AIM Lecture: Asymptotic Geometry of polynomials

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Statistical algebraic geometry

We are interested in asymptotic geometry as the degree $N \to \infty$ of zeros of polynomial systems

$$\begin{cases} p_1(z_1, \dots, z_m) = 0 \\ p_2(z_1, \dots, z_m) = 0 \\ \vdots \\ p_k(z_1, \dots, z_m) = 0. \end{cases}$$

We are interested both in complex (holomorphic) polynomials with $c_\alpha \in \mathbb{C}, z \in \mathbb{C}^m$ and real polynomials with $c_\alpha \in \mathbb{R}, x \in \mathbb{R}^m$.

More precisely, we are interested in the asymptotics as $N \to \infty$ of statistical properties of random polynomial systems.

Statistical Algebraic Geometry (2)

- Statistical algebraic geometry: zeros of individual polynomials define algebraic varieties. Instead of studying complexities of all possible individual varieties, study the expected (average) behaviour, the almost-sure behaviour.

- There are statistical patterns in zeros and critical points that one does not see by studying individual varieties, which are often ‘outliers’.

- Our methods/results concern not just polynomials, but holomorphic sections of any positive line bundle over a Kähler manifold.

Plan of talk

- Define 'Gaussian random polynomial' and Gaussian random polynomial with constrained spectra.

- Describe impact of the Newton polytope on the expected distribution of random polynomials with prescribed Newton polytope. Application to amoebas.

- Impact of the spectrum on the distribution of random complex fewnomials. Partial results on real fewnomials.

- Statements/pictures of results. Little discussion of proofs.

Complex polynomials in m variables

Some background on polynomials in m complex variables:

$$z = (z_1, \ldots, z_m) \in \mathbb{C}^m.$$

- Monomials: $\chi_\alpha(z) = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, $\alpha \in \mathbb{N}^m$.

- Polynomial of degree p (complex, holomorphic, not necessarily homogeneous):

$$f(z_1, \ldots, z_m) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \leq p} c_\alpha \chi_\alpha(z_1, \ldots, z_m).$$

- Homogenize to degree p: introduce new variable z_0 and put:

$$\widehat{\chi}_\alpha(z_0, z_1 \ldots, z_m) = z_0^{p-|\alpha|} z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

  We write $F = \widehat{f}_\alpha(z_0, z_1 \ldots, z_m)$ for the homogenized f.

Spectrum (support) and Newton polytope

- The *spectrum* (= *support*) of a polynomial p is the set $S = S_p$ of exponents of its non-zero monomials:

$$p(z) = \sum_{\alpha \in S} c_\alpha z^\alpha.$$

- Newton polytope Δ_p of p is the convex hull of S_p.

Spaces of (complex) polynomials

- Space of polynomials with spectrum *contained in* S is denoted by

$$Poly(S) = \{p(z) = \textstyle\sum_{\alpha \in S} c_\alpha z^\alpha, \;\; S \subset \mathbb{N}^m\}.$$

- Space of polynomials with Newton polytope Δ: $\mathcal{P}_\Delta = Poly(\Delta)$.

- Space of polynomials of degree N in m variables is denoted

$$\mathcal{P}_N^m = Poly(N\Sigma) = \{p(z) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha)\},$$

where $\Sigma \subset \mathbb{R}_+^m$ is the unit simplex and $N\Sigma$ denotes its dilate by N.

Gaussian random $SU(m+1)$ complex polynomials

Random polynomial: a probability measure on the space $\mathcal{P}_N^m$

Gaussian random:

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, z^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \bar{\lambda}_\beta) = \delta_{\alpha\beta}.$$

In coordinates λ_α:

$$d\gamma_p(f) = \frac{1}{\pi^{k_N}} e^{-|\lambda|^2} d\lambda \text{ on } \mathcal{P}_N^m.$$

Gaussian measure versus inner product

The Gaussian measure above comes from the Fubini-Study inner product on the space $\mathcal{P}_N^m$ of polynomials of degree N. Indeed,

$$||z^\alpha||_{FS} = \binom{N}{\alpha}^{-1/2}, \;\; \langle z^\alpha, z^\beta \rangle = 0, \; \alpha \neq \beta.$$

Namely, let $F(z_0, \dots, z_m) = z_0^N f(z'/z_0)$ homogenize f. Then
$||f||_{FS}^2 = \int_{S^{2m+1}} |F|^2 \, d\sigma$, (Haar measure).

