Answers to Math 214-2 Common Final Fall 2002

1. Let $u = \sin x$, then $du = \cos x\, dx$. The Substitution Rule gives

$$\int e^{\sin x} \cos x\, dx = \int e^u\, du = e^u + C = e^{\sin x} + C$$

2. Integration by parts gives

$$\int x \tan^{-1} x\, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx.$$

If we perform long division, we get

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

Then

$$\int \frac{x^2}{1+x^2}\, dx = x - \tan^{-1} x + C$$

Combining these calculations, we have

$$\int x \tan^{-1} x\, dx = \frac{x^2+1}{2} \tan^{-1} x - \frac{x}{2} + C$$

3. We write the integrand as a sum of partial fractions:

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Multiplying both sides by $(x+1)(x+2)$, we obtain

$$x - 1 = A(x+2) + B(x+1) = (A+B)x + (2A+B)$$

Equating the coefficients, we have

$$A + B = 1 \quad \text{and} \quad 2A + B = -1$$

Thus $A = -2$, $B = 3$, and so

$$\begin{aligned} \int \frac{x-1}{x^2+3x+2}\, dx &= -2 \int \frac{1}{x+1}\, dx + 3 \int \frac{1}{x+2}\, dx \\ &= -2 \ln|x+1| + 3 \ln|x+2| + C \end{aligned}$$

4. Integration by parts gives

$$\begin{aligned}
&\int \sin(\ln t)\,dt \\
=\;& t\sin(\ln t) - \int t\cos(\ln t)\frac{1}{t}\,dt \\
=\;& t\sin(\ln t) - t\cos(\ln t) - \int t\sin(\ln t)\frac{1}{t}\,dt \\
=\;& t\sin(\ln t) - t\cos(\ln t) - \int \sin(\ln t)\,dt
\end{aligned}$$

Solving this equation for $\int \sin(\ln t)\,dt$ yields

$$\int \sin(\ln t)\,dt = \frac{t}{2}\big[\sin(\ln t) - \cos(\ln t)\big] + C$$

5. Let $u = \ln x$, then $du = \frac{1}{x}dx$. Thus the Substitution Rule gives

$$\int_2^t \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \ln u\Big|_{\ln 2}^{\ln t} = \ln(\ln t) - \ln(\ln 2)$$

As $t \to \infty$, $\ln(\ln t) \to \infty$. So the integral diverges.

6. Using Simpson's Rule with $n = 4$ and $\Delta x = 0.25$, we obtain

$$\begin{aligned}
\int_0^1 f(x)\,dx &\approx \frac{\Delta x}{3}\big[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\big] \\
&= \frac{1}{3}0.25\big[2 + 4 + 4 + 12 + 5\big] = \frac{9}{4}
\end{aligned}$$

7. We first need to find the points of intersection of the two curves. This gives $5x - x^2 = x$, or $x^2 - 4x = 0$. Thus $x = 0$ or 4. So the total area bounded by the curves is

$$A = \int_0^4 (5x - x^2 - x)\,dx = \int_0^4 (4x - x^2)\,dx = \frac{32}{3}$$

8. (a) $V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left[\left(\frac{\pi}{2}\right)^2 - \left(\sin^{-1}(\sqrt{x})\right)^2\right]\,dx$

(b) $V = \int_0^{\frac{\pi}{2}} 2\pi y(\sin y)^2\,dy$

9. By separation method, we get

$$\int \frac{dy}{y^2+1} = \int 3x^2\,dx$$

which yields $\tan^{-1} y = x^3 + C$. Then $y = \tan(x^3 + C)$. By the initial condition $y(0) = 1$, we have $1 = \tan C$. Thus $C = \pi/4$. Therefore the solution is

$$y = \tan(x^3 + \pi/4).$$

10. a) This is an alternating series. Note $1/\ln(\ln n) \to 0$ as $n \to \infty$. Let

$$f(x) = \frac{1}{\ln(\ln x)}.$$

Then

$$f'(x) = \frac{-\frac{1}{x\ln x}}{[\ln(\ln x)]^2} < 0$$

for $x \geq 3$. Then f is decreasing eventually. By alternation series test, the series is convergent.

