# Extra problems for
# Introduction to Dynamical Systems: Discrete and Continuous

by Clark Robinson

Published by Prentice Hall, 2004

**2.3.5** Find the general solution of

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

with $\mathbf{x}(0) = (1, 3)^T$ by using the variation of parameters formula.

**4.3.1** : An easier example to analyze is

$$\dot{x} = y - x^2$$
$$\dot{y} = x - y.$$

**4.3.3** Consider the system of differential equations

$$\dot{x} = -2x + y - x^3$$
$$\dot{y} = -y + x^2.$$

a. Determine the fixed points.
b. Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.
c. Using the information from parts (a) and (b), sketch by hand a rough phase portrait. Explain and justify your sketch.

**4.3.4** Consider the system of differential equations

$$\dot{x} = y - x^3$$
$$\dot{y} = -y + x^2.$$

a. Determine the fixed points.
b. Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.
c. Using the information from parts (a) and (b), sketch by hand a rough phase portrait. Explain and justify your sketch.

**4.5.5** Consider a competitive market of two commodities whose prices (against a numeraire) are $p_1$ and $p_2$. Let $E_i(p_1, p_2)$ be the excess demand (demand minus supply) of the $i^{\text{th}}$ commodity. For $i = 1, 2$, assume that the price adjustment is given by $\dot{p}_i = k_i E_i(p_1, p_2)$ for constants $k_i > 0$. Based on economic assumptions, Shone (on pages 353-357 of *Economic Dynamics*) gives the following information about the nullclines and vector field in the $(p_1, p_2)$-space: Both nullclines $E_i = 0$ have positive slope in the $(p_1, p_2)$-space and cross at one point $\mathbf{p}^* = (p_1^*, p_2^*)$ with both $p_i^* > 0$. The nullcline $E_1 = 0$ crosses from below $E_2 = 0$ to above at $\mathbf{p}^*$ as $p_1$ increases. The partial derivatives satisfy $\dfrac{\partial E_1}{\partial p_1} < 0$ and $\dfrac{\partial E_2}{\partial p_2} < 0$.

**a.** Draw a qualitative picture of the nullclines and give the signs of $\dot{p}_1$ and $\dot{p}_2$ in the different regions of the first quadrant of the $(p_1, p_2)$-space.

**b.** Argue that for any initial conditions $(p_{1,0}, p_{2,0})$ for prices with $p_{1,0}, , p_{2,0} >$ 0 have solutions that converge to $\mathbf{p}^*$.

**5.1.3** Consider the system of differential equations given by

$$\dot{x} = xy^2$$
$$\dot{y} = -4yx^2.$$

Show that there is a real-valued function $G(x, y)$ that is constant along trajectories. Hint: Consider $\dot{y}/\dot{x}$.

**5.2.5** Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = x^3 + x^2 - 2x.$$

(a) Find the potential function $V(x)$ and draw its graph.
(b) Find the fixed points of the system of differential equations and classify the type of each.
(c) Draw the phase portrait for the system of differential equations. Pay special attention to the location of the stable and unstable manifolds of the saddle fixed points.

**5.2.6** Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x),$$

where the potential function is given in Figure 1. Draw the phase portrait for the system of differential equations. Pay special attention to the location of the stable and unstable manifolds of the saddle fixed points.

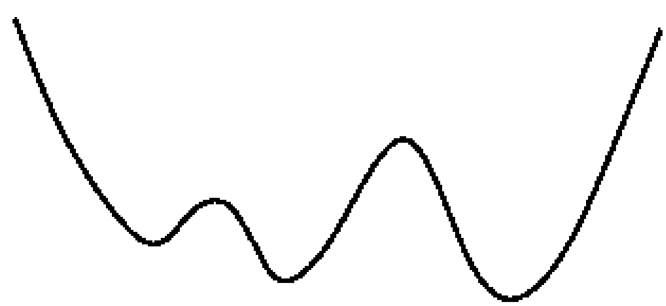

FIGURE 1. For Exercise 5.2.6

**6.2.9** Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -x - 2x^3 + y(4 - x^2 - 4y^2).$$

Show that the system has a periodic orbit. Hint: Find an "energy function" $L(x, y)$ by disregarding the terms involving $y$ in the $\dot{y}$ equation. Find the maximum and minimum of $L$ on the curve $\dot{L} = 0$. Use those values to find a region that is positively invariant without any fixed point.

