Scalar Ordinary Differential Equations

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In these notes we denote $\dfrac{dx}{dt}$ by $\dot{x}$ and $\dfrac{d^2x}{dt^2}$ by $\ddot{x}$.

The differential equation $\dot{x} = ax$ is usually considered in calculus courses, where a is a fixed parameter. An explicit expression for the solution is $x(t) = x_0 e^{at}$ where $x(0) = x_0$. An important property of the solution is that $x(t)$ has unlimited growth if $a > 0$ and decays to zero if $a < 0$. This differential equation is *linear* in x. An equation such as $\dot{x} = t^3 x$ is also linear, even though it is nonlinear in t. An example of a nonlinear scalar equation is the logistic equation $\dot{x} = x(1-x)$. A scalar differential equations that only involve one derivative with respect to time is called a *first order differential equation*.

A general first order scalar differential equation is given by $\dot{x} = f(t,x)$, where $f(t,x)$ can be a function of possibly both x and t and is called the *rate function*. If the rate function depends only x, $\dot{x} = f(x)$, the differential equation is called *autonomous*; it is called *nonautonomous* if the rate function depends explicitly on t. The differential equation $\dot{x} = t\,x^4$ is an example of a nonautonomous nonlinear first order scalar differential equation. A *solution* to the differential equation $\dot{x} = f(t,x)$, is a differentiable path $x(t)$ in $\mathbb{R}$ such that $\dot{x}(t) = f(t, x(t))$. We often specify the *initial condition* x_0 at some time t_0, i.e., we seek a solution $x(t)$ such that $x(t_0) = x_0$.

A *second order* differential equations is of the form $\ddot{x} = f(t, x, \dot{x})$. Examples of *second order* linear equations are $\ddot{x} = \text{-}a^2 x$ and $t^2\ddot{x} + t\dot{x} + (t^2 - n^2)x = 0$. The latter is an example of a *Bessel equation*, and we will consider it in winter quarter.

We shall concentrate on the following two questions:

(1) When possible, how can we find a solution to a given differential equation with the given initial condition $x(t_0) = x_0$? Note that we often cannot or do not find an explicit form of a solution for nonlinear differential equations.
(2) What are the long term properties of a solution as t goes to infinity?

In these notes, we consider the case of scalar differential equations, leaving to [5] the case of systems of differential equations (in some $\mathbb{R}^n$), such as the nonlinear predator-prey system $\dot{x} = x(1-y)$ and $\dot{y} = y(x-1)$. A good reference for the scalar equations is Boyce and DiPrima [2]. The book by Borrelli and Coleman [1] is a good reference for modeling applied situations. Finally, Strogatz's book [6] had many applications of ordinary differential equations.

1. Linear Scalar Differential Equations

For continuous functions $a, g : \mathbb{R} \to \mathbb{R}$, $\dot{x} = a(t)x + g(t)$ is a *nonhomogeneous linear differential equation* and $\dot{x} = a(t)x$ is the corresponding *homogeneous linear differential equation*.

Consider the homogeneous equation $\dot{x} = a(t)\,x$ with $x(t_0) = x_0$. Letting $b(t) = \int_{t_0}^{t} a(s)\,ds$,

$$\frac{\dot{x}}{x} = a(t),$$

$$\ln(x(t)) - \ln(x_0) = \int_{t_0}^{t} a(s)\,ds = b(t),$$

$$\frac{x(t)}{x_0} = e^{b(t)}.$$

This solution, $x(t) = x_0\,e^{b(t)}$, is usually found in calculus courses (at least for a constant $a = a(t)$).

Theorem 1 (Variation of Parameters). *Let* $\mathbf{I} \subset \mathbb{R}$ *be an interval and* $a,\, g : \mathbf{I} \to \mathbb{R}$ *be given continuous functions. Then the solution* $x(t) = \phi(t; t_0, x_0)$ *of the nonhomogeneous differential equation* $\dot{x} = a(t)x + g(t)$ *with* $t_0 \in \mathbf{I}$ *and* $x(t_0) = x_0$ *is defined for* $t \in \mathbf{I}$ *and is given by*

$$x(t) = e^{b(t)} x_0 + e^{b(t)} \int_{t_0}^{t} e^{-b(s)} g(s)\, ds, \tag{1}$$

where $b(t) = \int_{t_0}^{t} a(s)\, ds$*. This formula gives both existence and uniqueness of solutions.*

Proof. Let $x(t)$ be any solution of the nonhomogeneous differential equation with $x(t_0) = x_0$. For each t, $x(t)$ will lie on the solution curve for the homogeneous equation with initial condition z at $t = t_0$. Thus, $z(t)$ is defined by $x(t) = z(t)\, e^{b(t)}$. The nonhomogeneous term $g(t)$ forces the solution $x(t)$ from one of these curves to another as t progresses. Note that $b(t_0) = 0$, so $e^{b(t_0)} = 1$ and $z(t_0) = x_0$. We can derive the differential equation that this coefficient or parameter $z(t)$ must satisfy:

$$\begin{aligned} a(t)\, z(t)\, e^{b(t)} + g(t) = \dot{x}(t) &= z(t)\, \frac{d}{dt}\left(e^{b(t)}\right) + e^{b(t)}\, \dot{z}(t) = z(t)\, a(t)\, e^{b(t)} + e^{b(t)}\, \dot{z}(t), \\ \dot{z}(t) &= e^{-b(t)} g(t). \end{aligned}$$

Since $g(t)$ is given and $b(t)$ is determined by integration of $a(t)$, we know the derivative of $z(t)$. Integrating $\dot{z}(t) = e^{-b(t)} g(t)$ from t_0 to t,

$$\begin{aligned} z(t) &= z(t_0) + \int_{t_0}^{t} \dot{z}(t)\, ds = x_0 + \int_{t_0}^{t} e^{-b(s)} g(s)\, ds \qquad \text{and} \\ x(t) &= e^{b(t)} z(t) = e^{b(t)}\, x_0 + e^{b(t)} \int_{t_0}^{t} e^{-b(s)} g(s)\, ds. \end{aligned}$$

This derivation shows both that any solution must satisfy (1) and that (1) is a solution. □

Second Derivation. The function $b(t)$ is the antiderivative of $a(t)$, so $b'(t) = a(t)$ and $\frac{d}{dt} e^{-b(t)} = -a(t)\, e^{-b(t)}$. Therefore,

$$\begin{aligned} \frac{d}{dt}\left[e^{-b(t)} x(t)\right] &= e^{-b(t)}\, \dot{x}(t) - a(t)\, e^{-b(t)} x(t) \\ &= e^{-b(t)}\, [\dot{x}(t) - a(t) x(t)] \\ &= e^{-b(t)} g(t). \end{aligned}$$

Integrating from t_0 to t, we get

$$\begin{aligned} e^{-b(t)} x(t) - e^{-b(t_0)} x_0 &= \int_{t_0}^{t} e^{-b(s)} g(s)\, ds \\ x(t) &= x_0\, e^{b(t)} + e^{b(t)} \int_{t_0}^{t} e^{-b(s)} g(s)\, ds \end{aligned}$$

since $b(t_0) = 0$. □

Remark. Because the derivative of $e^{-b(t)} x(t)$ is a function of t alone, the differential equation can be solved by integrals and $e^{-b(t)}$ is called an *integrating factor*.

Notice that (i) $e^{b(t)}\, x_0$ is solution of the associated homogeneous equation with $x(t_0) = x_0$ and (ii) $y(t) = e^{b(t)} \int_{t_0}^{t} e^{-b(s)} g(s)\, ds$ is a particular solution of the nonhomogeneous equation with initial condition $y(t_0) = 0$.

Example 1. Consider the equation $t\,\dot{x} = -x + t^2$ or $\dot{x} = -x/t + t$. The solution of the homogeneous equation is $e^{\int -1/s\,ds} = e^{-\ln(t)} = t^{-1}$. If we set $x(t) = z(t)\,\frac{1}{t}$ with $z(t)$ the coefficient of the solution of the homogeneous equation, then

$$\begin{aligned}
\dot{x} &= -\frac{x}{t} + t = -\frac{z}{t^2} + t \quad \text{and} \\
\dot{x} &= \frac{d}{dt}\left(z(t)\,\frac{1}{t}\right) = \dot{z}\,\frac{1}{t} - z\,\frac{1}{t^2}, \quad \text{so} \\
\dot{z} &= t^2, \\
z(t) &= \tfrac{1}{3}\,t^3 + C, \\
x(t) &= \tfrac{1}{3}\,t^2 + \frac{C}{t}, \\
x(1) &= \tfrac{1}{3} + C, \\
C &= x(1) - \tfrac{1}{3}, \\
x(t) &= \tfrac{1}{3}\,t^2 + \frac{3\,x(1) - 1}{3\,t}.
\end{aligned}$$

The plots of $x(t) = \phi(t; \pm 1, x_0)$ for $x_0 = 0,\ 1/3,\ 2/3$ are shown in Figure 1. ■

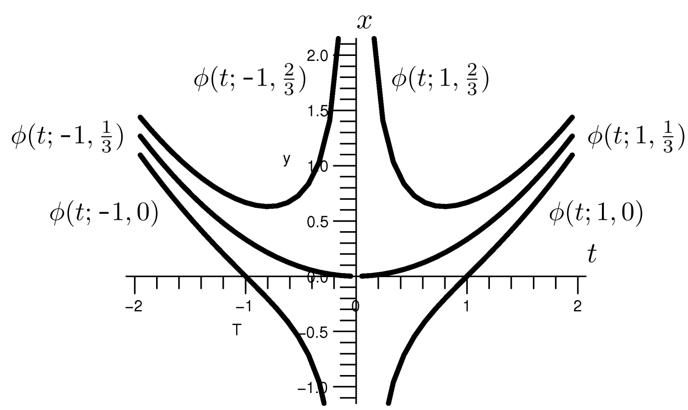

FIGURE 1. Solutions $\phi(t; \pm 1, x_0)$ of $t\,\dot{x} = -x + t^2$ for $x_0 = 0,\ 1/3,\ 2/3$.

Example 2. Consider the differential equation $\dot{x} = -\cos(t)\,x + 2\,\cos(t)$, with $x(0) = 3$.

