Kähler quantization and the homogeneous complex Monge Ampère equation on a toric variety

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Overview

1. This talk is about geodesics in the infinite dimensional symmetric space of Kähler metrics in a fixed Kähler class á la Donaldson-Semmes.

2. These geodesics are solutions of a homogeneous complex Monge-Ampère equation in 'space-time'. One would like to know existence, regularity...

3. Phong-Sturm proved that one can construct weak solutions by special polynomial approximations. The purpose of this talk is to study the geodesics and the polynomial approximation on a toric variety.

Space $\mathcal{H}$ of Kähler metrics in the class $[\omega]$

Let $L \to M$ be an ample holomorphic line bundle over a compact Kähler manifold (M, ω_0) with $\frac{1}{2\pi}\omega_0 \in H^{(1,1)}(M, \mathbb{Z})$ and with $c_1(L) = [\omega_0]$, the class of ω_0. Put $m = \dim M$.

Let h_0 be the unique Hermitian metric on L with curvature $(1,1)$ form ω_0. Any hermitian metric h with curvature in $[\omega_0]$ may be written $h_\varphi = e^{-\varphi} h_0$, with φ in the space

$$\mathcal{H} = \{\varphi \in C^\infty(M) : \omega_\varphi = \omega_0 + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi > 0\}.$$

$\mathcal{H}$ is a symmetric space

We endow $\mathcal{H}$ with a Riemannian metric:

Identify the tangent space $T_\varphi\mathcal{H}$ at $\varphi \in \mathcal{H}$ with $C^\infty(M)$, let $\psi \in T_\varphi\mathcal{H} \simeq C^\infty(M)$ and define

$$||\psi||_\varphi^2 \;=\; \int_M |\psi|^2 \, \omega_{\varphi^k} \quad .$$

With this Riemannian metric, $\mathcal{H}$ is an infinite dimensional negatively curved symmetric space (Mabuchi, Semmes, Donaldson).

Formally, $\mathcal{H} = \mathcal{G}_\mathbb{C}\backslash\mathcal{G}$ where $\mathcal{G}$ is the group of symplectic diffeomorphisms of (M, ω_0).

Geodesics of $\mathcal{H}$

The energy of a path φ_t of metrics is then energy functional

$$E = \int_0^1 \int_M \dot{\varphi}_t^2 \omega_{\varphi_t}^m dt.$$

The Euler Lagrange equations are

$$\ddot{\varphi} - |\partial \dot{\varphi}|^2_{\omega_\varphi} = 0.$$

This equation may be interpreted as a degenerate complex Monge-Ampère equation on $A \times M$ where $A = \{w \in \mathbb{C} : 1 \leq |w| \leq e\}$ is an annulus. Let $\Phi(z, w) = \varphi_{\log |w|}(z)$. Then

$$(\omega_0 + \frac{i}{2}\partial\bar{\partial}\Phi)^{m+1} = 0, \text{ on } A \times M.$$

Why study geodesics?

The geometry of $\mathcal{H}$ is relevant to the study of the relations between

1. Stability of the polarized Kähler manifold (M, ω_0, L).

2. Existence of canonical metrics in $[\omega_0]$, i.e. metrics of constant scalar curvature.

The first is an algebro-geometric notion, the second is transcendental (differential geometric). Donaldson and others are developing transcendental analogues of GIT to relate (1) and (2). Geodesics are the transcendental analogues of 1 PS (one-parameter subgroups), i.e. they are formally the 1 PS of $\mathcal{G}_{\mathbb{C}}$ (which does not exist).

Main problems about geodesics

- Existence/Uniqueness: does there exist a unique geodesic between two given metrics φ_0, φ_1 in $\mathcal{H}$? For which initial tangent vectors $(\varphi_0, \dot{\varphi}_0)$ does there exist an infinite geodesic ray φ_t with the given initial tangent vector?

- Regularity: How smooth are the solutions of the endpoint and/or initial value problem?

