mathbb{C}) \oplus \mathbb{C} \to Sym(m,\mathbb{C})$$

depending only on z, ∇ and $\Pi_{\mathcal{S}}$ such that the expected density of critical points with respect to Lebesgue measure dz in local coordinates is *given by*

$$\mathcal{K}^{\mathsf{crit}}_{\mathcal{S},\gamma,\nabla}(z) = \frac{1}{\pi^{\binom{m+2}{2}} \det A(z) \det \Lambda(z)}$$
$$\times \int_{\mathbb{C}} \int_{\mathsf{Sym}(m,\mathbb{C})} \left| \det \begin{pmatrix} H' & x\,\Theta(z) \\ \bar{x}\,\bar{\Theta}(z) & \bar{H}' \end{pmatrix} \right|$$
$$e^{-\left\langle \Lambda(z)^{-1}(H' \oplus x), H' \oplus x \right\rangle} \, dH' \, dx \,,$$

where $\Theta(z) = [\Theta_{i\bar{j}}]$ *is the curvature matrix of* ∇ *in the coordinates* $(z_1, \ldots, z_m)$. *The expected distribution of zeros* $\mathcal{K}^{\mathsf{crit}}_{\mathcal{S},\gamma,\nabla}(z)dz$ *is independent of the choice of frame and coordinates.*

Density of critical points

The formulae for $A(z)$ and $\Lambda(z)$ in adapted local normal coordinates are as follows. Let $F_{\mathcal{S}}(z,w)$ be the local expression for $\Pi_{\mathcal{S}}(z,w)$ in the frame e_L. Then $A(z)$ is () and $\Lambda = C - B^* A^{-1} B$, where

$$A = \left(\frac{\partial^2}{\partial z_j \partial \bar{w}_{j'}} F_{\mathcal{S}}(z,w)|_{z=w} \right),$$
$$B = \left[\left(\frac{\partial^3}{\partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}}) F_{\mathcal{S}}|_{z=w} \right) \quad \left((\frac{\partial}{\partial z_j} F_{\mathcal{S}}|_{z=w} \right) \right],$$
$$C = \left[\begin{array}{cc} \left(\frac{\partial^4}{\partial z_q \partial z_j \partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}}|_{z=w} \right) & \left(\frac{\partial^2}{\partial z_j \partial z_q} F_{\mathcal{S}} \right) \\ \left(\frac{\partial^2}{\partial \bar{w}_{q'} \partial \bar{w}_{j'}} F_{\mathcal{S}} \right)|_{z=w} & F_{\mathcal{S}}(z,z) \end{array} \right],$$

$$1 \leq j \leq m\,, 1 \leq j \leq q \leq m\,, 1 \leq j' \leq q' \leq$$

In the above, A, B, C are $m \times m$, $m \times n$, $n \times n$ matrices, respectively, where $n = \frac{1}{2}(m^2 + m + 2)$. T

Expected number of critical points for positive line bundles We assume that $\omega = \frac{i}{2}\Theta_h$ so that $c_1(L) = [\frac{1}{\pi}\omega]$ or $-[\frac{1}{\pi}\omega]$. More precisely, the Kähler form is given by

$$\omega = \pm\frac{i}{2}\Theta_h = \pm\frac{i}{2}\partial\bar{\partial}K, \qquad K = -\log h.$$

The volume form is then assumed to be

$$dV = \frac{\omega^m}{m!}$$

(and thus the total volume of M is $\frac{\pi^m}{m!}\,|c_1(L)^m|$). Given one such metric h_0 on L, the other metrics have the form $h_\varphi = e^\varphi h$ and $\Theta_h = \Theta_{h_0} - \partial\bar{\partial}\varphi$, with $\varphi \in C^\infty(M)$. As a consequence of Lemma 1, we obtain the following integral formula for K^{crit} in these cases:

Lemma **1** *Let* $(L, h) \to M$ *denote a positive or negative holomorphic line bundle, and suppose that* $H^0(M, L)$ *contains a finite-dimensional subspace* $\mathcal{S}$ *with the* 1*-jet spanning property. Give*

*M the volume form $dV = \frac{1}{m!}\left(\pm\frac{i}{2}\Theta_h\right)^m$ induced from the curvature of L. Then there exist positive Hermitian matrices $A(z), \Lambda(z)$ (cf. (**??**)) depending on h and z such that*

$$K^{\text{crit}}_{h,\mathcal{S}}(z) = \frac{1}{\pi^{\binom{m+2}{2}}}$$

$$\det A \det \Lambda \int_{\text{Sym}(m,\mathbb{C})\times\mathbb{C}} \left|\det(H'H'^{*} - |x|^2 I)\right.$$

$$\left| e^{-\langle \Lambda(z)^{-1}(H',x),(H',x)\rangle} \, dH' \, dx \, . \right.$$

Here, $H' \in Sym(m,\mathbb{C})$ is a complex symmetric matrix, and the matrix Λ is a Hermitian operator on the complex vector space $Sym(m,\mathbb{C})\times\mathbb{C}$.