The solution of the homogeneous equation is $e^{-\int \cos(t)\,dt} = e^{-\sin(t)}$. If $x(t) = z(t)\,e^{-\sin(t)}$, then

$$\begin{aligned}
-\cos(t)\,z(t)\,e^{-\sin(t)} + 2\,\cos(t) = \dot{x} &= \dot{z}\,e^{-\sin(t)} - z(t)\,e^{-\sin(t)}\cos(t), \\
\dot{z} &= 2\,\cos(t)\,e^{\sin(t)}, \\
z(t) &= \int 2\,e^{\sin(t)}\,\cos(t)\;dt + C = 2\int e^u\,du + C \\
&= 2\,e^u + C = 2\,e^{\sin(t)} + C \\
x(t) &= 2 + C\,e^{-\sin(t)}.
\end{aligned}$$

This is the general solution with parameter C. Using the initial conditions, $3 = 2 + C$, $C = 1$, and $x(t) = 2 + e^{-\sin(t)}$. ■

Example 3 (Periodically Forced). Consider the equation $\dot{x} = -x + \sin(t)$ with $x(0) = x_0$, which has a periodic forcing term. The solution of the homogeneous equation is $e^{-\int 1 dt} = e^{-t}$. If

$x(t) = z(t)\, e^{-t}$, then

$$\begin{aligned} -z(t)\, e^{-t} + \sin(t) &= \dot{x} = \dot{z}\, e^{-t} - z(t)\, e^{-t}, \\ \dot{z} &= e^{t} \sin(t), \\ z(t) - x_0 &= \int_0^t e^s \sin(s)\; ds \\ &= \tfrac{1}{2} \left[e^t \sin(t) - e^t \cos(t) + 1 \right] \\ x(t) &= x_0\, e^{-t} + \tfrac{1}{2} \left[\sin(t) - \cos(t) + e^{-t} \right]. \end{aligned}$$

The first term, $x_h(t) = x_0\, e^{-t}$, is a solution of the homogeneous equation with initial condition x_0 and decays with time (is a transient). The second group of terms is a response to the forcing term and is a particular solution of the nonhomogeneous equation with initial condition 0, $x_{\rm nh}(t) = ({}^1\!/\!{}_2) \left[\sin(t) - \cos(t) + e^{-t}\right]$. Note that $x_{\rm nh}(t) - ({}^1\!/\!{}_2)e^{-t}$ oscillates with the period of the forcing term. Figure 2 contains a plot of the two terms and the whole solution; the solution of the homogeneous equation with initial condition $x_0 = 2$ is dashed, the plot of the particular solution of the nonhomogeneous equation is shown with a dot-dash curve, and the solution of the nonhomogeneous solution with initial condition $x_0 = 2$ is shown as solid curve.

Because the solution of the homogeneous equation decays to zero, solutions of the nonhomogeneous equation with different initial conditions are asymptotic to each other as t goes to infinity. See Figure 3. ■

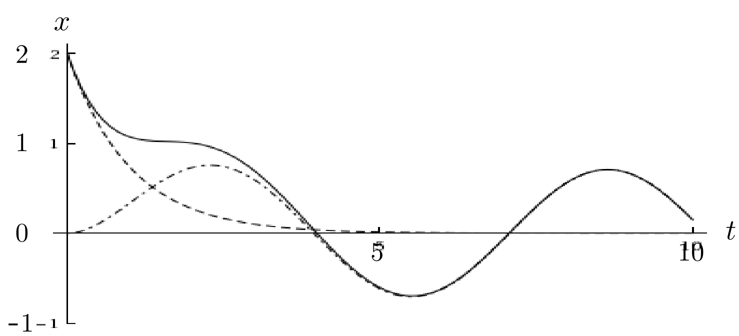

FIGURE 2. Example 3: $x(t)$ (solid), $x_{\rm h}(t)$ (dashed), and $x_{\rm nh}(t)$ (dot-dash)

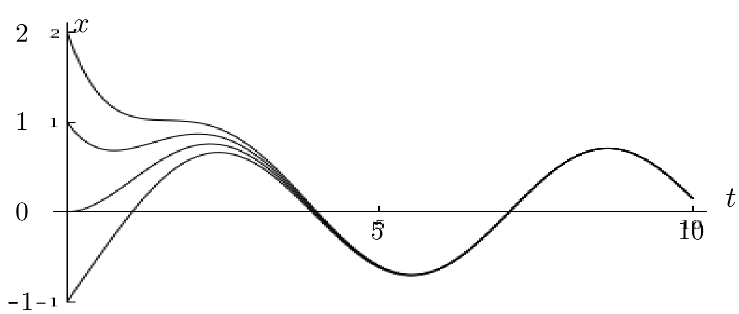

FIGURE 3. Example 3: Several initial conditions

Example 4 (**Compound Interest**)**.** An initial amount of money x_0 is put in an account with an annual rate of interest of $r > 0$. If the interest is compounded n times a year than the amount at the end of a year would be $(1 + {}^r\!/\!{}_n)^n\, x_0$. As n goes to infinity, this converges to $e^r x_0$, which is the solution of the differential equation $\dot{x} = r\, x$ at $t = 1$. The solution of this differential equation is called *continuous compounding*, which has an effective annual rate of e^r. (For $r = 0.05$, daily compounding has an effective rate of .051267 and continuous an effective rate of .051271096.)

Now, assume that money is added continuously to the account at the constant rate g, giving the differential equation governing the amount of money in the account of $\dot{x} = r\,x + g$. The solution is determined as follows: $a(t) = r$, $b(t) = rt$, $e^{b(t)} = e^{rt}$, and

$$\begin{aligned} x(t) &= x_0\, e^{rt} + e^{rt} \int_0^t e^{-rs} g\; ds = x_0\, e^{rt} + e^{rt} \left[-\frac{g}{r} e^{-rs} \Big|_0^t \right] \\ &= x_0\, e^{rt} + e^{rt}\, \frac{g}{r} \left[1 - e^{-rt} \right] = \left[x_0 + \frac{g}{r} \right] e^{rt} - \frac{g}{r}. \end{aligned}$$

In the long term, the amount in the account approaches the amount that would result from an initial deposit of $x_0 + \dfrac{g}{r}$ with no money added. ■

Example 5 (Cooling Body). This example is an example of cooling of a body relative to the temperature of the surrounding air. The cooling is determined by Newton's law of cooling which says that the rate of change of the temperature of the body is proportional to the difference of the temperature of the body and the surrounding air.

Let $T(t)$ and T_0 be the temperature of the body at times t and 0 and A be the temperature of the air. For a positive parameter k the differential equation is

$$\frac{dT}{dt} = -k\,(T - A) = -k\,T + k\,A.$$

The solution of the homogeneous equation is e^{-kt}. If $T(t) = z(t)\, e^{-kt}$, then

$$\begin{aligned} -k\, z(t)\, e^{-kt} + k\, A = \dot{T} &= \dot{z}\, e^{-kt} - z(t)\, k\, e^{-kt}, \\ \dot{z} &= k\, A\, e^{kt}, \\ z(t) - T_0 &= k\, A \int_0^t e^{ks}\; ds = A\,(e^{kt} - 1), \\ T(t) &= A + (T_0 - A)\; e^{-k\,t}. \end{aligned}$$

The cooling parameter k can be determined by measuring the air temperature and the temperature of the body at two times, $T(0) = T_0$ and $T(t_1) = T_1$:

$$\begin{aligned} T_1 - A &= (T_0 - A)\; e^{-k\, t_1}, \\ e^{k\, t_1} &= \frac{T_0 - A}{T_1 - A}, \\ k &= \frac{1}{t_1} \left[\ln(T_0 - A) - \ln(T_1 - A) \right]. \end{aligned}$$

If the air temperature is $A = 68$ and it takes 2 hours to cool from 85 to 74, then

$$k = \frac{1}{2} \ln \left(\frac{85 - 68}{74 - 68} \right) = 12 \ln \left(\frac{17}{6} \right) \approx 0.5207 \text{ degrees per hour.}$$

Given that value of k, if the body has cooled from 98.6 to 85, then the time is

$$t_c = \frac{1}{k} \ln \left(\frac{T_0 - A}{T_1 - A} \right) = \frac{1}{0.5207} \ln \left(\frac{98.6 - 68}{85 - 68} \right) \approx 1.129 \text{ hours.}$$

■

Example 6 (Falling Body). Suppose that an object is thrown straight up with initial velocity v_0 and initial height y_0. Assume that the friction is proportional to the velocity so the acceleration satisfies

$$m\, \ddot{y} = -m\, g - c\, \dot{y},$$

where m is the mass and g is the force of gravity. Since the equation only involves the velocity $v = \dot{y}$ and its derivative $\dot{v} = \ddot{y}$, we get a first order linear equation for the velocity

$$\dot{v} = -\frac{c}{m}\, v - g.$$

The solution of the homogeneous equation is $e^{-c' t}$ where $c' = c/m$, to simplify the expressions in the calculation. If $v(t) = z(t)\, e^{-c' t}$, then

$$\begin{aligned} -c'\, z(t)\, e^{-c' t} - g &= \dot{z}\, e^{-c' t} - z(t)\, c'\, e^{-c' t}, \\ \dot{z} &= -g\, e^{c' t}, \\ z(t) &= v_0 + \frac{g}{c'} \left(1 - e^{c' t}\right) \\ v(t) &= \left(v_0 + \frac{g}{c'}\right) e^{-c' t} - \frac{g}{c'}. \end{aligned}$$

This has a terminal velocity $v_\infty = -g/c' = -gm/c$.

A large, light object has small mass m and large coefficient of friction c, so large c'. Thus, the term $e^{-c' t}$ decays rapidly to zero and the object comes close to its terminal velocity quickly. ■

Example 7 (Radioactive Decay). (Based on Borrelli and Coleman) Consider a situation where material contains uranium-234 that undergoes radioactive decay to thorium-230 with a half-life of approximately $\tau_1 = 2 \times 10^5$ years. In turn, thorium-230 undergoes radioactive decay with a half-life of approximately $\tau_2 = 8 \times 10^4$. Let x be the amount of uranium-234 and y be the amount of thorium-230. Since the decay rate is proportional to the amount of substance present, $\dot{x} = -k_1\, x$. The rate of change of thorium-230 is determined by the amount created from uranium-234, $k_1\, x$, and the amount lost through decay, $-k_2\, y$. Thus, the situation is modeled by a cascade of differential equations,

$$\begin{aligned} \dot{x} &= -k_1\, x, \\ \dot{y} &= k_1\, x - k_2\, y, \end{aligned}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$. (We discuss below how the rate constants are determined in terms of the half-life.)

$$\begin{aligned} x(t) &= x_0\, e^{-k_1 t}, \qquad \text{then} \\ \dot{y} &= -k_2\, y + k_1 x_0\, e^{-k_1 t}, \\ y(t) &= y_0\, e^{-k_2 t} + e^{-k_2 t} \int_0^t e^{k_2 s} k_1 x_0\, e^{-k_1 s}\, ds \\ &= y_0\, e^{-k_2 t} + k_1 x_0\, e^{-k_2 t} \int_0^t e^{(k_2 - k_1)\, s}\, ds \\ &= y_0\, e^{-k_2 t} + \frac{k_1 x_0}{k_2 - k_1} e^{-k_2 t} \left[e^{(k_2 - k_1)\, t} - 1\right] \\ &= \left[y_0 - \frac{k_1 x_0}{k_2 - k_1}\right] e^{-k_2 t} + \frac{k_1 x_0}{k_2 - k_1} e^{-k_1\, t} \end{aligned}$$

For uranium-234, the half-life is the time when half the original amount is left:

$$\begin{aligned} \frac{1}{2} x_0 &= x_0\, e\text{-}k_1 \tau_1, \\ k_1 \tau_1 &= \ln(2). \end{aligned}$$

In the same way, $k_2 \tau_2 = \ln(2)$.