- Behavior of functionals (e.g. Mabuchi K-energy) along an infinite geodesic ray.

Background results

The endpoint problem is a Dirichlet problem for the homogeneous complex MA equation on $A \times M$. When the boundary data are C^∞, then the solution is at least $C^{1,1}$. (X.X. Cheng, using work of B. Guan and J. Spruck).

The initial value problem is the MA equation in a punctured disc. There are no general results.

Donaldson observed that one can formally solve the initial value problem as follows: Let $\exp tH_{\dot{\varphi}_0}$ be the Hamiltonian flow w.r.t. ω_0 of $\dot{\varphi}_0$. Complexify t to it. Then $(\exp itH_{\dot{\varphi}_0})^*\omega_0 - \omega_0 = i\partial\bar{\partial}\varphi_t$.

Phong-Sturm approximations

Phong-Sturm construct geodesic segments and infinite rays as limits of of 1 PS geodesics in certain symmetric spaces $\mathcal{B}_k \subset \mathcal{H}$, known as spaces of Bergman (or Fubini-Study) metrics.

The main idea is Monge-Ampère geodesics are 1 PS of $\mathcal{G}_{\mathbb{C}}$. So they should be approximated by 1 PS of the finite dimensional symmetric spaces $\mathcal{B}_k \subset \mathcal{H}$.

Bergman metrics

Let $d_k+1 = \dim H^0(M, L^k)$ and let $\mathcal{B}H^0(M, L^k)$ denote the manifold of all bases $\underline{s} = \{s_0, \ldots, s_{d_k}\}$ of $H^0(M, L^k)$. Given a basis, we define the Kodaira embedding

$$\Phi_{\underline{s}} : M \to \mathbb{CP}^{d_k}, \;\; z \to [s_0(z), \ldots, s_{d_k}(z)].$$

A Bergman (hermitian) metric of height k is a metric of the form

$$(1) \quad h_{\underline{s}} := (\Phi_{\underline{s}}^* h_{FS})^{1/k} = \frac{h_0}{\left(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}\right)^{1/k}},$$

where h_{FS} is the Fubini-Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. We then define

$$(2) \qquad \mathcal{B}_k = \{h_{\underline{s}}, \;\; \underline{s} \in \mathcal{B}H^0(M, L^k)\}.$$

Hilbert maps from $\mathcal{H} \to \mathcal{B}_k$

Basic idea of Yau, Tian...$\mathcal{B}_k$ is a very close approximation to $\mathcal{H}_k$. There is a specific correspondence

$$Hilb_k : \mathcal{H} \to \mathcal{B}_k, \;\; h \to \; h(k) = (\Phi_{S_k}^* h_{FS})^{1/k},$$

$S_k =$ an orthonormal basis of $H^0(M, L^k)$ for h.

The metric $h(k)$ is independent of the choice of orthonormal basis.

Then $h(k) \to h$ in C^∞ and has a complete asymptotic expansion in k^{-1}. (Tian-Yau-Z-(Catlin); Boutet de Monvel-Sjöstrand parametrix).

Bergman Kähler potentials

We defined $\mathcal{H}$ is the space of Kähler potentials of Kähler metrics in the fixed class. The Kähler potential (relative to h_0) corresponding to $h_{\underline{s}}$ is

$$\varphi_{\underline{s}}(z) = \frac{1}{k} \log \sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k}. \tag{3}$$

Bergman geodesics

We note that $\mathcal{B}_k = GL(d_k+1,\mathbb{C})/U(d_k+1)$ is a symmetric space, since $GL(d_k+1,\mathbb{C})$ acts transitively on the set of bases, while $\Phi_{\underline{s}}^* h_{FS}$ is unchanged if we replace the basis $\underline{s}$ by a unitary change of basis.