Thus, the same ensemble could be written:

$$f = \sum_{|\alpha| \leq N} \lambda_\alpha \; \frac{z^\alpha}{||z^\alpha||_{FS}},$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \bar{\lambda}_\beta) = \delta_{\alpha\beta}.$$

Why the Fubini-Study $SU(m+1)$-ensemble?

One could use any inner product in defining a Gaussian measure: Write

$$s = \sum_j c_j S_j, \quad \langle S_j, S_k \rangle = \delta_{jk}$$

with $\mathbf{E}(c_j) = 0 = \mathbf{E}(c_j c_k)$, $\mathbf{E}(c_j \overline{c_k}) = \delta_{jk}$.

We use the Fubini-Study because the expected distribution of zeros (or critical points etc.) of 'typical' polynomials become uniform over $\mathbb{CP}^m$. Thus, the ensemble is natural for projective geometry. Taking $\sum c_\alpha z^\alpha$ with c_α normal biases the zeros towards the torus $|z_j| = 1$.

‘Distribution of complex zeros’

Easiest to define for full systems of m polynomials in m variables, since their simultaneous zeros form a discrete set.

We define the distribution of zeros of a full system $(f_1, \ldots, f_m)$ by

$$Z_{f_1,\ldots,f_m} = \sum_{\{z_j : f_1(z_j) = \cdots = f_m(z_j) = 0\}} \delta_{z_j}.$$

Here, $\delta(z)$ is the Dirac point mass at z. I.e. $\int \psi \delta(z) = \psi(z)$.

Note that $Z_{f_1,\ldots,f_m}$ is not normalized, i.e. its mass is the number of zeros.

Distribution of zeros

More generally, we define the 'delta' function on the joint zero set of one or several polynomials.

Given $f_1, \ldots, f_k$, $k \leq m$, put $Z_{f_1,\ldots,f_k} = \{z \in (C^*)^m : f_1(z) = \cdots = f_k(z) = 0\}$. $Z_{f_1,\ldots,f_k}$ defines a (k,k) current of integration:

$$\langle \psi, Z_{f_1,\ldots,f_k} \rangle = \int_{Z_f} \psi.$$

By Wirtinger's formula, the integral of a scalar function φ over Z_f can be defined as

$$\int_{Z_{f_1,\ldots,f_k}} \varphi \frac{\omega_{FS}^{n-k}}{(n-k)!}.$$

Expected zero distributions = density of zeros: Definition

It is the average $\mathbf{E}_N(Z_{f_1,\dots,f_k},\varphi)$ of the measures $(Z_{f_1,\dots,f_k},\varphi)$ over the ensemble of polynomials f.

For a random full system of m polynomials by $\mathbf{E}_N(Z_{f_1,\dots,f_m},\varphi)$,e.g. we have:

$$\mathbf{E}_N(Z_{f_1,\dots,f_m})(U) = \int d\gamma_p(f_1)\cdots\int d\gamma_p(f_m)$$

$$\times\Big[\#\{z\in U : f_1(z)=\cdots=f_m(z)=0\}\Big]\,,$$

for $U\subset\mathbb{C}^{*m}$, where the integrals are over $\mathcal{P}_N^{\mathbb{C}}$.

The definition is analogous for systems of k polynomials in m variables.

Expected distribution of zeros in the $SU(m+1)$ ensemble

- $SU(m+1)$ ensemble: $\mathbf{E}(Z_f) = \frac{N^m}{Vol(\mathbb{CP}^m)} dVol_{\mathbb{CP}^m}$. It is a volume form on $\mathbb{CP}^m$ invariant under $SU(m+1)$, so is a constant multiple of the invariant volume form. The constant is determined by integrating, which gives the expected number of zeros. This must equal the Bezout number N^m, the product of the degrees of the f_j's.