Because we have aggregated the amount of each of the two radioactive materials, this model is considered a *two compartmental model.* ■

Problem 1. Find the solution of the differential equation $\dot{x} = 2x$, which satisfies the following initial conditions: **a.** $x(0) = 0$; **b.** $x(0) = -1$; **c.** $x(2) = 3$.

Problem 2. Find the solution of the differential equation $\dot{x} = 2tx$, which satisfies the initial condition $x(0) = 1$.

Problem 3. Find the solution of the differential equation $\dot{x} = \cos(t)\,x$, which satisfies the initial condition $x(-\pi/2) = -1$.

Problem 4. Find the solution of the differential equation $\dot{x} = 2tx + t$, which satisfies the condition $x(0) = 0$. *Hint:* Use the formula of Theorem 1.

Problem 5. For $k = -2, -1, 0, 1, 2$ find the solution (for $t > 0$) of the differential equation $\dot{x} = x/t + t$, which satisfies the initial condition $x(1) = k$. Graph these solutions in the (t, x) plane. What happens to these solutions when $t \to 0$? Notice that at $t = 0$ the function $a(t) = 1/t$ is undefined!

Problem 6. Consider the nonhomogeneous linear equation (NH) $\dot{x} = a(t)x + g(t)$ with the associated homogeneous linear equation (H) $\dot{x} = a(t)x$.

- **a.** If $x_p(t)$ is one (particular) solution of the nonhomogeneous equation (NH) and $x_h(t)$ is a solution of the homogeneous equation (H), show that $x_p(t) + x_h(t)$ is a solution of the nonhomogeneous equation (NH).
- **b.** Assume that $x_p(t)$ is a particular solution of the nonhomogeneous equation (NH). Show that the general solution of the nonhomogeneous equation (NH) is $x_p(t) + Ce^{b(t)}$ where $b(t)$ is given as in Theorem 1 and C is an arbitrary constant. *Hint*: For any solution $x(t)$ of (NH), show that $x(t) - x_p(t)$ satisfies (H).

Problem 7. For the supply $q_s = a + bp$ and demand $q_d = c - ep$ with scalars a, b, c, e, $f > 0$, consider a continuous price adjustment given by

$$\frac{dp}{dt} = f(q_d - q_s) = -f(e + b)p + f(c - a).$$

- **a.** Assuming that $c > a$, find the equilibrium p^* where $\dot{p} = 0$.
- **b.** Find an explicit expression for the solution $p(t)$ such that $p(0) = p_0 > 0$.

2. Existence of Solutions for Nonlinear Equations

Before considering a few solution methods for nonlinear equations, we discuss the existence and uniqueness of solutions. The rate function $f(t, x)$ is often define for all t and x, but sometimes that is not the case. So we define an open set. (Also see page 98 in [4].)

Definition. A subset $\mathscr{D}$ of $\mathbb{R}^2$ is *open* provided that for each $(t_0, x_0) \in \mathscr{D}$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $\{\,(t, x) : t_0 - \delta_1 < t < t_0 + \delta_1,\ \ x_0 - \delta_2 < x < x_0 + \delta_2\,\} \subset \mathscr{D}$.

The following theorem generalizes Theorem 3.1 in [5] to include the case for time dependent differential equations. It states the existence and uniqueness of solutions, as well as the continuity and differentiability of solutions with respect to initial conditions.

Theorem 2 (Existence and Uniqueness for Scalar Differential Equations). *Consider a scalar differential equation $\dot{x} = f(t, x)$, where $f : \mathscr{D} \to \mathbb{R}$ is a continuous function on an open subset $\mathscr{D}$ of $\mathbb{R}^2$ such that $\dfrac{\partial f}{\partial x}(t, x)$ is also continuous.*

a. *For an initial condition $(t_0, x_0) \in \mathscr{D}$, there exists a solution $x(t)$ to $\dot{x} = f(t, x)$ such that $(t, x(t)) \in \mathscr{D}$ for some time interval $t_0 - \delta < t < t_0 + \delta$ and $x(t_0) = x_0$. Moreover, the solution is unique in the sense, that if $x(t)$ and $y(t)$ are two such solutions with $x(t_0) = x_0 = y(t_0)$, then they must be equal on the largest interval of time about $t = t_0$ where both solutions are defined. Let $\phi(t; t_0, x_0) = x(t)$ be this unique solution with $\phi(t_0; t_0, x_0) = x(t_0) = x_0$.*

b. *The solution* $\phi(t; t_0, x_0)$ *depends continuously on the initial conditions* (t_0, x_0)*. Moreover, let (i)* $T > 0$ *be a time for which* $\phi(t; t_0, x_0)$ *is defined for* $t_0 - T \le t \le t_0 + T$ *and (ii) let* $\epsilon > 0$ *be any bound on the distance between solutions. Then, there exists a* $\delta > 0$ *which measures the distance between allowable initial conditions, such that if* $|x_0' - x_0| < \delta$ *and* $|t_0' - t_0| < \delta$*, then* $\phi(t; t_0', x_0')$ *is defined for* $t_0 - T \le t \le t_0 + T$ *and*

$$\left|\phi(t; t_0', x_0') - \phi(t; t_0, x_0)\right| < \epsilon \qquad \text{for } t_0 - T \le t \le t_0 + T.$$

c. *In fact, the solution* $\phi(t; t_0, x_0)$ *depends differentiably on the initial condition,* x_0*.*

The proof follows of this theorem follows from the multidimensional version of the preceding theorem in in Section 3.3 of [5]. Note for a linear equation, the solution exists and is unique on the interval **I** on which the coefficients $a(t)$ and $b(t)$ exist and are continuous.

Example 3.3 in [5] discusses the example $f(x) = \sqrt[3]{x}$, for which the solutions are not unique; both $x_1(t) = \left(2t/3\right)^{\frac{3}{2}}$ and $x_2(t) \equiv 0$ are solutions for $t \ge 0$ with $x_1(0) = 0 = x_2(0)$. Note that $f'(0)$ does not exist.

In the case when the solutions exist and are unique and the right hand side depends only on x, $\dot{x} = f(x)$, we write $\phi(t; x_0)$ for $\phi(t; 0, x_0)$ and call it the *flow* of the differential equation. The uniqueness of solutions implies that the flow satisfies the group property $\phi(t; \phi(t_1; x_0) = \phi(t+t_1; x_0)$. See Figure 2 of Section 3.1.1 of [5].

2.1. **Solutions Tangent to Slope Field.** If a solution $x(t)$ of $\dot{x} = f(t, x)$ is plotted in (t, x)-plane, the slope of a tangent line at a point $(t, x(t))$ is $f(t, x(t))$. Therefore, if we plot vectors at points (t, x) with slopes $f(t, x)$, a solution curve is tangent to these vectors. Figure 4 gives such a slope field and several solution curves for the logistic equations $\dot{x} = x(1 - x)$. Note that the uniqueness of solutions implies that the plot of two different solution curves cannot cross in the (t, x)-plane.

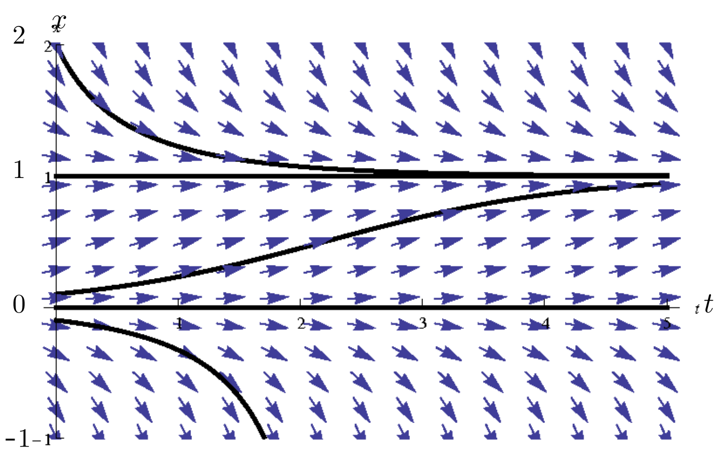

FIGURE 4. Slope field and several solutions for $\dot{x} = x(1 - x)$

3. **Separation of Variables**

The solution of a linear differential equation can be found by means of an integral. In this section, we consider another class of equations that can be solved by means of integrals.

A nonlinear equation of the form $\dot{x} = f(x)g(t)$ is called *separable* because it can be written as

$$\frac{1}{f(x)}\,\frac{dx}{dt} = g(t),$$

where the left side depends only on x and the right side depends only on t. Integrating with respect to t and applying the change of variables to the left hand side, the term $\frac{dx}{dt}\,dt$ changes the integral into one with respect to x,

$$\int \left(\frac{1}{f(x)}\right) dx = \int g(t)\,dt.$$

The result of these integrals gives an implicit solution with some function of x equal to some function of t. In general, it is difficult to get an explicit solution by solving this implicit relation for x.

Example 8. Consider the equation

$$\dot{x} = t\,x^4.$$

We solve this equation by the method of separation of variables, that converts it into a problem of integrals: taking the term involving x to left side by dividing by x^2 we get

$$\frac{\dot{x}}{x^4} = t.$$

Integrating with respect to t, the term $\dot{x}\,dt$ changes it to and integral with respect to x:

$$\frac{1}{2}t^2 + C = \int t\,dt = \int \frac{\dot{x}}{x^4}\,dt = \int \frac{1}{x^4}\,dx = -\frac{1}{3x^3}.$$

Solving for x in terms of t yields

$$x^3 = \frac{-1}{\frac{3}{2}\,t^2 + 3C} \qquad \text{and} \qquad x(t) = \left[\frac{1}{-\frac{3}{2}\,t^2 - 3C}\right]^{\frac{1}{3}}.$$

The initial conditions at $t = 0$ satisfies $x_0 = {}^{-1}/(3C)^{\frac{1}{3}}$ or $-3C = {}^{1}/x_0^3$, so

$$x(t) = \left[\frac{1}{\frac{1}{x_0^3} - \frac{3}{2}\,t^2}\right]^{\frac{1}{3}} = \left[\frac{2\,x_0^3}{2 - 3\,t^2\,x_0^3}\right]^{\frac{1}{3}}.$$

Note that for $x_0 > 0$, this becomes undefined for $t = \pm\left[{}^{2}/3x_0^3\right]^{\frac{1}{2}}$. Thus the solution is not defined for all time. So while linear equations with coefficients defined for all time have solutions that are defined for all time, this is not true for nonlinear equations. ■

Example 9. Consider $\dot{x} = \dfrac{3 + 2t}{4x^3 - 2x - 5}$. The solutions satisfy

$$\int 4\,x^3 - 2\,x - 5\;dx = \int 3 + 2t\;dt + C$$

$$x^4 - x^2 - 5\,x = 3\,t + t^2 + C.$$

This is an implicit solution and cannot be solved for x in terms of t. This is another difference between linear and nonlinear equations: Linear differential equations always have explicit solutions while we cannot always find one for a nonlinear differential equation. ■

Problem 8. Solve the nonlinear differential equation $\dot{x} = 3\sqrt{tx}$ by separation of variables.