Geodesics in $\mathcal{B}_k$ = 1 PS (one-parameter subgroups) e^{tA} of $GL(d_k,\mathbb{C})$. Given two endpoint bases $\underline{\widehat{s}}^{(0)}, \underline{\widehat{s}}^{(1)}$ we may assume the change of basis matrix is diagonal and write $A = Diag(\lambda_j)$ so that the 1PS geodesic between the endpoint Bergman metrics is

$$\varphi_k(t;z) \;=\; \frac{1}{k}\log\left(\sum_{j=0}^{N} e^{2\lambda_j t}|\widehat{s}_j^{(0)}(z)|^2_{h_0^k}\right).$$

Phong-Sturm problem: Convergence of Bergman space geodesics to Monge Ampere geodesics

Let $\varphi_0, \varphi_1 \in \mathcal{H}$ and let φ_t be the Monge-Ampere geodesic from φ_0 to φ_1.

Let $\varphi_0(k) = Hilb_k(\varphi_0), \varphi_1(k) = Hilb_k(\varphi_1)$ be the Bergman metrics of level k

Let $\varphi_k(t)$ be the Bergman geodesic from $\varphi_0(k)$ to $\varphi_1(k)$.

Problem Show that $\varphi_k(t) \to \varphi_t$ in a good sense as $k \to \infty$?

Results of D.H. Phong- J. Sturm on geodesic segments

Theorem **1** *Let* $h_t = e^{-\varphi_t} h_0$ *be the unique* $C^{1,1}$ *metric joining* h_0 *to* h_1*. Then,*

$$\varphi_t = \lim_{\ell \to \infty} \{\sup_{k \geq \ell} \varphi_k(t)\}^* \text{ uniformly as } \ell \to \infty$$

where for $u : X \times [0,1] \to \mathbb{R}$

$$u^*(z_0) = \lim_{\epsilon \to 0} \sup_{|z - z_0| < \epsilon} u(z)$$

is the upper envelope of u*.*

Further, $\varphi_k(t) = \lim_{\ell \to \infty}\{\sup_{k \geq \ell} \varphi_k(t)\}$ almost everywhere.

Results of D.H. Phong- J. Sturm on geodesic rays and test configurations

The most important geodesics are infinite geodesic rays. The only known construction is Phong-Sturms's construction using test configurations in the sense of Donaldson.

Test configurations are special 1PS degenerations. We define them later for toric varieties, where they are elementary.

Phong-Sturm defined TC geodesic rays as limits of certain Bergman geodesic rays. They proved that the limits are weak solutions of MA. Question: what are these solutions? how regular? what is weak?

Convergence problem on a toric Kähler manifold

For the rest of this talk, we assume (M, ω) is a toric Kähler manifold and $L \to M$ is the toric line bundle.

Toric variety: a compactification of $(\mathbb{C}^*)^m$ such that $(\mathbb{C}^*)^m$ acts holomorphicaly on M with an open orbit.

Let $\mathbf{T}^m$ be the underlying real torus. Let

$$\mu_0 : M \to P$$

be the moment map wrt ω_0; we assume P is a Delzant polytope.

Define the toric hermitian metrics in a fixed Kähler class by

$$\mathcal{H}_{\mathbf{T}^m} = \{\omega \in \mathcal{H} : \omega \text{ is invariant under } \mathbf{T}^m\}.$$

Monge Ampère on a toric variety is linearized by the Legendre transform

Following Guillemin and Abreu, we let φ denote the full Kähler potential of $\omega \in \mathcal{H}_{\mathbf{T}^m}$ in the open orbit. It is a functional only of the variables $|z_j|^2 = e^{\rho_j}$. The moment map for ω_φ equals $\nabla_\rho \varphi(\rho)$.

The symplectic potential dual to φ is its Legendre transform:

$$u(x) = \sup_\rho \left(\langle x, \rho \rangle - \varphi(\rho) \right).$$

The curve of symplectic potentials corresponding to a MA geodesic φ_t equals $u_0 + t(u_1 - u_0)$ when the endpoint potentials are $u_0 u_1$.

