Math 313-1 **Final March 2003** **R.C. Robinson**

No books, no notes. Calculators are allowed.
Show all your work in your bluebook.
Show all your work in your bluebook. Start each problem on a new page.

1. (30 Points) Let

$$f(x) = -\frac{3}{2}\,x^2 + \frac{5}{2}\,x + 1.$$

Notice that $f(0) = 1$, $f(1) = 2$, and $f(2) = 0$ is a period-3 orbit.

a. Is the orbit $\mathcal{O}^+_f(0)$ attracting or repelling?

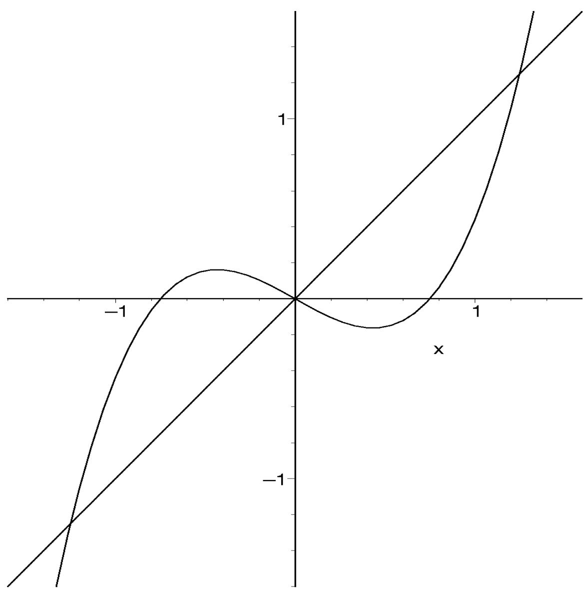
b. What is the Lyapunov exponent of $x_0 = 0$?

2. (50 Points) Let

$$f(x) = x^3 - \frac{9}{16}\,x.$$

Notice that

$$f'(x) = 3\,x^2 - \frac{9}{16} \geq -\frac{9}{16} > -1.$$



a. Find the fixed points and determine their stability type as attracting or repelling.

b. Find the critical points, where $f'(x) = 0$. Call the critical points $x_c^{\pm}$.

c. Show that for x in $[x_c^-, x_c^+]$, $|f'(x)| < 1$, so by the Mean Value Theorem

$$|f(x) - f(0)| < |x - 0|.$$

(If you do not see how to show this, you can still use this result in the next part of the problem.)

d. What is the basin of attraction of 0?

e. Show the Schwarzian derivative of f is negative. Note:

$$S_f(x) = \frac{f'''(x)\,f'(x) - \frac{3}{2}f''(x)^2}{f'(x)^2}.$$

3. (20 Points) Let

$$f(y) = 1 - \frac{3}{4}\, y^2 \quad \text{and} \quad g(x) = 3\, x\, (1 - x).$$

Show that $y = C(x) = 4\, x - 2$ is a conjugacy between f and g. Notice, that x is the variable for g and y is the variable for f.

4. (30 Points) Assume f is a continuous map on the real line which has a period-6 orbit $f(1) = 5$, $f(2) = 6$, $f(3) = 4$, $f(4) = 1$, $f(5) = 2$, and $f(6) = 3$. Label the interval between these points with symbols a through e by $\mathbf{I}_a = [1, 2]$, $\mathbf{I}_b = [2, 3]$, etc.

- **a**. Give the transition graph for these intervals.
- **b**. What periods are forced to exist by this transition graph? Notice that most orbits alternate between the intervals $[1, 3]$ and $[4, 6]$.

5. (20 Points) Let f be the map

$$f(x) = \frac{4}{\pi}\, \arctan(x).$$

Note that $f(0) = 0$, $f(1) = 1$, and $f(-1) = -1$. Give all the attracting sets for f. Which of these sets are attractors? Why are there no chaotic attractors? Remember that

$$f'(x) = \frac{4}{\pi\,(1 + x^2)}.$$

6. (50 Points) Let

$$f(x) = \begin{cases} 3 + \dfrac{3}{2}\,(x - 1) & \text{for } 0 \le x \le 1 \\ 3 - 2\,(x - 1) & \text{for } 1 \le x \le 2 \\ 1 + \left(\dfrac{1 + \sqrt{5}}{2}\right)\,(x - 2) & \text{for } 2 \le x. \end{cases}$$

- **a**. Show that $[1, 3]$ has a trapping region.
- **b**. Show that f has a Markov partition on $[1, 3]$. (Don't worry about the image of points outside $[1, 3]$ like $x = 0$ or $x = 4$.
- **c**. Is f topologically transitive on $[1, 3]$? If so, why?
- **d**. Does f have a chaotic attractor? If so, why?
- **e**. What estimate can you give for the Lyapunov exponents of orbits in $[1, 3]$ which do not pass through the points 1 and 2 where the derivative does not exist? Give an estimate like $A \le \ell(x_0; f) \le B$ where A and B are specific values.