Problem 9. Consider the equation $\dot{x} = \dfrac{t^2}{x^2(1 + t^3)}$ with $x(0) = 2$. Solve by separation of variables. Then, solve for x in terms of t. What interval of t (that includes $t = 0$) is the solution defined? *Note* the rate function becomes infinite when $x(t) = 0$ but the explicit form of the solution continues through this value.

3.1. **Growth Models.** We consider some population growth models and related problems that can be solved by separation of variables.

Example 10 (Logistic Equation). Consider a single population measured by the variable x. If there is competition within a population for resources (crowding), then the *growth rate per unit population*, $\dot{x}/x$, would decrease as the population increases. For the simplest model with crowding, this rate decreases linearly, or $\dot{x}/x = r - cx = r\,(1 - cx/r)$ with $r,\, c > 0$. Letting $K = r/c > 0$, we get the nonlinear scalar differential equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right),$$

which is called the *logistic differential equation.* The time derivative $\dot{x} = 0$ for $x = 0$ and K, with corresponding constant solutions $\phi(t;0) \equiv 0$ and $\phi(t;K) \equiv K$. These points where $\dot{x} = 0$ are called *fixed points* or *equilibria* or *steady states* of the differential equation. For $x \neq 0, K,$ applying separation of variables, we take all the terms involving x to the left side, we get

$$\frac{K\,\dot{x}}{x\,(K - x)} = r.$$

Using the method of partial fractions with constants A and B to be determined,

$$\frac{K}{x\,(K-x)} = \frac{A}{x} + \frac{B}{K-x} = \frac{AK - Ax + Bx}{x\,(K-x)},$$

so $K = AK$ and $0 = B - A,$ or $B = A = 1.$ Thus we get the differential equation

$$\frac{\dot{x}}{x} + \frac{\dot{x}}{K-x} = r.$$

Integrating with respect to t, the term $\dot{x}\,dt$ changes it to and integral with respect to x:

$$\begin{aligned} \int \frac{1}{x}\,dx + \int \frac{1}{K-x}\,dx &= \int r\,dt, \\ \ln(|x|) - \ln(|K-x|) &= rt + C_1, \qquad \text{and} \\ \frac{|x|}{|K-x|} &= C\,e^{rt}, \qquad \text{where } \; C = e^{C_1}. \end{aligned}$$

Assuming $0 < x < K$ so we can drop the absolute value signs, we can solve for x:

$$\begin{aligned} x &= CKe^{rt} - Ce^{rt}x, \\ (1 + Ce^{rt})x &= CKe^{rt}, \qquad \text{and} \\ x &= \frac{CKe^{rt}}{1 + Ce^{rt}} = \frac{CK}{C + e^{-rt}}. \end{aligned}$$

If x_0 is the initial condition at $t = 0$, then some more algebra shows that $C = x_0/(K - x_0),$ so

$$\phi(t;x_0) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}.$$

A direct calculation shows that this form of the solution is valid for any x_0 and not just those with $0 < x_0 < K$. See Figure 5.

A solution $\phi(t;x_0)$ can be continued for a maximal interval of definition $t^-_{x_0} < t < t^+_{x_0}$. Given the form of the solution, it can be continued until the denominator becomes zero. For $0 \leq x_0 \leq K,$ the denominator is never zero and $\phi(t;x_0)$ is defined for all $t,$ $t^-_{x_0} = -\infty < t < \infty = t^+_{x_0}$. On the

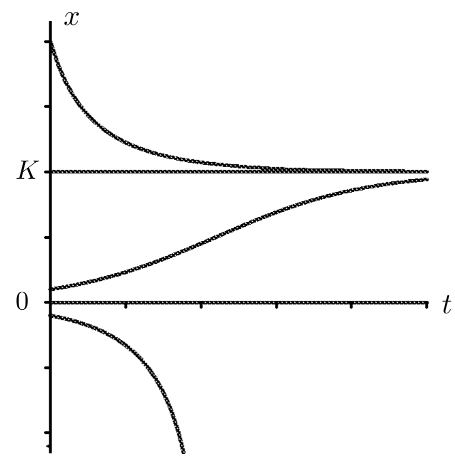

FIGURE 5. Logistic Equation: plot of $x = \phi(t; x_0)$ as a function of t for several x_0

other hand, if $x_0 > K > 0$, then the denominator is zero for

$$e^{rt} = \frac{x_0 - K}{x_0} < 1 \qquad \text{and}$$

$$t^-_{x_0} = \frac{1}{r}\left(\ln(x_0 - K) - \ln(x_0)\right) < 0,$$

while $t^+_{x_0} = \infty$. Finally, if $x_0 < 0$, then $t^-_{x_0} = -\infty$ and

$$t^+_{x_0} = \frac{1}{r}\left(\ln(K - x_0) - \ln(|x_0|)\right) > 0.$$

Thus, some solutions are defined for all time, others for a bounded forward time, and others for a bounded backward time.

Some of the long term forward behavior can be determined by looking at only the form of the rate function and not using the solution itself. For an initial condition $0 < x_0 < K$, $\dot{x} > 0$ along the solution $\phi(t; x_0)$, and $\phi(t; x_0)$ increases toward K. Also, for $x_0 > K$, $\dot{x} < 0$ along the solution, and $\phi(t; x_0)$ decrease toward K. So we can conclude that for any $x_0 > 0$, even without solving the differential equation, $\phi(t; x_0)$ tends toward K as t goes to infinity. For this reason, K is called the *carrying capacity*.

Notice that the fact that $f'(K) = r - 2r = -r < 0$ ensures that $\dot{x} > 0$ for $x < K$ and x near K, and that $\dot{x} < 0$ for $x > K$ and x near K; therefore, the fact that $f'(K) < 0$ is enough to ensure that the fixed point $x = K$ is attracting from both sides. ■

Definition. Assume that the rate function $f(x)$ is continuous with a continuous partial derivative with respect to x. Since any two solutions of $\dot{x} = f(x)$ with the same initial condition x_0 agree on their common interval of definition, we can continue $\phi(t; x_0)$ to some *maximal interval of definition* $t^-_{x_0} < t < t^+_{x_0}$. As the logistic equation illustrates, sometimes $t^+_{x_0}$ is infinity and other times it can be a finite positive value. Similarly, sometimes $t^-_{x_0}$ is minus infinity and other times it can be a finite negative value.

The only way that a solution is not defined for all time is that the solution goes to infinity or a point where the differential equation is not defined. See the following theorem. Then, if the solution is defined for all time and is bounded it must converge to a fixed point by the argument used in the next to last paragraph of the example of the logistic equation. (See Theorem 4.4 of [5].)

Theorem 3. *Consider a scalar autonomous differential equation $\dot{x} = f(x)$ on $\mathbb{R}$, for which $f(x)$ has a continuous derivative. Assume that $x(t) = \phi(t; x_0)$ is the solution, with initial condition x_0. Assume that the maximum interval containing 0 for which it can be defined is (t^-, t^+).*

a. *Further assume that the solution $\phi(t;x_0)$ is bounded for $0 \le t < t^+$, i.e., there is a constant $C > 0$ such that $|\phi(t;x_0)| \le C$ for $0 \le t < t^+$. Then as t converges to t^+, $\phi(t;x_0)$ must converge either to a fixed point or to a point where $f(x)$ is undefined.*

b. *Similarly, if the solution $\phi(t;x_0)$ is bounded for $t^- < t \le 0$, then, as t converges to t^-, $\phi(t;x_0)$ must converge either to a fixed point or to a point where $f(x)$ is undefined.*

c. *Assume that $f(x)$ is defined for all x in $\mathbb{R}$. (i) If $f(x_0) > 0$, assume that there is a fixed point $x^* > x_0$, and in fact, let x^* be the smallest fixed point larger than x_0. (ii) If $f(x_0) < 0$, assume that there is a fixed point $x^* < x_0$, and in fact, let x^* be the largest fixed point less than x_0. Then, $t^+ = \infty$ and $\phi(t;x_0)$ converges to x^* as t goes to infinity.*

The consequence of the sign of the derivative at a fixed point is given in the following theorem. The proof is basically that given in the See Theorem 4.5 of [5].

Theorem 4. *Assume that x^* is a fixed point for the autonomous scalar differential equation $\dot{x} = f(x)$, where f and f' are continuous.*

a. *If $f'(x^*) < 0$, then x^* is an attracting fixed point.*
b. *If $f'(x^*) > 0$, then x^* is a repelling fixed point.*
c. *If $f'(x^*) = 0$, then the derivative does not determine the stability type.*

Example 11 (**Harvesting**)**.** Assume that from a population that is governed by the logistic equation that the population is decreases at a rate of $-h < 0$ due to external aspects, e.g., from harvesting or fishing:

$$\dot{x} = r\,x\left[1 - \frac{x}{K}\right] - h = f(x).$$

The fixed points satisfy $rx - (^r/_K)\,x^2 - h = 0$ or $rx^2 - rKx + hK = 0$, and are

$$x_\pm = \frac{rK \pm \sqrt{r^2K^2 - 4rhK}}{2r} = \frac{1}{2}\,K \pm \frac{1}{2r}\sqrt{r^2K^2 - 4rhK}.$$

If $rK > 4h$, then both roots are real with $0 < x_- < x_+$ and

$$f(x) = \begin{cases} < 0 & \text{for } \; x < x_- \\ > 0 & \text{for } \; x_- < x < x_+ \\ < 0 & \text{for } \; x_+ < x. \end{cases}$$

By Theorem 4.4(c) in [5], if $\phi(t;x_0)$ is bounded then it must converge to a fixed point. Without solving the equation explicitly, we can see that

(i) if $0 \le x_0 < x_-$, then $x(t)$ decreases down to 0 in finite time as t increases,
(ii) if $x_- < x_0 < x_+$, then $x(t)$ increases up to x_+ as $t \to \infty$,
(ii) if $x_+ < x_0$, then $x(t)$ decreases down to x_+ as $t \to \infty$.

Therefore, $x = x_-$ is a threshold for survival and $x = x_+$ is the steady state population for a population large enough to survive, i.e. $x_0 > x_-$. ■

A discussion of the effect of harvesting with other models of population growth is given in Chapter 1 of [3] by Brauer and Castillo-Chávez.