Universal limit theorem

Our main result gives an associated asymptotic expansion for $K_N^{\text{crit}}(z)$:

Theorem **2** *For any positive Hermitian line bundle $(L,h) \to (M,\omega)$ over any compact Kähler manifold, the critical point density has an asymptotic expansion of the form*

$$N^{-m} K_N^{\text{crit}}(z) \sim \Gamma_m^{\text{crit}} + a_1(z) N^{-1} + a_2(z) N^{-2} + \cdots ,$$

where Γ_m^{crit} is a universal constant depending only on the dimension m of M. Hence the expected total number of critical points on M is

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!} \Gamma_m^{\text{crit}}\, c_1(L)^m N^m + O(N^{m-1}) .$$

The leading constant in the expansion is given by the integral formula

$$\Gamma_m^{\text{crit}} = \left(2\pi^{\frac{m+3}{2}}\right)^{-m} \int_0^{+\infty} \int_{\text{Sym}(m,\mathbb{C})} |\det(SS^* - tI)|$$

$$e^{-\frac{1}{2}\|S\|_{\text{HS}}^2 - t}\, dS\, dt ,$$

Riemann surfaces The universality of the principal term may seem rather surprising in view of the fact that the number of critical points depends on the metric. In the case of Riemann surfaces, we can explicitly evaluate the leading coefficient:

Corollary **3** *For the case where M is a Riemann surface, we have $\Gamma_1^{\text{crit}} = \frac{5}{3\pi}$, and hence the expected number of critical points is $\mathcal{N}(h^N) = \frac{5}{3}c_1(L)N + O(\sqrt{N})$. The expected number of saddle points is $\frac{4}{3}N$ while the expected number of local maxima is $\frac{1}{3}N$.*

There are $\sim N$ critical points of a polynomial of degree N in the classical sense, all of which are saddle points.

Metrics with minimal number of critical points

It is natural to wonder which hermitian metrics produce the minimal expected number of critical points. To put the question precisely, let $L \to (M, [\omega])$ be a holomorphic line bundle over any compact Kähler manifold with $c_1(L) = [\omega]$, and consider the space of Hermitian metrics h on L for which the curvature form is a positive $(1,1)$ form:

$$P(M, [\omega]) = \{h : \frac{i}{2}\Theta(h) \text{ is a positive } (1,1)\text{– form }\}.$$

Definition: We say that $h \in P([\omega])$ is asymptotically minimal if

(13)

$$\exists N_0 : \forall N \geq N_0, \ \mathcal{N}(h^N) \leq \mathcal{N}(h_1^N), \quad \forall h_1 \in P([\omega]).$$

Density of critical points on Riemann surfaces

We measure the critical point density with respect to the volume form $\pm\frac{i}{2}\Theta_h$. Put:

$$Q = \begin{pmatrix} -1 & 0 \\ & \\ 0 & 1 \end{pmatrix}$$

and denote the eigenvalues of $\Lambda(z)Q$ by μ_1, μ_2. We observe that μ_1, μ_2 have opposite signs since $\det Q\Lambda = -\det\Lambda < 0$. Let $\mu_2 < 0 < \mu_1$.

Theorem **4** *let $(L,h) \to M$ be a positive or negative Hermitian line bundle on a (possibly non-compact) Riemann surface M with volume form $dV = \pm\frac{i}{2}\Theta_h$, and let $\mathcal{S}$ be a finite-dimensional subset of $H^0(M,L)$ with the 1-jet spanning property. Then:*

$$K_h^{\text{crit}}(z) = \frac{1}{\pi A(z)} \frac{\mu_1^2 + \mu_2^2}{|\mu_1| + |\mu_2|} = \frac{1}{\pi A(z)} \frac{Tr\,\Lambda^2}{Tr|\Lambda^{\frac{1}{2}}Q\Lambda^{\frac{1}{2}}|}\,,$$

where $\mu_1,\ \mu_2$ are as above.

Exact formula on $\mathbb{CP}^1$

Theorem **5** *The expected number of critical points of a random section* $s_N \in H^0(\mathbb{CP}^1, \mathcal{O}(N))$ *(with respect to the Gaussian measure on* $H^0(\mathbb{CP}^1, \mathcal{O}(N$ *induced from the Fubini-Study metrics on* $\mathcal{O}(N)$ *and* $\mathbb{CP}^1$*) is*

$$\frac{5N^2 - 8N + 4}{3N - 2} = \frac{5}{3} N - \frac{14}{9} + \frac{8}{27} N^{-1} \cdots .$$

Asymptotically minimal number of critical points

Integrating the density of critical points, we find that the expected total number of critical points has the expansion

$$\mathcal{N}(h^N) = \frac{\pi^m}{m!}\Gamma_m^{\mathrm{crit}}\; c_1(L)^m\, N^m$$

$$+ \int_M \rho dV_\omega N^{m-1}$$

$$+ C_m \int_M \rho^2 dV_\Omega N^{m-2} + O(N^{m-3})\,.$$

The leading order term is universal, so the metric with asymptotically minimal $\mathcal{N}(h^N)$ is the one with minimal $\int_M \rho^2 dV_\omega$, at least assuming that it is non-universal.