**6.2.10** A special case of the Rowenzweig model for a predator-prey system had the following system of differential equations

$$\begin{aligned}\dot{x} &= x(6 - x) - y\,x^{\frac{1}{2}}\\ \dot{y} &= y\left(x^{\frac{1}{2}} - 1\right)\end{aligned}$$

**a.** Find all the fixed points.
**b.** Show that the fixed point $(x^*, y^*)$ with $x^*, y^* > 0$ is repelling and the other (two) fixed points are saddles.
**c.** Assume that every orbit in this first quadrant is bounded for $t \geq 0$, prove that there must be a limit cycle. *Note:* This model gives a more stable periodic limit cycle for a predator-prey system, than the periodic orbits for earlier models considered.

**6.4.5** Mas-Colell gave a Walrasian price and quantity adjustment that undergoes an Andronov-Hopf bifurcation. Let $Y$ denote the quantity produced and $p$ the price. Assume that the marginal wage cost is $\psi(Y) = 1 + 0.25\,Y$ and the demand function is $D(p) = -0.025\,p^3 + 0.75\,p^2 - 6p + 48.525$. The model assumes that the price and quantity adjustment are determined by the system of differential equations

$$\begin{aligned}\dot{p} &= 0.75\,[D(p) - Y]\\ \dot{Y} &= b\,[p - \psi(Y)]\,.\end{aligned}$$

**a.** Show that $p^* = 11$ and $Y^* = 40$ is a fixed point.
**b.** Show that the eigenvalues have zero real part for $b^* = 4.275$. Assuming the fixed point is weakly attracting for $b^*$, show that this is an attracting limit cycle for $b$ near $b^*$ and $b < b^*$.

**6.4.6** Consider the Holling-Tanner predator-prey system given by

$$\begin{aligned}\dot{x} &= x\left(1 - \frac{x}{6} - \frac{5y}{3(1+x)}\right)\\ \dot{y} &= \mu y\left(1 - \frac{y}{x}\right).\end{aligned}$$

**a.** Find the fixed point $(x^*, y^*)$ with $x^* > 0$ and $y^* > 0$.
**b.** Find the value of $\mu^* > 0$ for which the eigenvalues for the fixed point $(x^*, y^*)$ have zero real part.
**c.** Assuming the fixed point is weakly attracting for $\mu^*$, show that this is an attracting limit cycle for $\mu$ near $\mu^*$ and $\mu < \mu^*$.

**6.6.5** Consider the system of differential equations

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - y + x^2 + y^2.\end{aligned}$$

Using the scalar function $g(x, y) = e^{-2x}$ in the Dulac Criterion, show that the system does not have a periodic orbit.

**6.6.6** Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = x - 2x^3 - y,$$

with flow $\phi(t; \mathbf{x})$.

a. What is the divergence of the system?
b. If $\mathscr{D}$ is a region in the plane with positive area, what is the area of $\mathscr{D}(t)$ in terms of the area of $\mathscr{D}$?

**6.8.2** Consider the system of Van der Pol equations with parameter given by

$$\dot{x} = y + \mu\, x - x^3$$
$$\dot{y} = -x.$$

Show that the system as an Andronov Hopf bifurcation at $\mu = 0$. Is the bifurcation subcritical or supercritical? Is the periodic orbit orbitally attracting or repelling?