Example 12 (**Economic Growth**)**.** The Solow-Swan model of economic growth is given by

$$\begin{aligned} \dot{K} &= s\,A\,K^a L^{1\text{-}a} - \delta\,K \\ \dot{L} &= n\,L, \end{aligned}$$

where K is the capital, L is the labor force, $A\,K^a L^{1\text{-}a}$ is the production function with $0 < a < 1$ and $A > 0$, $0 < s \le 1$ is the rate of reinvestment of income, $\delta > 0$ is the rate of depreciation of capital, and $n > 0$ is the rate of growth of the labor force. A new variable $x = {}^{K}\!/_{L}$ is introduced that is the capital per capita (of labor). The differential equation that x satisfies is as follows:

$$\begin{aligned}\dot{x} &= \frac{1}{L}\,\dot{K} - \frac{K}{L^2}\,\dot{L} \\ &= \frac{1}{L}\,sA\,K^a L^{1-a} - \frac{1}{L}\,\delta\,K - \frac{K}{L^2}\,n\,L \\ &= sA\,x^a - (\delta + n)\,x \\ &= x^a\left[sA - (\delta + n)\,x^{1-a}\right].\end{aligned}$$

Notice the similarity to the logistic equation. The equilibrium where $\dot{x} = 0$ and $x > 0$ occurs for

$$sA = (n+\delta)x^{1-a}, \qquad \text{or}$$

$$x^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-a}}.$$

For $0 < x < x^*$, $\dot{x} > 0$ and x increases toward x^*. For $x > x^*$, $\dot{x} < 0$ and x decreases toward x^*. It can be shown that for any initial capital $x_0 > 0$, the solution $x(t; x_0)$ limits to x^* as t goes to infinity. Therefore in this model, all solutions with $x_0 > 0$ tend to the steady state x^* of the capital to labor ratio. ■

Problem 10. Consider $\dot{x} = x^2 - 1 = (x+1)(x-1)$.

- **a.** Solve the the nonlinear differential equation by separation of variables. *Hint:* Imitate the solution method of Example 10.
- **b.** Discussion the limit of $x(t)$ as t goes to infinity for different ranges of $x_0 > 0$.the nonlinear differential equation

Problem 11. Consider the differential equations

$$\dot{x} = r\,x\left(1 - \frac{x}{K}\right)\left(\frac{x}{T} - 1\right).$$

with $0 < T < K$ and $r > 0$. The factor $\frac{x}{T} - 1$ is negative for $x < T$, so adds a threshold to the population growth model

- **a.** Find the fixed points and the sign of $\dot{x}$ between the fixed points.
- **b.** Discussion the limit of $x(t)$ as t goes to infinity for different ranges of $x_0 > 0$.

3.2. **The Modeling Process: Differential Systems.** (This section is extracted from Section 1.4 of [1], with a few word changes to reflect the models we have considered.) The essential aspects of a mathematical model using ordinary differential equations is as follows.

Natural Variables: A natural process is described by a collection of *natural variables* that depend on a single independent variable. In all our examples, time t is the independent variable. The natural variables we have considered are (i) amount of money (Example 4), (ii) temperature (Example 5), (iii) height, velocity, and acceleration (Example 6), (iv) population (Example 10), and (v) capital and labor (Example 12).

Natural Laws: A natural process evolves in time according to *natural lows* or *principles* involving the natural variables. Sometimes the laws arise empirically (e.g., the acceleration of a following body in Example 6), and sometimes they arise from some deep scientific theory of the laws of nature. Sometimes, the evolution is a model that has some of the properties of observed phenomenon (e.g., the situation for Economic Growth in Example 12).

Forcing or Driving Terms: In some cases, there are factors in the external environment that affect the rate of change. These can depend of time making the differential equation nonautonomous. In Example 4, money was added at a constant rate g. In Example 11, the populations is harvested at a rate -h. Example 3 has a periodic forcing terms but was not derived by modeling a physical situation.

Natural Parameters: Natural laws often contain *parameters* that are constants which someone must experimentally determine (e.g., the half-life of a radioactive substance or the gravitational constant.)

4. **Exact Equations**

In this section, we consider differential equations in a little different form,

$$M(t,x) + N(t,x)\,\dot{x} = 0, \tag{2}$$

where $M(t,x)$ and $N(t,x)$ are functions of both t and x. This differential equation is called *exact* provided that there is a function $G(t,x)$ such that $G_t(t,x) = M(t,x)$ and $G_x(t,x) = N(t,x)$.

Note that if an equation is exact and a curve $\mathbf{r}(t) = (t, x(t))$ is on a level set of G, where we consider x as a function of t, then

$$0 = \frac{d}{dt}G(\mathbf{r}(t)) = G_t(t,x) + G_x(t,x)\,\dot{x} = M(t,x) + N(t,x)\,\dot{x},$$

and $x(t)$ is a solution of the differential equation.

Recall from vector calculus that a vector field $\mathbf{F}$ is a gradient (conservative), $\nabla G = \mathbf{F}$, if and only if $\nabla \times \mathbf{F} = \mathbf{0}$. See Theorem 3.5 of Chapter 6 in [4]. In $\mathbb{R}^2$ with $\mathbf{F}(t,x) = M(t,x)\,\mathbf{i} + N(t,x)\,\mathbf{j}$, this condition is $N_t(t,x) - M_x(t,x) \equiv 0$. (In this section we write subscripts for the partial derivatives with respect to the variable indicated.) This criterion for a gradient vector field calculus gives us the following theorem.

Theorem 5. *Let the function $M(t,x)$, $N(t,x)$, $M_x(t,x)$, and $M_t(t,x)$ be continuous, where the subscripts denote partial derivatives. Then equation (2) is exact if and only if*

$$M_x(t,x) = N_t(t,x) \tag{3}$$

at all points (t,x).

Besides referring to the theorem on gradient vector fields, we sketch a proof after giving an example that introduces the way of finding the conserved function G, and so implicit solutions for exact equations.

Example 13. Consider the scalar differential equation

$$2t + x^2 + 2tx\,\dot{x} = 0.$$

This differential equation is neither linear nor separable, so we check whether it is exact:

$$\frac{\partial}{\partial x}(2t + x^2) = 2x = \frac{\partial}{\partial t}(2tx).$$

The function G needs to satisfy

$$\begin{aligned} \frac{\partial G}{\partial t} &= 2t + x^2, \qquad \text{so} \\ G(t,x) &= \int 2t + x^2\,dt \quad + \, h(x) = t^2 + t\,x^2 + h(x), \end{aligned}$$

where $h(x)$ is a function of only x. (The constant of integration is independent of t and only depends on x.) The partial derivative of G with respect to x needs to satisfy

$$\begin{aligned} \frac{\partial G}{\partial x} &= 0 + 2tx + h'(x) = 2tx, \qquad \text{so} \\ h'(x) &= 0, \end{aligned}$$

and $h(x)$ is a constant. Taking the constant equal to zero, the solutions lie on the level curves of $G(t,x) = t^2 + t\,x^2$. A solution satisfies $t\,x^2(t) = C - t^2$, or

$$x(t) = \pm\sqrt{\frac{C}{t} - t}.$$

■

Proof of theorem. If the equation is exact and $G(t,x)$ exists with $G_t = M$ and $G_x = N$, then

$$M_x = G_{tx} = G_{xt} = N_t,$$

and the terms satisfy equation (3).

If the terms satisfy equation (3), then we imitate the solution method of the example. Let

$$G_t = M(t,x), \qquad \text{so}$$
$$G(t,x) = \int M(t,x)\,dt + h(x).$$

To be exact, we need

$$N(t,x) = G_x = \int M_x(t,x)\,dt + h'(x), \qquad \text{so}$$
$$h'(x) = N(t,x) - \int M_x(t,x)\,dt.$$

Since the partial derivative of the right hand side with respect to t is zero, $0 = N_t(t,x) - M_x(t,x)$, this equation can be solved for $h(x)$ as a function of x alone. □

Problem 12. Using the theorem, determine whether the following equations are exact. If they are exact, solve for the function $G(t,x)$ that is constant along solutions.

a. $(3t^2 - 2tx + 2) + (6x^2 - t^2 + 3)\,\dot{x} = 0$
b. $(2t + 4x) + (2t - 2x)\,\dot{x} = 0$
c. $(at - bx) + (bt + cx)\,\dot{x} = 0$
d. $(at + bx) + (bt + cx)\,\dot{x} = 0$
e. $2t\sin(x) + t^2\cos(x)\,\dot{x} = 0$
f. $tx^2 + (2t^2x + 2x)\,\dot{x} = 0$

4.1. **Integrating Factors.** For linear equations, we often had to multiply the equation by of function of t in order the change the problem into one we could solve by integrals. In the same way, we can sometimes find a function μ of x and t, called an *integrating factor*, such that $\mu\,M + \mu\,N\,\dot{x} = 0$ becomes exact.

Example 14. Consider $x^2 - t^2\dot{x} = 0$. This equation is not exact since

$$\frac{\partial}{\partial x}(x^2) = 2x \neq \text{-}2t = \frac{\partial}{\partial t}(\text{-}t^2).$$

We want an expression μ such that $\frac{\partial}{\partial x}(\mu\,x^2) = \frac{\partial}{\partial t}(\text{-}\mu\,t^2)$. If we divide by t^2x^2, as we would do using the method of separation of variables, then

$$\frac{\partial}{\partial x}(t^{\text{-}2}) = 0 = \frac{\partial}{\partial t}(\text{-}x^{\text{-}2}).$$

Therefore $\mu = t^{\text{-}2}x^{\text{-}2}$ is an integrating factor, and $t^{\text{-}2} - x^{\text{-}2}\dot{x} = 0$ is exact. Applying the method as before, we find that $G(t,x) = t^{\text{-}3} - x^{\text{-}3}$ is constant along solutions. ■

Even when an integrating factor exists, it is not easy to find. It needs to satisfy

$$\begin{aligned} \frac{\partial}{\partial t}(\mu\, N) &= \frac{\partial}{\partial x}(\mu\, M) \\ \mu_t\, N + \mu\, N_t &= \mu_x\, M + \mu\, M_x \\ \mu\, (N_t - M_x) &= M\, \mu_x - N\, \mu_t. \end{aligned}$$

We often try for a μ that is a function of only t or only x.

If μ is a function of only t, then $\mu_x = 0$ and

$$\begin{aligned} \frac{\mu_t}{\mu} &= \frac{M_x - N_t}{N} \\ \ln(\mu) &= \int \frac{M_x - N_t}{N}\, dt. \end{aligned}$$

For this expression to yield of a function of only t, the integrand must be independent of x.

If μ is a function of only x, then $\mu_t = 0$ and

$$\begin{aligned} \frac{\mu_x}{\mu} &= \frac{N_t - M_x}{M} \\ \ln(\mu) &= \int \frac{M_x - N_t}{N}\, dx. \end{aligned}$$

For this expression to yield of a function of only x, the integrand must be independent of t.