- We now start to look at how constraining the Newton polytope biases the distribution from uniformity.

Polynomials with fixed Newton polytope

We now ask: how is the distribution of zeros affected by the Newton polytope Δ of a polynomial? Recall

- **Bezout's theorem**: m generic homogeneous polynomials $F_1, \ldots, F_m$ of degree p have exactly p^m simultaneous zeros; these zeros all lie in $\mathbb{C}^{*m}$, for generic F_j.

- **Bernstein-Kouchnirenko Theorem** The number of joint zeros in $\mathbb{C}^{*m}$ of m generic polynomials $\{f_1, \ldots, f_m\}$ with given Newton polytope Δ equals $m!\mathrm{Vol}(\Delta)$.

- More generally, the f_j may have different Newton polytopes Δ_j; then, the number of zeros equals the 'mixed volume' of the Δ_j.

Theme

The Newton polytopes Δ_j of a polynomial system $f_1, \ldots, f_m$ with $\Delta_{f_j} = \Delta_j$ also have a crucial influence on the *spatial distribution* of *zeros* $\{f_1 = \cdots = f_m = 0\}$ and critical points $\{df = 0\}$. For simplicity we assume $\Delta_j = \Delta$:

- There is a *classically allowed region*

  $$\mathcal{A}_\Delta = \mu_\Sigma^{-1}(\frac{1}{p}\Delta)$$

  region where the zeros or critical points concentrate with high probability;

- in its complement, the *classically forbidden region*, they are usually sparse.

  Here,

  $$\mu_\Sigma(z) = \left(\frac{|z_1|^2}{1 + \|z\|^2}, \ldots, \frac{|z_m|^2}{1 + \|z\|^2}\right)$$

  is the moment map of $\mathbb{CP}^m$.

Asymptotic and Statistical

These results are statistical and asymptotic:

- Not *all* polynomials $f \in \mathcal{P}_{\Delta}^{m}$ have this behaviour; but typical ones. We will endow $\mathcal{P}_{\Delta}^{m}$ with a Gaussian probability measure, and show that the above patterns form the expected behaviour of random polynomials.

- The variance is small compared to the expected value: i.e. the statistics are 'self-averaging' in the limit $N \to \infty$. Here, as $N \to \infty$, we dilate $\Delta \to N\Delta$.

Random polynomials with $\Delta_f \subset \Delta$

Definition of the ensemble: Let $Poly(\Delta)$ denote the space of polynomials with $\Delta_f \subset \Delta$.

Endow $Poly(\Delta)$ with the *conditional probability measure* $\gamma_p|_\Delta$:
(1)

$$d\gamma_{p|\Delta}(s) = \frac{1}{\pi^{\#\Delta}} e^{-|\lambda|^2} d\lambda, \quad s = \sum_{\alpha \in \Delta} \lambda_\alpha \frac{z^\alpha}{\|z^\alpha\|},$$

where the coefficients $\lambda_\alpha =$ independent complex Gaussian random variables with mean zero and variance one. Denote conditional expectation by $\mathbf{E}_{|\Delta}$.

Asymptotics of expected distribution of zeros

Let $\mathbf{E}_{|\Delta}(Z_{f_1,\dots,f_m})$ = expected distribution of simultaneous zeros of $(f_1,\dots,f_m)$, chosen independently from $Poly(\Delta) = \mathcal{P}^m_\Delta$. We will determine the asymptotics of the expected density as the polytope is dilated $\Delta \to N\Delta, N \in \mathbb{N}$.