**6.8.3** (Fitzhugh-Nagumo model for neurons) The variable $v$ is related to the voltage and represents the extent of excitation of the cell. The variable is scaled so $v = 0$ is the resting value, $v = a$ is the value at which the neuron fires, and $v = 1$ is the value above which the amplification turns to damping. It is assumed that $0 < a < 1$. The variable $w$ denotes the strength of the blocking mechanism. Finally, $J$ is the extent of external stimulation. The Fitzhugh-Nagumo system of differential equations are

$$\dot{v} = -v(v - a)(v - 1) - w + J$$
$$\dot{w} = \epsilon(v - bw),$$

where $0 < a < 1$, $0 < b < 1$, and $0 < \epsilon \ll 1$. For simplicity, we take $\epsilon = 1$.

a. Take $J = 0$. Show that $(0, 0)$ is the only fixed point. (Hint: $4/b > 4 > (1 - a)^2$.) Show that $(0, 0)$ is asymptotically stable.
b. Let $(v_J, w_J)$ the the fixed point for a value of $J$. Using the fact that $J > 0$ lifts the graph of $w = -v(v - a)(v - 1) + J$, explain why the value of $v_J$ increases as $J$ increases.
c. Take $a = 1/2$ and $b = 3/8$. Find the value of $v^* = v_{J^*}$ for which the fixed point $(v_{J^*}, w_{J^*})$ is a linear center. Hint: Find the value of $v$ and not of $J$.
d. Keep $a = 1/2$ and $b = 3/8$. Using Theorem 6.4.4, show that there is a supercritical Andronov-Hopf bifurcation of an attracting periodic orbit. Remark: The interpretation is that when the stimulation exceeds this threshold, then the neuron becomes excited.

**7.1.3** The exercise repeats example 7.1.9. A better problem is as follows: Let $V(x) = x^6/6 - 5x^4/4 + 2x^2$, and $L(x, y) = V(x) + y^2/2$. Notice that $V'(x) = x^5 - 5x^3 + 4x$ and $0 = V'(0) = V'(\pm 1) = V'(\pm 2)$. Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x) - y\,[L(x, y) - L(1, 0)]\,.$$

a. Show that $\dot{L} = -y^2\,[L(x, y) - L(1, 0)]$.

b. Draw the phase portrait.
c. What are the attracting sets and attractors for this system of differential equations.

**7.1.5** Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = x - 2x^3 - y,$$

with flow $\phi(t; \mathbf{x})$. Let $V(x) = \frac{x^4 - x^2}{2}$ and $L(x, y) = V(x) + \frac{y^2}{2}$.

a. Show that $\mathscr{D} = L^{-1}((-\infty, 1])$ is a trapping region and $\mathbf{A} = \bigcap_{t \geq 0} \phi(t; \mathscr{D})$ is an attracting set. Why are $W^u(\mathbf{0})$ and $\pm\sqrt{1/2})$ contained in $\mathbf{A}$?
b. What is the divergence? What is the area of $\phi(t; \mathscr{D})$ in terms of the area of $\mathscr{D}$? What is the area of $\mathbf{A}$?

**7.2.2** The exercise repeats example 7.2.5. A better problem is as follows: Let $V(x) = x^6/6 - 5x^4/4 + 2x^2$, and $L(x, y) = V(x) + y^2/2$. Notice that $V'(x) = x^5 - 5x^3 + 4x$ and $0 = V'(0) = V'(\pm 1) = V'(\pm 2)$. Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x) - y\,[L(x, y) - L(1, 0)]\,.$$

Let $C_0 = L(1, 0)$.

a. Does the system have sensitive dependence on initial conditions at points in the set $L^{-1}(C_0)$? Explain your answer.
b. Does the system have sensitive dependence on initial conditions when restricted to the set $L^{-1}(C_0)$? Explain your answer.
c. Is the set $L^{-1}(C_0)$ a chaotic attractor?

**7.2.3** Consider the system of differential equations given by

$$\dot{\tau} = 1 + \frac{1}{2}\sin(\theta) \qquad (\bmod 2\pi)$$
$$\dot{\theta} = 0 \qquad (\bmod 2\pi).$$

Show that the system has sensitive dependence on initial conditions at all points.