Example 15. Consider $(3t^2x + 2tx + x^3) + (t^2 + x^2)\,\dot{x} = 0$. Then $M_x - N_t = (3t^2 + 2t + 3x^2) - (2t) = 3t^2 + 3x^2 \neq 0$, so the equation is not exact. If we tried for an integrating factor that is a function of x alone,

$$\frac{\mu_x}{\mu} = \frac{N_t - M_x}{M} = \frac{\text{-}(3t^2 + 3x^2)}{3t^2x + 2tx + x^3}$$

is not a function of x along, so this is not possible. To see if we can find an integrating factor that is a function of t alone,

$$\begin{aligned} \frac{\mu_t}{\mu} &= \frac{M_x - N_t}{N} = 3, \qquad \text{so} \\ \ln(\mu) &= 3t, \\ \mu &= e^{3t}. \end{aligned}$$

Thus, we have found an integrating factor and need

$$\begin{aligned} G_x &= e^{3t}\,(t^2 + x^2), \\ G &= t^2x\, e^{3t} + \tfrac{1}{3}x^3e^{3t} + h(t), \\ G_t &= 2tx\, e^{3t} + 3t^2x\, e^{3t} + x^3e^{3t} + h'(t) = e^{3t}\left(3t^2x + 2tx + x^3\right), \\ h'(t) &= 0. \end{aligned}$$

Therefore, $G(t,x) = t^2x\, e^{3t} + \frac{1}{3}x^3e^{3t}$ is constant along solutions. ■

Problem 13. For the following differential equations, find an integrating factor that is a function of a single variable.

a. $x + (2t - xe^x)\,\dot{x} = 0.$

b. $(3t^2x + 2tx + x^3) + (t^2 + x^2)\,\dot{x} = 0.$

c. $\left[\frac{4t^3}{x^2} + \frac{3}{x}\right] + \left[\frac{3t}{x^2} + 4x\right]\dot{x} = 0.$

d. $1 + \left(\frac{t}{x} - \sin(x)\right)\dot{x} = 0.$

5. **Second Order Scalar Equations**

This section depends on the solutions of linear systems with constant coefficients and Sections 2.1 – 2.2 from [5] should be covered first. We start by showing how a second order scalar linear differential equations can be changed into a system of linear differential equations.

Consider

$$a\,\ddot{y} + b\,\dot{y} + c\,y = 0, \tag{4}$$

where a, b, and c are constants with $a \neq 0$. "Solve" means we are looking for a C^2 function $y(t)$ which satisfies the above equation. This equation is called *second order* since it involves derivatives up to order two. Assume that $y(t)$ is a solution (4), set $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, and consider the vector $\mathbf{x}(t) = (x_1(t), x_2(t))^T = (y(t), \dot{y}(t))^T$. Then

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{b}{a}x_2(t) - \frac{c}{a}x_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

since $\ddot{y}(t) = -\left({}^{b}\!/\!{}_{a}\right)\dot{y}(t) - \left({}^{c}\!/\!{}_{a}\right)y(t) = -\left({}^{b}\!/\!{}_{a}\right)x_2(t) - \left({}^{c}\!/\!{}_{a}\right)x_1(t)$. We have shown that if $y(t)$ is a solution of the equation (4) then $\mathbf{x}(t) = (x_1(t), x_2(t))^T = (y(t), \dot{y}(t))^T$ is a solution of the equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}. \tag{6}$$

Notice that the characteristic equation of (6) is $\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0$ or $a\lambda^2 + b\lambda + c = 0$, which has a simple relationship with the original second order equation (4). Since we have to specify initial conditions of both $x_1(t_0)$ and $x_2(t_0)$ for the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ given by (6), we have to specify initial conditions of both $y(t_0) = x_1(t_0)$ and $\dot{y}(t_0) = x_2(t_0)$ for the second order equation $\ddot{y} + a\dot{y} + by = 0$, i.e., both position and velocity.

The following theorem relates the solutions of the linear system to the solutions of the second order scalar equation.

Theorem 6. *For $a \neq 0$, consider the second order differential equation*

$$a\,\ddot{y} + b\,\dot{y} + c\,y = 0.$$

- **a.** *If r_1 and r_2 are (real) distinct roots of the characteristic equation $a\,\lambda^2 + b\,\lambda + c = 0$, then $e^{r_1 t}$ and $e^{r_2 t}$ are two independent solutions of the second order scalar differential equation.*
- **b.** *If r is a repeated real root of $a\,\lambda^2 + b\,\lambda + c = 0$, then e^{rt} and te^{rt} are two independent solutions of the second order scalar differential equation.*
- **c.** *If $r = \beta + i\,\omega$ is a complex root (with $\omega \neq 0$), then $e^{\beta t}\cos(\omega t)$ and $e^{\beta t}\sin(\omega t)$ are two real independent solutions.*

Proof. (**a**) By the discussion above, r_1 and r_2 are roots of the characteristic equation for the linear system. The corresponding eigenvectors are solutions of the homogeneous equation for

$$\begin{bmatrix} -r_j & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r_j \end{bmatrix},$$

and so must be $(1, r_j)^T$. Thus

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \mathbf{x}(t) = \begin{bmatrix} 1 \\ r_j \end{bmatrix} e^{r_j t}$$

is a solution of the linear system. So,

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{array}\right] \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \qquad \text{so}$$

$$\ddot{y} = -\frac{c}{a}\,y - \frac{b}{a}\dot{y},$$

and $y(t) = e^{r_j t}$ is a solution of the second order scalar equation.

We calculate the Wronskian to see whether the solutions are independent:

$$W(t) = \det\left[\begin{array}{cc} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{array}\right] = (r_2 - r_1)e^{(r_1+r_2)t} \neq 0,$$

so the solutions are linearly independent by the result for linear systems.

(**b**) By part (a), e^{rt} is one solution. In a manner similar to the derivation of variable of parameters, we look for a second solution in the form $y(t) = u(t)\,e^{rt}$. Note that

$$\begin{aligned} \dot{y} &= u'\,e^{rt} + r\,u\,e^{rt} \\ \ddot{y} &= u''\,e^{rt} + 2\,r\,u'\,e^{rt} + r^2\,u\,e^{rt}. \end{aligned}$$

Then $y(t)$ is a solution if and only if

$$\begin{aligned} 0 &= a\,\left[u''\,e^{rt} + 2\,r\,u'\,e^{rt} + r^2\,u\,e^{rt}\right] + b\,\left[u'\,e^{rt} + r\,u\,e^{rt}\right] + c\,u\,e^{rt} \\ &= e^{rt}\left[a\,u'' + u'(2a\,r + b) + u\,(a\,r^2 + b\,r + c)\right] \\ &= e^{rt}\left[a\,u'' + u'(2a\,r + b)\right]. \end{aligned}$$

Because r is a repeated root, $r = {}^{-b}\!/_{2a}$ and $2ar + b = 0$. Thus, $u'' = 0$, $u' = k_1$ (a constant), and $u = k_1\,t + k_0$. Thus, all the solutions are of the form $k_0 e^t + k_1 t\,e^t$. The second solution is $t\,e^t$.

The Wronskian is

$$W(t) = \det\left[\begin{array}{cc} e^{rt} & t\,e^{rt} \\ re^{rt} & (1+rt)e^{rt} \end{array}\right] = (1 + rt - rt)e^{2rt} = e^{2rt} \neq 0,$$

so the solutions are independent.

(**c**) If $r = \beta + i\,\omega$ is a complex root, then $e^{(\beta + i\,\omega)t} = e^{\beta t}\cos(\omega t) + i\,e^{\beta t}\sin(\omega t)$ is a complex solution and the real and imaginary parts are real solutions, i.e., $e^{\beta t}\cos(\omega t)$ and $e^{\beta t}\sin(\omega t)$.

The two complex solutions $z_1(t) = e^{(\beta + i\,\omega)t}$ and $z_2(t) = e^{(\beta - i\,\omega)t} = e^{\beta t}\cos(\omega t) - i\,e^{\beta t}\sin(\omega t)$ are independent complex solutions by part (a), and

$$\begin{aligned} y_1(t) &= e^{\beta t}\cos(\omega t) = \tfrac{1}{2}\left[\,z_1(t) + z_2(t)\,\right] \qquad \text{and} \\ y_2(t) &= e^{\beta t}\sin(\omega t) = \tfrac{1}{2i}\left[\,z_1(t) - z_2(t)\,\right]. \end{aligned}$$

It follows that $y_1(t)$ and $y_2(t)$ are independent solutions. □

Remark. Since an equation of the form (4) can be converted to a system of the type (5) and vice versa, we do not need to consider equations of type (4) separately: We can use the solutions of (5) to give solutions (4). However, Theorem 6 shows that it is easier to find roots r_1 and r_2 of $a\,\lambda^2 + b\,\lambda + c = 0$ and then two solutions of (4) are $e^{r_j t}$ for $j = 1, 2$.

Example 16. Consider $\ddot{y} + 3\dot{y} + 2y = 0$. The characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ has roots $\lambda = -1$ and -2, so two independent solutions are e^{-t} and e^{-2t}.

Example 17. Consider $\ddot{y} + 4\dot{y} + 4y = 0$. The characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has a repeated roots $\lambda = -2, -2$. By the theorem, two independent solutions are e^{-2t} and $t\,e^{-2t}$.

Example 18. Consider $\ddot{y} + 9y = 0$. The characteristic equation $\lambda^2 + 9 = 0$ has complex roots $\lambda = \pm 3\,i$, and complex solutions are $e^{\pm 3it} = \cos(3t) \pm i\,\sin(3t)$. Taking the real and imaginary parts, $\cos(3t)$ and $\sin(3t)$ are two real solutions.

Problem 14. Consider the second order scalar differential equation $\ddot{y} - 5\dot{y} + 4y = 0$, with initial conditions $y(0) = 3$, $\dot{y}(0) = 6$.

- **a.** Write down the corresponding 2×2 linear system and solve it for the general vector solution and the solution with the given initial conditions. Use this vector solution to get the general scalar solution for second order scalar differential equation and the solution with the given initial conditions.
- **b.** Solve the second order scalar differential equation by a second direct way as follows: One solutions is $y_1(t) = e^t$. Find a second independent solution by looking for a solution of the form $y_2(t) = u(t)\, y_1(t)$. Find a second order differential equation that $u(t)$ satisfies by taking the derivatives of $u(t)\, y_1(t)$ and substitute them into the original second order differential equation. Find the general form of $u(t)$ by solving this diffential equation. What is the second independent solution?

Problem 15. Consider the second order scalar differential equation $\ddot{y} - 2\dot{y} + y = 0$.

- **a.** Write down the corresponding 2×2 linear system and solve it for the general vector solution. Use this general vector solution to get the general scalar solution for the second order scalar differential equation. Note that two solutions of the scalar equation are of the form e^{rt} and te^{rt} for the correct choice of r.
- **b.** Solve the second order scalar differential equation by a second direct way as follows: One solutions is $y_1(t) = e^t$. Find a second independent solution by looking for a solution of the form $y_2(t) = u(t)\, y_1(t)$. Find a second order differential equation that $u(t)$ satisfies by taking the derivatives of $u(t)\, y_1(t)$ and substitute them into the original second order differential equation. Find the general form of $u(t)$ by solving this diffential equation. What is the second independent solution?
- **c.** Find a solution of the scalar equation which satisfies the initial conditions $y(0) = 2$, $\dot{y}(0) = 5$.

Problem 16. Consider the differential equation $\ddot{y} - 4\dot{y} + 25y = 0$. Find two real solutions by looking for solutions of the form e^{rt}. *Hint*: For $r = a + ib$ complex, what are the real and imaginary parts of e^{at+ibt}? Do not write down the corresponding linear system.