Theorem **1** *(Shiffman-Z) Suppose that Δ is a simple polytope in $\mathbb{R}^m$. Then, as Δ is dilated to $N\Delta$,*

$$\frac{1}{(Np)^m}\mathbf{E}_{|N\Delta}(Z_{f_1,\dots,f_m}) \to \begin{cases} \omega^m_{\mathrm{FS}} & \text{on } \mathcal{A}_\Delta \\ 0 & \text{on } \mathbb{C}^{*m} \backslash \mathcal{A}_\Delta \end{cases}.$$

Thus, the simultaneous zeros of m polynomials with Newton polytope Δ concentrate in the allowed region and are uniform there, giving a quantitative BK result.

Zeros of one polynomial

Instead of m polynomials, we could consider $1 \leq k < m$ polynomials. For each face F of Δ let $\mathcal{R}_F$ denote the flow-out of F w.r.t. the $\mathbb{R}_+^m$ action.

Theorem **1** *Let Δ be a convex integral polytope. Then there exists a closed semipositive $(1,1)$-form ψ_Δ on $\mathbb{C}^{*m}$ with piecewise $\mathcal{C}^\infty$ coefficients such that:*

i) $N^{-1}\mathbf{E}_{|N\Delta}(Z_f) \to \psi_\Delta$ *in* $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{C}^{*m})$.

ii) $\psi_\Delta = p\omega_{\mathrm{FS}}$ *on the classically allowed region* $\mu^{-1}(\frac{1}{p}\Delta^\circ)$.

iii) On each region $\mathcal{R}_F^\circ$, the $(1,1)$-form ψ_Δ is $\mathcal{C}^\infty$ and has constant rank equal to $\dim F$*; in particular, if $v \in \Sigma^\circ$ is a vertex of $\frac{1}{p\Delta}$, then $\psi_\Delta|_{\mathcal{R}_v^\circ} = 0$.*

Discussion

There is an explicit formula for ψ_Δ in terms of a function $b_\Delta(z)$ which (roughly) measures the 'distance' of z to Δ. We only stated some properties of ψ_Δ for simplicity.

The expected volume of the simultaneous zero set of k polynomials, has the following exotic distribution law:

For any open set $U \subset\subset \mathbb{C}^{*m}$,

$$\frac{1}{N^k}\mathbf{E}_{|N\Delta}\mathrm{Vol}(|Z_{f_1,\dots,f_k}| \cap U)$$
$$\to \frac{1}{(m-k)!}\int_U \psi_\Delta \wedge \cdots \wedge \psi_\Delta \wedge \omega_{\mathrm{FS}}^{m-k} \, .$$

Application to 2D Amoebas and tentacles

Let $f(z_1, z_2) \in \mathcal{P}^2_\Delta$. One defines:

Compact Amoeba of $f = \mu_\Sigma(Z_f)$;

Amoeba of $f = \mathrm{Log}\,(Z_f)$, where Log $: \mathbb{C}^{*m} \to \mathbb{R}^m$, is $(z_1, \ldots, z_m) \mapsto (\log|z_1|, \ldots, \log|z_m|)$.

Tentacles of Amoeba:= its ends on $\partial\Sigma$. For a generic 2-D amoeba with polytope Δ, lattice points in $\partial\Delta \iff$ unbounded components of the complement. Each tentacle corresponds to a segment connecting 2 adjacent lattice points on $\partial\Delta$. Hence # tentacles of $A = \#\partial\Delta \cap \mathbb{Z} =$ *length* of $\partial\Delta$.

Free tentacles

We can decompose $\partial\Delta$ into two pieces:

$$\partial^\circ\Delta = \partial\Delta \cap p\Sigma^\circ, \quad \text{versus } \partial^e\Delta = \Delta \cap \partial(p\Sigma).$$

Tentacles corresponding to segments of $\partial^\circ\Delta$ end (in the compact picture Σ) at a vertex of Σ, and tentacles corresponding to segments of $\partial^e\Delta$ are free to end anywhere on the face of Σ containing the segment. We call the latter *free tentacles*.