b) By the definition of the absolutely convergence, we have

$$\sum \left|(-1)^n \cdot \frac{1}{\ln(\ln n)}\right| = \sum \frac{1}{\ln(\ln n)}.$$

Note $\ln x < x$ for $x \geq 3$. Then $\ln(\ln x) < \ln x$ for $x \geq 3$ since $y = \ln x$ is increasing. Then we see

$$\frac{1}{n} < \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}.$$

By the p-series test and comparison test, we see

$$\sum \frac{1}{\ln(\ln n)}$$

is divergent. So the given series is not convergent absolutely.

11. Let

$$a_n = \frac{(x+1)^n}{n \cdot 2^n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = |x+1|\,\frac{n}{2(n+1)} \to \frac{|x+1|}{2}$$

as $n \to \infty$. By the Ratio Test, the series converges if $|x+1| < 2$. Then the radius of convergence is 2.

12. Note

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_0^\infty (-1)^n x^n$$

for $|x| < 1$ and $[\ln(x+1)]' = 1/(1+x)$. Then

$$\begin{aligned} \ln(x+1) &= \int \frac{1}{1+x}\,dx \\ &= \int \sum_0^\infty (-1)^n x^n\,dx = C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots . \end{aligned}$$

To determine C, let $x = 0$ on both sides. Then $C = 0$. Thus

$$x\ln(x+1) = \sum_1^\infty (-1)^{n+1}\,\frac{x^{n+1}}{n}$$

for $|x| < 1$.

13. Note

$$[\ln(1-x)]' = -\frac{1}{1-x} = -\sum_0^\infty x^n \quad \text{for} \quad |x| < 1.$$

Then

$$-\ln(1-x) = \int \frac{1}{1-x}\,dx = C_0 + \sum_1^\infty \frac{x^n}{n}.$$

By letting $x = 0$, we get $C_0 = 0$. Thus

$$\ln(1-x) = -\sum_1^\infty \frac{x^n}{n}.$$

Then

$$\ln(1+x^5) = \ln[1-(-x^5)] = -\sum_1^\infty \frac{1}{n}(-x^5)^n = \sum_1^\infty (-1)^{n+1}\frac{1}{n}x^{5n}.$$

Therefore

$$\int \ln(1+x^5)\,dx = \int \sum_1^\infty (-1)^{n+1}\frac{1}{n}x^{5n} = C + \sum_1^\infty (-1)^{n+1}\frac{x^{5n+1}}{n(5n+1)}.$$

14. Note the derivatives of $y = \cos x$ repeat. Then we see the coefficients

$$f(\pi/3) = \frac{1}{2},\ f'(\pi/3) = -\frac{\sqrt{3}}{2},\ f''(\pi/3) = -\frac{1}{2},\ f^{(3)}(\pi/3) = \frac{\sqrt{3}}{2}$$

will repeat. Thus the Taylor series for the function is

$$f(x) = \sum_0^\infty (-1)^n \frac{1}{2}\frac{(x-\pi/3)^{2n}}{(2n)!} + \sum_0^\infty (-1)^{n+1}\frac{\sqrt{3}}{2}\frac{(x-\pi/3)^{2n+1}}{(2n+1)!}.$$

15. a)

$$\frac{dT}{dt} = k\,\frac{1}{T}.$$

b) By the separation method,

$$\int T\,dT = \int k\,dt.$$

Then

$$\frac{1}{2}T^2 = kt + C.$$

If we consider $T \geq 0$, then

$$T(t) = \sqrt{2kt + C}.$$

c) Let us assume the time is measured since the midnight. Then $T(0) = 1$ and $T(6) = 2$. Then we can determine the constants k and C. Thus $k = 1/4$ and $C = 1$. Now we want time t such that

$$T(t) = \sqrt{\frac{1}{2}t + 1} = 3.$$

Then $t = 16$.