Problem 17. Consider the 2×2 linear systems $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

- **a.** Write down the corresponding second order equation.
- **b.** Find the solutions of the second order equation directly as in problem 16.
- **c.** Compare the solutions of part (b) with the first coordinates of the solutions found using the exponential given in Example 2.3 in [5].
- **d.** Find the solution of the second order scalar equation that satisfies the initial conditions $y(0) = 2$, and $\dot{y}(0) = -1$.

Problem 18. Consider a mass m that is connected to a spring that has a restoring force proportional to the displacement from equilibrium $m\,\ddot{x} = -k\,x$, with $k,\, m > 0$. Let $\omega_0^2 = k/m$. If friction is added that is proportional to the velocity but with a force in the opposite direction, then the equations become $m\,\ddot{x} = -c\,\dot{x} - k\,x$ with $c > 0$, or

$$m\,\ddot{x} + c\,\dot{x} + k\,x = 0.$$

- **a.** For $c = 0$ and $k,\, m > 0$ find the general solution. What is the period before the motion come back to the original position and velocity?
- **b.** What is the form of the solution for (i) $0 < c < 2\sqrt{km}$, (ii) $c = 2\sqrt{km}$, and $c > 2\sqrt{km}$? Descibe the manner in which solutions converge toward the equilibrium with $x = 0 = \dot{x}$.

5.1. **Nonhomogeneous Second Order Scalar Equations.** The material depends on Section 2.3 in [5]. Consider the following nonhomogeneous second order scalar differential equation

$$\ddot{y} + b\,\dot{y} + c\,y = f(t). \tag{7}$$

and the corresponding nonhomogeneous linear system

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}. \tag{8}$$

For two independent solutions $y_1(t)$ and $y_2(t)$, a fundamental matrix solutions is

$$\mathbf{M}(t) = \begin{bmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{bmatrix},$$

with Wronskian

$$W(t) = \det(\mathbf{M}(t)) = y_1(t)\,\dot{y}_2(t) - y_2(t)\,\dot{y}_1(t)$$

and inverse

$$\mathbf{M}(t)^{-1} = \frac{1}{W(t)} \begin{bmatrix} \dot{y}_2 & -y_2 \\ -\dot{y}_1 & y_1 \end{bmatrix}.$$

By variation of parameters for linear systems (Theorem 2.7 in [5]),

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{M}(t)\mathbf{M}(0)^{-1}\mathbf{x}_0 + \mathbf{M}(t)\int_0^t \mathbf{M}(s)^{-1} \begin{bmatrix} 0 \\ f(s) \end{bmatrix}\, dt \\
&= \mathbf{M}(t)\mathbf{M}(0)^{-1}\mathbf{x}_0 + \mathbf{M}(t)\int_0^t \frac{1}{W(s)} \begin{bmatrix} -y_2(s)\,f(s) \\ y_1(s)\,f(s) \end{bmatrix}\, ds.
\end{aligned}$$

The first coordinate of the integral terms gives a particular scalar equation for the nonhomogeneous equation,

$$y_p(t) = -y_1(t)\int_0^t \frac{y_2(s)\,f(s)}{W(s)}\, ds + y_2(t)\int_0^t \frac{y_1(s)\,f(s)}{W(s)}\, ds.$$

Example 19. If two solutions are $y_1(t) = e^{\lambda t}$ and $y_2(t) = e^{\mu t}$, then

$$\begin{aligned}
W(s) &= \det \begin{bmatrix} e^{\lambda s} & e^{\mu s} \\ \lambda\, e^{\lambda s} & \mu\, e^{\mu s} \end{bmatrix} = (\mu - \lambda)\, e^{(\lambda+\mu)\,t}, \qquad \text{and} \\
y_p(t) &= -e^{\lambda t}\int_0^t \frac{e^{\mu s}\, f(s)}{(\mu-\lambda)\, e^{(\lambda+\mu)\,s}}\, ds + e^{\mu t}\int_0^t \frac{e^{\lambda s}\, f(s)}{(\mu-\lambda)\, e^{(\lambda+\mu)\,s}}\, ds \\
&= \left[\frac{1}{\mu-\lambda}\right]\int_0^t \left[e^{\mu t}e^{-\mu s} - e^{\lambda t}e^{-\lambda s}\right] f(s)\, ds.
\end{aligned}$$

■

Example 20. Consider

$$t^2\ddot{y} + t\,\dot{y} + \left(t^2 - \tfrac{1}{4}\right) y = t^{-1/2}\sin(t),$$

which has $y_1(t) = t^{-1/2}\cos(t)$ and $y_2(t) = t^{-1/2}\sin(t)$ as solutions of the homogeneous equation.

The Wronskian is

$$\begin{aligned}
W(t) &= \det \begin{bmatrix} t^{-1/2}\cos(t) & t^{-1/2}\sin(t) \\ -t^{-1/2}\sin(t) - \frac{1}{2}t^{-3/2}\cos(t) & t^{-1/2}\cos(t) - \frac{1}{2}t^{-3/2}\sin(t) \end{bmatrix} \\
&= t^{-t}\cos^2(t) - \tfrac{1}{2}t^{-2}\cos(t)\sin(t) + t^{-1}\sin^2(t) + \tfrac{1}{2}t^{-2}\cos(t)\sin(t) \\
&= t^{-1}.
\end{aligned}$$

If we write $y_p(t) = y_1(t)\,u_1(t) + y_2(t)\,u_2(t)$, then

$$\begin{aligned} u_1(t) &= \int_0^t -\frac{y_2(s)\,f(s)}{W(s)}\;ds = \int_0^t -s\left[s^{-1/2}\sin(s)\right]\left[s^{-1/2}\sin(s)\right]\;ds \\ &= \int_0^t -\sin^2(s)\;ds \\ &= -\tfrac{1}{2}t + \tfrac{1}{2}\sin(t)\cos(t). \end{aligned}$$

Also,

$$\begin{aligned} u_2(t) &= \int_0^t \frac{y_1(s)\,f(s)}{W(s)}\;ds = \int_0^t s\left[s^{-1/2}\cos(s)\right]\left[s^{-1/2}\sin(s)\right]\;ds \\ &= \int_0^t \sin(s)\cos(s)\;ds \\ &= \tfrac{1}{2}\sin^2(t). \end{aligned}$$

Therefore the general solution is

$$c_1 t^{-1/2}\cos(t) + c_2 t^{-1/2}\sin(t) - \tfrac{1}{2}t^{1/2}\cos(t) + \tfrac{1}{2}t^{-1/2}\sin(t)\cos^2(t) + \tfrac{1}{2}t^{-1/2}\,\sin^3(t).$$

■

5.2. **Undetermined Coefficients or Judicious Guessing.** Because the variation of parameters equation is so involved, it is often easier to guess a solution related to the time dependent forcing term.

Example 21. Consider

$$\ddot{y} - \dot{y} - 2y = 2e^t.$$

The characteristic equation is $0 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. Thus two solutions of the homogeneous equation are e^{2t} and e^{-t}. Therefore, the forcing term $2e^t$ is not a solution of the homogeneous equations. Therefore, we guess a solution of the form $y_p(t) = A\,e^t$ with an unspecified coefficient A. Taking the derivatives $y_p = y_p' = y_p'' = A\,e^t$ and substituting into the differential equation, we need

$$\begin{aligned} &A\,e^t - A\,e^t - 2\,A\,e^t = 2e^t, \qquad \text{so} \\ &-2A = 2, \qquad \text{or} \\ &A = -1. \end{aligned}$$

Thus a particular solution is $-e^t$ and the general solution is $-e^t + c_1 e^{2t} + c_2 e^{-t}$. ■

Example 22. Consider

$$\ddot{y} - \dot{y} - 2y = 2e^{-t}.$$

In this case, $2e^{-t}$ is a solution of the homogeneous equation so $A\,e^{-t}$ cannot be used to find a solution of the nonhomogeneous equation. So we try $y_p(t) = A\,t\,e^{-t}$. Then $\dot{y}_p(t) = A\,e^{-t} - At\,e^{-t}$ and $\ddot{y}_p(t) = -2Ae^{-t} + At\,e^{-t}$. Thus we need

$$\begin{aligned} 2e^{-t} &= A\left[-2e^{-t} + t\,e^{-t}\right] - A\,\left[e^{-t} - t\,e^{-t}\right] - 2A\,t\,e^{-t}, \\ &= -3\,A\,e^{-t}, \\ A &= -\frac{2}{3}. \end{aligned}$$

The general solution is $-\tfrac{2}{3}\,t\,e^{-t} + c_1 e^{2t} + c_2 e^{-t}$. ■

The following chart indicates what is a good guess for different forms of $f(t)$ in the equation $\ddot{y} + a\,\dot{y} + b\,y = f(t)$.

$f(t)$	$f(t)$ is not a solution of H	$f(t)$ is a solution of H
e^{rt}	$A\,e^{rt}$	$A\,t\,e^{rt}$ or $A\,t^2e^{rt}$
$a_0 + a_1t + \cdots + a_nt^n$	$A_0 + A_1t + \cdots + A_nt^n$	
$(a_0 + a_1t + \cdots + a_nt^n)e^{rt}$	$(A_0 + A_1t + \cdots + A_nt^n)e^{rt}$	
$\sin(\omega t)$ or $\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$	$At\sin(\omega t) + Bt\cos(\omega t)$
$(a_0 + a_1t)\sin(\omega t)$	$(A_0 + A_1t)\sin(\omega t) + (B_0 + B_1t)\cos(\omega t)$	
$(b_0 + b_1t)\cos(\omega t)$	$(A_0 + A_1t)\sin(\omega t) + (B_0 + B_1t)\cos(\omega t)$	

Example 23. Consider $\ddot{y} - \dot{y} - 2y = t + t^2$.

$$\begin{aligned} y_p(t) &= A_0 + A_1t + A_2t^2 \\ \dot{y}_p(t) &= A_1 + 2A_2t \\ \ddot{y}_p(t) &= 2A_2, \\ t + t^2 &= (2A_2 - A_1 - 2A_0) + t(\text{-}2A_2 - 2A_1) + t^2(\text{-}2A_2). \end{aligned}$$

We need

$$\begin{aligned} 0 &= 2A_2 - A_1 - 2A_0 \\ 1 &= \text{-}2A_2 - 2A_1 \\ 1 &= \text{-}2A_2. \end{aligned}$$

The solution is $A_2 = {}^{-1}\!/_2$, $A_1 = 0$, and $A_0 = {}^{-1}\!/_2$. A particular solution is $y_p(t) = \text{-}\frac{1}{2}\,t^2 - \frac{1}{2}$. ■

Example 24 (Periodically Forced Oscillator). Consider $m\,\ddot{y} + k\,y = F_0\,\cos(\omega\,t)$. Let $\omega_0 = \sqrt{k/m}$ be the frequency of the homogeneous equation. Assume that $\omega \neq \omega_0$, so $m\omega^2 - k \neq 0$.

$$\begin{aligned} y_p(t) &= A\,\cos(\omega\,t) + B\,\sin(\omega\,t) \\ \dot{y}_p(t) &= \text{-}A\omega\,\sin(\omega\,t) + B\omega\,\cos(\omega\,t) \\ \ddot{y}_p(t) &= \text{-}A\omega^2\,\cos(\omega\,t) - B\omega^2\,\sin(\omega\,t), \\ F_0\,\cos(\omega\,t) &= \cos(\omega\,t)\left[\text{-}Am\omega^2 + kA\right] + \sin(\omega\,t)\left[\text{-}Bm\omega^2 + kB\right]. \end{aligned}$$

So, $B = 0$ and $F_0 = A\,(k - m\omega^2) = Am\,(\omega_0^2 - \omega^2)$ or $A = {}^{F_0}\!/_{m\,(\omega_0^2 - \omega^2)}$. Thus, a particular solution is

$$y_p(t) = \frac{F_0}{m\,(\omega_0^2 - \omega^2)}\,\cos(\omega\,t).$$

with period of the forcing term ${}^{2\pi}\!/_{\omega}$. The general solution is obtained by adding the general solution of the homogeneous equation, $c_1\cos(\omega_0t) + c_2\sin(\omega_0t) + y_p(t)$.