Classically allowed tentacles

We say that a free tentacle is a *classically allowed tentacle* if its end is in the classically allowed region $\mathcal{A}_\Delta$. For an amoeba A, we let $\nu_{\mathsf{AT}}(A)$ denote the number of classically allowed tentacles of A. It is clear from the above that

$$\nu_{\mathsf{AT}}(A) \leq \#\{\text{free tentacles}\} = \text{Length}(\partial^e \Delta)$$

and that this bound can be attained for any polytope Δ. Here, 'Length' means the length in the above sense; i.e., the diagonal face of $p\Sigma$ is scaled to have length p. Our result is that the maximum is asymptotically the average:

Corollary **2** *For a convex lattice polytope Δ, we have*

$$\frac{1}{N}\mathbf{E}_{|N\Delta}\Big(\nu_{\mathsf{AT}}\Big(\text{Log}\,(Z_f)\Big)\Big) \to \text{Length}(\partial^e \Delta)\ .$$

Mass asymptotics

A key ingredient is the mass asymtotics of random sections:

Theorem **3**

$$\mathbf{E}_{\nu_{N\Delta}}\left(|f(z)|^2_{\mathsf{FS}}\right) \sim \begin{cases} \frac{\omega^m}{\mathrm{Vol}(P\Delta)} + O(N^{-1}), \\ \textit{for} \quad z \in \mathcal{A}_\Delta = \mu^{-1}(\frac{1}{p}\Delta^\circ) \\ \\ N^{-s/2} e^{-Nb(z)} \left[c_0^F(z) + O(N^{-1})\right], \\ \textit{for } z \in (\mathbb{C}^*)^m \backslash \mathcal{A}_\Delta \end{cases}$$

where c_0^F *and* $b|_{\mathcal{R}_F^\circ}$ *are positive.*

b is a kind of Agmon distance, giving decay of ground states away from the classically allowed region.

Ideas and Methods of Proofs

- For each ensemble, we define the two-point function
$$\Pi_N(z,w) = \mathbf{E}_N(f(z)\overline{f(w)}).$$
It is the Bergman-Szegö (reproducing) kernel for the inner product space of polynomials or sections of degree N.

- All densities and correlation functions for zeros may be expressed in terms of the joint probability density (JPD) of the random variables $X(f) = f(z_0), \Xi(f) = df(z_0)$. For critical points, we also need $Hf(z_0) = Hessian f(z_0)$.

- For Gaussian ensembles, the JPD is a Gaussian with covariance matrix depending only on $\Pi_N(z,w)$ and its derivatives.

Bergman-Szegö kernels

More precisely:

- Expected distribution of zeros: $\mathbf{E}_N(Z_f) = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Pi_N(z,z) + \omega$.

- Joint probability distribution (JPD) $D_N(x^1,\dots,x^n;\xi^1,\dots,\xi^n;z^1,\dots,z^n)$ of random variables $x^j(s) = s(z^j)$, $\xi^j(s) = \nabla s(z^j)$, $=$ function of Π_N and derivatives.

- Correlation functions in terms of JPD

  $$K^N(z^1,\dots,z^n) = \int D_N(0,\xi,z)\prod_{j=1}^{n}\left(\|\xi^j\|^2 d\xi^j\right)d\xi.$$

Fixed Newton polytope

We need exponentially accurate asymptotics of the conditional Bergman- Szegö kernel

$$\Pi_{|N\Delta}(z,w) = \sum_{\alpha \in N\Delta} \frac{z^\alpha \overline{w}^\alpha}{||z^\alpha||_{FS}||w^\alpha||_{FS}}.$$

This projection sifts out terms with $\alpha \in \Delta$ from the simple Szegö projector of $\mathbb{CP}^m$.

We need asymptotics of $\Pi_{|N\Delta}(z,w)$. For this we use the Khovanskii-Pukhlikov (Brion-Vergne, Guillemin) Euler MacLaurin sum formula.

Non-convex polytopes and fewnomials

Our methods extend without much modification to non-convex polytopes Δ, possibly with empty interior. We could thus let $\Delta = \{\alpha_1, \ldots, \alpha_f\}$ be some spectrum and consider its dilates $N\Delta$. $Poly(N\Delta)$ is then a space of fewnomials. The density of zeros of polynomials in $Poly(N\Delta)$ is very singular as $N \to \infty$.