Convergence results on a toric variety:endpoint problem

The question is whether the Phong-Sturm Bergman endpoint geodesics converge to the MA geodesics. The answer is...

Theorem **2** *(Song-Z, 2007) Let $L \to M$ be a very ample toric line bundle over a smooth compact toric variety M. Let $\mathcal{H}_T$ denote the space of toric Hermitian metrics on L. Let $h_0, h_1 \in \mathcal{H}_T$ and let h_t be the Monge-Ampère geodesic between them. Let $h_k(t)$ be the Bergman geodesic between $Hilb_k(h_0)$ and $Hilb_k(h_1)$ in $\mathcal{B}_k$. Let $h_k(t) = e^{-\varphi_k(t,z)} h_0$ and let $h_t = e^{-\varphi_t(z)} h_0$. Then*

$$\lim_{k \to \infty} \varphi_k(t, z) = \varphi_t(z)$$

in $C^2(\mathbb{R} \times M)$.

Convergence results on a toric variety: test configuration initial value problem

A toric test configuration is defined by a piecewise linear convex function f on P with rational coefficients. Pick $R \in \mathbb{N}$ larger than $\max f$ and think of the graph of $R - f(x)$ as a roof over P, defining a new polytope Q of one higher dimension. As one moves from bottom P to top $(R - f)$ one degenerates the toric variety. Phong-Sturm construct an infinite ray from it:

Theorem **3** *(Song-Z, 2007): Let $L \to X$ be a very ample toric line bundle over a toric Kähler manifold. Let $h_0 \in \mathcal{H}_{\mathbf{T}^m}$ and let T a test configuration. Then the Phong-Sturm rays $\psi_k(t; z)$ converge in C^1 to a $C^{1,1}$ geodesic ray $\psi_t(z)$ in $\mathcal{H}_{\mathbf{T}^m}$. It is not C^2 and ω_{ψ_t} has null directions on certain open sets.*

Sketch of proofs

The ingredients are:

- Explicit formulae for the Bergman geodesic rays, both in the endpoint and test configuration cases.

- They are sums over lattice points in P. But very nonstandard ones with exponentially growing/decaying coefficients in k.

- We use a mixture of Bergman kernel asymptotics, large deviations methods, and ad hoc boundary estimates to prove convergence. Usual microlocal methods do not work.

Explicit formula for geodesic segments in $\mathcal{B}_k$

The $\mathcal{B}_k$-geodesic segments between $Hilb_k(h_0)$ and $Hilb_k(h_1)$ are given by

$$\varphi_k(t,z) = \frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{Q^k_{h_0}(\alpha)}{Q_{h^k}(\alpha)} \right)^t \frac{||s_\alpha(z)||^2_{h_0^k}}{||s_\alpha||^2_{h_0^k}}.$$

Here, $\{s_\alpha\}$ are the *monomials* of degree k = joint eigenfunctions of the torus action on $H^0(M, L^k)$. The joint eigenvalues $\{\alpha\}$ run over lattice points in the polytope P corresponding to M = the image of M under a moment map for the Hamiltonian $\mathbf{T}^m$ action. $Q_{h^k}(\alpha) = ||s_\alpha||^2_{h^k}$.

Norms of monomials in different Hermitian metrics

The L^2-norms of the monomials $\chi_\alpha(z) = z^\alpha$ in the inner product on $H^0(M, L^k)$ determined by the hermitian metric h are

$$Q_{h^k}(\alpha) = ||s_\alpha||^2_{h^k} := \int_{\mathbb{C}^m} |z^\alpha|^2 e^{-k\varphi(z)} \omega_\varphi^m / m!$$

Symplectic potential formula for norming constants: push the integral forward to the polytope P under μ_φ:

$$Q_{h^k}(\alpha) = \int_P e^{-k\left(u_\varphi(x) + \langle \frac{\alpha}{k} - x, \nabla u_\varphi(x) \rangle\right)} dx,$$