**7.2.4** Let $V(x) = -2x^6 + 15x^4 - 24x^2$, for which $V'(x) = -12x^5 + 60x^3 - 48x$, $V'(0) = V'(\pm 1) = V'(\pm 2) = 0$, $V(0) = 0$, $V(\pm 1) = -11$, and $V(\pm 2) = 16$. Also, $V(x)$ goes to $-\infty$ as $x$ goes to $\pm\infty$. Let $L(x, y) = V(x) + y^2/2$.

a. Plot the potential function $V(x)$ and sketch the phase portrait for the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x).$$

Notice that $L(\pm 2, 0) > L(0, 0)$.

b. Sketch the phase portrait for the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x) - y\,L(x, y).$$

Pay special attention to the stable and unstable manifolds of saddle fixed points.

c. Let $\mathbf{A} = \{(x, y) \in L^{-1}(0) : -2 < x < 2\}$. Does the system of part (b) have sensitive dependence on initial conditions at points of the set $\mathbf{A}$? Explain why. Does it have have sensitive dependence on initial conditions when restricted to $\mathbf{A}$? Explain why.

d. What are the attracting sets and attractors for the system of part (b)?

**7.2.4** Consider the system of differential equations given by

$$\begin{aligned}
\dot{\tau} &= 1 + \frac{1}{2}\sin(\theta) \qquad (\text{ mod } 2\pi\ )\\
\dot{\theta} &= 0 \qquad\qquad\qquad\quad (\text{ mod } 2\pi\ )\\
\dot{x} &= y\\
\dot{y} &= x - 2x^3 - y\,(-x^2 + x^4 + y^2)
\end{aligned}$$

and the test function $L(\tau, \theta, x, y) = \dfrac{-x^2 + x^4 + y^2}{2}$.

a. Show that the system has sensitive dependence on initial conditions when restricted to $\mathbf{A} = L^{-1}(0)$.

b. Discuss why $\mathbf{A} = L^{-1}(0)$ is a chaotic attractor. Does it seems chaotic?

**7.7.2** Consider the system of differential equations given by

$$\begin{aligned}
\dot{\tau} &= 1 + \frac{1}{2}\sin(\theta) \qquad (\text{ mod } 2\pi\ )\\
\dot{\theta} &= 0 \qquad\qquad\qquad\quad (\text{ mod } 2\pi\ )\\
\dot{x} &= y\\
\dot{y} &= x - 2x^3 - y\,(-x^2 + x^4 + y^2).
\end{aligned}$$

Let $\Gamma = \{(\tau, \theta, x, y) : -x^2 + x^4 + y^2 = 0\ \}$.

a. For any point $(\tau_0, \theta_0, x_0, y_0)$ in $\Gamma$, explain why there are two Lyapunov exponents equal to zero and two nonzero Lyapunov exponents with one positive exponent.

b. Discuss why this example satisfies our definition of a chaotic attractor but does not pass the test for a chaotic attractor.

**11.2.7** Consider the map

$$f(x) = \begin{cases} \dfrac{x}{3} & \text{for } x \le 0\\ 3x & \text{for } 0 \le x \le 1\\ 3(2-x) & \text{for } 1 \le x \le 2\\ 3(x-2) & \text{for } 2 \le x \le 3\\ 2 + \dfrac{x}{3} & \text{for } 3 \le x. \end{cases}$$

**a.** Show that $f$ has a trapping region for an attracting set.

**b.** Show that the attracting set is a chaotic attractor (i.e., (i) show $f$ restricted to the attractor has sensitive dependence on initial condition, and (ii) f has a point whose limit set is the attractor).

**13.2.9** : What is the tranition graph for the map given in the Figure 2.

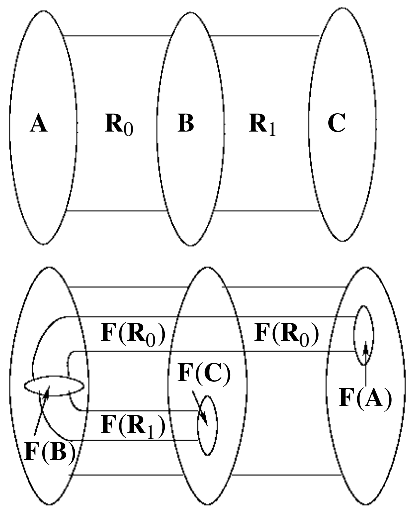


FIGURE 2. For Exercise 13.2.9