But rather than dilate one spectrum, we could just randomly choose a spectrum with f elements from $N\Sigma$. This gives a new theory–random fewnomials. We describe some of our work in progress, where we have fairly complete results for random complex fewnomials. Eventually, we would like to understand the expected number of real zeros of random real fewnomial systems.

Fewnomials

Given $S \subset \mathbf{N}^m$, we denote by $|S| = \#\{\alpha \in S\}$ the number of lattice points in S.

By the degree of S we mean the least N such that $S \subset N\Sigma$, i.e. the maximum of the degrees of the monomials in S.

Roughly speaking, p is a fewnomial if

$$\frac{|S_p|}{\deg p} << 1.$$

If $P = (p_1, \ldots, p_k)$ and if

$$f = \#\{\text{ monomials appearing in P}\}$$

We will refer to f as the *fewnomial number* of the system.

We consider asymptotics when $N \to \infty$ while $f = |S|$ is held fixed.

Khovanskii's theorem: preparation

We now state a basic result motivating our work. Notation:

- $P = (p_1, \ldots, p_m)$ = system of m complex polynomials on $(\mathbb{C}^*)^m$.

- $U \subset \mathbf{T}^m$ = open set of real m-torus $\mathbf{T}^m \subset (\mathbb{C}^*)^m$.

- $N(P, U)$ = number of zeros with arguments lying in U.

- $S(P, U) = N(P, \mathbf{T}^m) Vol(U)/Vol(\mathbf{T}^m)$.

$N(P, \mathbf{T}^m) = m! V(\Delta_1, \ldots, \Delta_m)$.

Khovanskii's theorem

$$|N(P,U) - S(P,U)| \leq \Pi(U, \Delta_1, \ldots, \Delta_m)\varphi(m,f),$$

where $\varphi(m,f)$ depends only on f, m and where

$$f = \#\{\text{ monomials appearing in P}\}$$

and where $\Pi(U, \Delta_1, \ldots, \Delta_m)$ is the smallest number of translates of Δ^* required to cover the boundary of U.

Khovanskii's bound:

$$\varphi(m,f) \leq 2^m 2^{f(f-1)/2}(m+1)^f. \tag{2}$$

Thus: arguments of the zeros of a fewnomial system are rather uniformly distributed in the torus $\mathbf{T}^m$, with an error bound determined solely by the fewnomial number rather than by their degrees N.

Application: Real zeros of Real fewnomials

Let $P = (p_1, \ldots, p_m)$ denote a system of m real polynomials on $(\mathbb{R}^*)^m$, and let $N_{\mathbb{R}}(p)$ denote the number of real zeros of the polynomial system The fewnomials result gives a bound

$$|N_{\mathbb{R}}(P)| \leq \varphi(m, f), \tag{3}$$

just in terms of the fewnomial number.

This bound is believed to be far from sharp. What is the correct order of magnitude? Our eventual goal is to find the *expected value* of $N_{\mathbb{R}}(P)$ as one averages over all fewnomial systems with f fixed.

The random fewnomial ensemble $\mathcal{F}(N, f)$

To define a random system of $k \leq m$ fewnomials of degree N:

- Fix the degree N and the numbers f_j of elements of each spectrum $S_j \subset N\Sigma$;

- Then choose the spectra S_j at random, and then choose the coefficients $c_{j\alpha}$ of $p \in Poly(S_j)$ at random.

- We use counting measure for the spectra, and the $SU(m+1)$-invariant Gaussian measure for the coefficients. Before defining the ensemble precisely, we state our problems and results.