Family of probability measures

$$\text{(4)} \qquad \mu_k^z = \frac{1}{\Pi_{h_0^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h_0^k}}{||s_\alpha||^2_{h_0^k}} \, \delta_{\frac{\alpha}{kd}},$$

where

$$\Pi_{h_0^k}(z,z) = \sum_{\alpha \in kP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|^2_{h_0^k}}{||s_\alpha||^2_{h_0^k}}$$

is the contracted Szegö kernel on the diagonal (or density of states);

Large deviations principle

Theorem **4** *For any $z \in M$, the probability measures μ_k^z satisfy a uniform Laplace large deviations principle with rate k and with convex rate functions $I^z \geq 0$ on P. defined as follows:*

- *If $z \in M^0$, the open orbit, then $I^z(x) = u_0(x) - \langle x, \log|z| \rangle + \varphi_{P^o}(z)$, where φ_{P^o} is the canonical Kähler potential of the open orbit and u_0 is its Legendre transform, the symplectic potential;*

- *When $z \in \mu_0^{-1}(F)$ for some face F of ∂P, then $I^z(x)$ restricted to $x \in F$ is a restricted version. On complement of $\bar{F}$ it is defined to be $+\infty$.*

Varadhan's Lemma

Varadhan's Lemma *Let $d\mu_k$ be probability measures on X which satisfy the LDP with rate k and rate function I on X. Let F be a continuous function on X which is bounded from above. Then*

$$\lim_{k\to\infty} \frac{1}{k} \log \int_X e^{kF(x)} d\mu_k(x) = \sup_{x\in X} [F(x) - I(x)].$$

This would give C^0 convergence of our ray

$$\varphi_k(t,z) = \frac{1}{k} \log \sum_{\alpha \in kP\cap\mathbb{Z}^m} \left(\frac{Q^k_{h_0}(\alpha)}{Q_{h^k}(\alpha)} \right)^t \frac{||s_\alpha(z)||^2_{h_0^k}}{||s_\alpha||^2_{h_0^k}}.$$

if $\left(\frac{Q^k_{h_0}(\alpha)}{Q_{h^k}(\alpha)} \right)^t$ had the form $e^{kF_t(\alpha)}$. This is true in the interior but false at the boundary...

Test configuration rays

In this case, the ray has the basic form

$$\varphi_k(t,z) = \frac{1}{k}\log \sum_{\alpha\in kP\cap\mathbb{Z}^m} e^{k(R-f(\frac{\alpha}{k}))}\frac{||s_\alpha(z)||^2_{h_0^k}}{||s_\alpha||^2_{h_0^k}}.$$

where f is a piecewise linear convex function and $R \in \mathbb{Z}, R >> 0$. The graph of $R-f$ is used to make a one higher dimensional polytope from P, which makes a toric degeneration of M.

Test configuration rays

One finds that the limit ray (over the open orbit) is $\psi_t = \mathcal{L}(u_0 + tf)$ where u_0 is the symplectic potential. Here, $\mathcal{L}$ is the Legendre transform. So the test ray is the Legendre transform of a piecewise smooth function.

The Legendre transforms smooths out the corners of f to C^1, but no further than $C^{1,1}$. ψ_t determines a moment map

$$\mu_t : M^o \to P, \;\; \mu_t(e^{\rho/2+i\theta}) = \nabla_\rho \psi_t(e^{\rho/2+i\theta}) \;\text{ on }\; M^o.$$

μ_t fails to be a homeomorphism from $M/\mathbf{T}^m$ to P as in the smooth case. Indeed, the usual inverse map defined by gradient of the symplectic potential pulls apart the polytope discontinuously into different regions. But it is a homeomorphism from the underlying real toric variety $M_{\mathbb{R}}$ to the graph of the subdifferential of $u + tf$.