The general solution of the homogeneous equation can be rewritten as

$$R\,\cos\left(\omega_0(t - \delta)\right),$$

where $R = \sqrt{c_1^2 + c_2^2}$, $c_1 = R\cos(\omega_0\delta)$, and $c_2 = R\sin(\omega_0\delta)$. The parameter R is the amplitude of the homogeneous solution; the quantity δ is the time at which the the solution of the homogeneous equation reaches its maximum. If $R \neq 0$ (i.e., $(c_1, c_2) \neq (0, 0)$), then the solution is a combination of solutions with periods ${}^{2\pi}\!/_{\omega}$ and ${}^{2\pi}\!/_{\omega_0}$. If ${}^{\omega}\!/_{\omega_0}$ is irrational, then the solution is not periodic. However, if $\dfrac{\omega}{\omega_0} = \dfrac{i}{j}$ (where i and j have no common factors), then the solution is T-periodic where $T = j \cdot \dfrac{2\pi}{\omega_0} = i \cdot \dfrac{2\pi}{\omega}$.

If $y(0) = 0 = \dot{y}(0)$, then $c_2 = 0$ and $c_1 = -A$, so

$$\begin{aligned} y(t) &= \frac{F_0}{m\,(\omega_0^2 - \omega^2)}\,[\cos(\omega\, t) - \cos(\omega_0 t)] \\ &= \frac{2\,F_0}{m\,(\omega_0^2 - \omega^2)}\, \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right). \end{aligned}$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega >> |\omega_0 - \omega|$ and the term $\sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right)$ oscillates much more rapidly than $\sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right)$. Thus, the motion is a rapidly oscillation by $\sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right)$ with a variable amplitude $\dfrac{2\,F_0}{m\,(\omega_0^2 - \omega^2)}\, \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right)$. See Figure 6.

If the initial condition is not $(y(0), \dot{y}(0)) \neq (0, 0)$, then the envelop does not pinch to zero. See Figure 7. ■

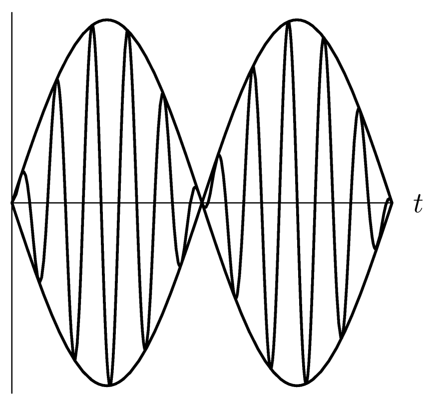

FIGURE 6. Example 24: Plot of solution and envelop $\pm\dfrac{2\,F_0}{m\,(\omega_0^2 - \omega^2)}\, \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right)$.

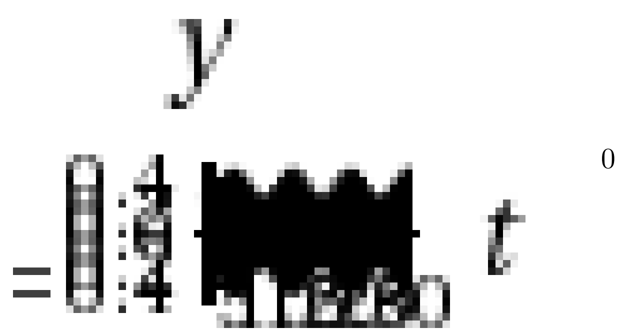

FIGURE 7. Example 24: Plot of solution with $y(0) = 0.25$ and $\dot{y}(0) = 0$

Example 25. Consider the *periodically force oscillator with damping*

$$m\,\ddot{y} + 2m\gamma\,\dot{y} + k\,y = F_0\,\cos(\omega t).$$

The characteristic equation for the homogeneous equation is $m\,\lambda^2 + 2m\gamma\,\lambda + k$, which has roots $-\gamma \pm i\,\sqrt{\omega_0^2 - \gamma^2}$ where ω_0 be the frequency of the undamped homogeneous equation, $\omega_0^2 = \dfrac{k}{m}$.

We assume that γ is small enough so that $\omega_0^2 - \gamma^2 > 0$, and set $\beta = \sqrt{\omega_0^2 - \gamma^2}$ so the general solution of the homogeneous equation is

$$c_1\, e^{-\gamma t} \cos(\beta t) + c_2\, e^{-\gamma t} \sin(\beta t).$$

Since this solution of the homogeneous equation goes to zero as t goes to infinity, any solution of the nonhomogeneous equation converges to the particular solution $y_p(t)$ of the nonhomogeneous equation, which we call the *steady-state solution.*

A particular solution $y_p(t)$ of the nonhomogeneous equation is of the form $A\,\cos(\omega t) + B\,\sin(\omega t)$. Differentiating and substituting into the equation, we need

$$F_0\,\cos(\omega t) = \cos(\omega t)\left[-Am\omega^2 + B2m\gamma\omega + Ak\right] + \sin(\omega t)\left[-Bm\omega^2 - A2m\gamma\omega + Bk\right].$$

The undetermined coefficients need so solve

$$\begin{aligned} \frac{F_0}{m} &= \left(\omega_0^2 - \omega^2\right) A + (2\gamma\omega) B \qquad \text{and} \\ 0 &= (\text{-}2\gamma\omega)\, A + \left(\omega_0^2 - \omega^2\right) B. \end{aligned}$$

Letting $\Delta = (\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2$,

$$\begin{aligned} \begin{bmatrix} A \\ B \end{bmatrix} &= \frac{1}{m} \begin{bmatrix} \omega_0^2 - \omega^2 & 2\gamma\omega \\ \text{-}2\gamma\omega & \omega_0^2 - \omega^2 \end{bmatrix}^{-1} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\ &= \frac{1}{m\Delta} \begin{bmatrix} \omega_0^2 - \omega^2 & \text{-}2\gamma\omega \\ 2\gamma\omega & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\ &= \frac{F_0}{m\Delta} \begin{bmatrix} \omega_0^2 - \omega^2 \\ 2\gamma\omega \end{bmatrix}. \end{aligned}$$

Thus, a particular solution is

$$y_p(t) = \frac{F_0(\omega_0^2 - \omega^2)}{m\Delta}\,\cos(\omega t) + \frac{F_0 2\gamma\omega}{m\Delta}\,\sin(\omega t).$$

This solution can also be written as

$$\begin{aligned} y_p(t) &= R\,\cos(\omega(t - \delta)) \qquad \text{where} \\ R &= \sqrt{\frac{F_0^2}{m^2\Delta^2}\left((\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right)} = \frac{F_0}{m\sqrt{\Delta}}, \\ \cos(\omega\delta) &= \frac{1}{R}\,\frac{F_0(\omega_0^2 - \omega^2)}{m\Delta} = \frac{\omega_0^2 - \omega^2}{\sqrt{\Delta}} \qquad \text{and} \\ \sin(\omega\delta) &= \frac{1}{R}\,\frac{F_0 2\gamma\omega}{m\Delta} = \frac{2\gamma\omega}{\sqrt{\Delta}}, \qquad \text{or} \\ \tan(\omega\delta) &= \frac{2\gamma\omega}{\omega_0^2 - \omega^2}. \end{aligned}$$

The ratio R/F_0 is the factor by which the amplitude of the forcing term is scaled in the steady-state solution. The quantity δ is the *time shift* that the maximum is shifted from the forcing term to the steady-state solution. The period T of the forcing satisfies $\omega T = 2\pi$, so $\dfrac{\delta}{T} = \dfrac{\omega\,\delta}{2\pi}$ is the fraction of the period of the time shift, and $\omega\,\delta = 2\pi\left(\dfrac{\delta}{T}\right)$ is the fraction of 2π and is called the *phase shift.*

If $m = k = 1$, $2\gamma = 0.125$, $F_0 = 3$, and $\omega = 0.3$, then $R \approx 3.27$ is greater than the forcing amplitude 3. Figure 8 shows the plot of the solution with $y(0) = 2$ and $\dot{y}(0) = 0$ together with the dashed plot of the forcing term. Notice that after the transient terms diminish, the solution has the same period as the forcing term with slightly larger amplitude and small phase shift.

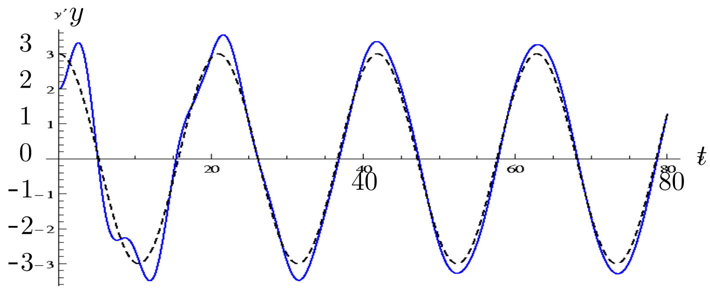

FIGURE 8. Damped forced oscillator and forcing term

For all the parameters fixed except for the forcing frequency ω, the amplitude of the response R is largest when Δ is smallest, so when

$$\omega_{\text{max}}^2 = \omega_0^2 - 2\,\gamma^2 \qquad \text{if} \;\; 2\,\gamma^2 < \omega_0^2.$$

The maximum response amplitude is

$$R_{\text{max}} = \frac{F_0}{2m\gamma\sqrt{\omega_0^2 - \gamma^2}}.$$

■

Problem 19. Consider the *resonance* case where the forcing has the same frequency as the natural oscillation, $m\,\ddot{x} + k\,x = F_0\,\cos(\omega_0)$ where $\omega_0 = \sqrt{k/m}$.

a. Find the general solution.
b. What happens to the solution as t goes to infinity?

Problem 20. In each of the following nonhomogeneous second order scalar equations, find the general solution.

a. $\ddot{y} - 2\dot{y} - 3y = 3e^{2t}$.
b. $\ddot{y} - 2\dot{y} - 3y = 3te^{2t}$.
c. $2\ddot{y} + 3\dot{y} + y = t^2 + 3\sin(t)$.
d. $\ddot{y} - 2\dot{y} + y = 5e^t$.

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