Expected density of zeros: Results for one polynomial

Theorem **2** *Let p be a random fewnomial of fewnomial number f and of degree N. Then:*

$$\mathbf{E}\, Z_p(z) \sim N K_f(z),$$

$$K_f(z) := \int_{\Sigma^f} \Gamma_f(\lambda, z) d\lambda^1 \cdots d\lambda^f, \quad \textit{with}$$

$$\Gamma_f(\lambda, z) = \partial_z \bar{\partial}_z \min_{j=1,\dots,f} [\langle \hat{\lambda}^j, \log \hat{\lambda}^j \rangle + \langle \log |z|, \hat{\lambda}^j \rangle].$$

$\Gamma_f(\lambda, z)$ and $K_f(z)$ are $(1,1)$ currents on $(\mathbb{C}^*)^m_z$. For fixed λ, the function

$$\min_{j=1,\dots,f} [\langle \hat{\lambda}^j, \log \hat{\lambda}^j \rangle + \langle \log |z|, \hat{\lambda}^j \rangle]$$

is piecewise linear in $\log |z|$, so for each fixed λ, $\Gamma_f(\lambda, z)$ is supported on the 'corner set' of the integrand in $\rho = \log |z|$ coordinates. Integration in λ smooths out this current.

Density of zeros for systems

Using the independence of the polynomials in a random system, we obtain a similar result for the higher codimension case:

Theorem **3** *Let* $(p_1, \ldots, p_k)$ *be a random fewnomial of* k *fewnomials with fewnomial numbers* $(f_1, \ldots, f_k)$ *and of degree* N *in the free* $SU(m+1)$ *ensemble. Then:*

$$\mathbf{E}_N Z_{p_1,\ldots,p_k}(z) = N^k\, K_f \wedge K_f \wedge \cdots \wedge K_f \;\; (ktimes)$$

$$= \int_{\Sigma^{f_1} \times \cdots \times \Sigma^{f_k}} \bigwedge_{n=1}^{k} \Gamma_{f_n}(\lambda_n, z) d\lambda.$$

For fixed λ_n, $\bigwedge_{n=1}^{k} \Gamma_{f_n}(\lambda_n, \rho)$ is a current supported on the intersection of the corner sets of the k factors. Also, $d\lambda = \Pi_{n=1}^{k} \lambda_n^1 \cdots d\lambda_n^{f_n}$.

Counting complex zeros in an open set

Integrating the current gives the count of the simeltaneous zeros in an open set $\Omega \subset (\mathbb{C}^*)^m$ with smooth boundary for a random fewnomial system of n fewnomials of fewnomial number $(f_1, \ldots, f_m)$ and degree N. We denote this number by $N((f_1, \ldots, f_m), \Omega)$.

Corollary **4**

$$\mathbf{E}N((f_1, \ldots, f_m), \Omega) \sim N^m \int_\Omega \int_{\Sigma^{f_1} \times \cdots \times \Sigma^{f_m}}$$

$$\bigwedge_{n=1}^m \Gamma_{f_n}(\lambda_n, z) d\lambda.$$

This is averaging over possible 'limit spectra' the number of intersection points of the intersection of the m corner sets.

Counting complex zeros with arguments in an open set

The same formula gives the number of simultaneous zeros with arguments in an open set $U \subset \mathbf{T}^m$. Indeed, the volume form depends only on the modulus $|z| = e^\rho$ of the zeros.

Corollary **5**

$$\mathbf{E}_{\mathcal{F}(N,f)} N_{arg}((f_1, \dots, f_m), U) \sim C_f N^m Vol(U),$$

$$C_f = \int_{arg^{-1}(U)} \int_{\Sigma^{f_1} \times \cdots \times \Sigma^{f_m}} \bigwedge_{n=1}^{k} \Gamma_{f_n}(\lambda_n, \rho) d\lambda\}.$$

Thus, as Khovanskii suggested, the expected number $N_{arg}((f_1, \dots, f_m), U)$ of simultaneous zeros with arguments in an open set $U \subset \mathbf{T}^m$ is a multiple $C_f N^m Vol(U)$ of the volume of U as a subset of the torus. Obviously the expected number of real roots is zero.

Expected density of real zeros of real fewnomial systems

The real case is still in progress. It is more difficult because one must take derivatives first and asymptotics second. And the asymptotics have to change by Khovanskii's theorem!

We define the ensembles $\mathcal{F}_{f,N}$ as in the complex case, but conditioned from the $O(N+1)$ polynomial ensemble.

Random real $O(m+1)$ polynomials

Let us define the most symmetrical real ensemble.

Let $Poly(N\Sigma)_{\mathbb{R}}$ be the space of real polynomials

$$p(x) = \sum_{|\alpha| \leq N} c_\alpha \chi_\alpha(x), \quad \chi_\alpha(x) = x^\alpha, \quad x \in \mathbb{R}^m, \alpha \in N\Sigma$$

of degree N in m real variables with real coefficients. Define the inner product

$$\langle \chi_\alpha, \chi_\beta \rangle = \delta_{\alpha,\beta} \frac{1}{\binom{N}{\alpha}}.$$

Define a random polynomial in the $O(m+1)$ ensemble as

$$f = \textstyle\sum_{|\alpha| \leq N} \lambda_\alpha \sqrt{\binom{N}{\alpha}}\, x^\alpha,$$

$$\mathbf{E}(\lambda_\alpha) = 0, \quad \mathbf{E}(\lambda_\alpha \lambda_\beta) = \delta_{\alpha\beta}.$$

Why $O(m+1)$?

If we homogenize the polynomials $Poly(N\Sigma)$, we obtain a representation of $O(m+1)$. The invariant inner product is

$$\langle P, Q\rangle := P(D)\bar{Q}(0) = \int_{\mathbb{R}^n} P(2\pi i\xi)\bar{\hat{Q}}(\xi)d\xi,$$

where $P(D)$ is the constant coefficient differential operator defined by the Fourier multiplier $P(2\pi i\xi)$.

We may regard the zeros as points of $\mathbb{R}P^m$. The expected distribution of zeros will be uniform there w.r.t. the natural volume form.

Expected distribution of zeros in the $O(m+1)$ ensemble

- $O(m+1)$ ensemble: $\mathbf{E}(Z_f) = \frac{N^{m/2}}{Vol(\mathbb{R}P^m)} dVol_{\mathbb{R}P^m}$. For the same reason, it must be a constant multiple of the invariant volume form. But this time the number of zeros is a random variable. Shub-Smale (1995) showed that the expected number of zeros is the square root of the Bezout number for complex roots.

Results on random real polynomial systems with fixed Newton polytope

The analogous result for the expected number of real roots and the density of real roots forn the conditional $O(m+1)$ ensemble, where we constrain all polynomials to have Newton polytope P:

Theorem **6** *(Shiffman-Zelditch, May 1, 2003)*

$$\mathbf{E}_{N\Delta}(Z_{f_1,\ldots,f_m})(x) = \begin{cases} a_m N^{m/2}, & x \in \mathcal{A}_\Delta \\ \\ O(N^{(m-1)/2}), & x \in \mathbb{R}P^m \backslash \mathcal{A}_\Delta. \end{cases},$$

where $a_m = Vol_{\mathbb{R}P^m}(\mathcal{A}_\Delta)$. *The coefficient* a_m *is NOT the square root of the BKK number of complex roots.*

Result

A preliminary result:

Proposition **7** *We have:*

$$K_f^N(x) = \tfrac{1}{\pi^{knm}} \tfrac{Q}{|\mathcal{S}(N,f)|} \textstyle\sum_{S \in \mathcal{S}(N,f)}$$

$$\frac{\sqrt{\det \nabla_x \nabla_y \log \Pi_{N|S}(x,y)|_{x=y}}}{[\sqrt{\Pi_{N|S}(x,x)}]^m},$$

where $Q := \int_{\mathbb{R}^m} |\xi| \exp\left(-\langle \xi, \xi \rangle\right) d\xi$, *and where* $\Pi_{N|S}$ *is the Szegö kernel for the spectrum* S

$$\Pi_{N|S}(x,y) = \sum_{\beta \in S} \binom{N}{\beta} x^\beta y^\beta.$$