SOLUTIONS OF THE REVIEW PROBLEMS

Problem 1.

a) Convergence. See example 3 in Sec 5.10.

b) Convergence. See example 8 in Sec 5.10.

c) Divergence. See example 7 in Sec 5.10.

d) Convergence.

$$\int_e^\infty \frac{dx}{x(\ln x)^2} = \int_1^\infty \frac{dt}{t^2}.$$

Note $\int_1^\infty \frac{dt}{t^2}$ is convergent by p-test ($p = 2 > 1$).

e) Divergence.

$$\int_0^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int_0^\infty e^u\, du = 2 \lim_{T\to\infty} \int_0^T e^u\, du = \infty.$$

f) Convergence

$$\int_1^\infty \frac{dx}{x+e^{2x}} \leq \int_1^\infty \frac{dx}{e^{2x}} = \lim_{T\to\infty} \int_1^T e^{-2x}\, dx = \frac{1}{2e^2}.$$

Problem 2.

$$V = \pi \int_0^{1/2} (14x)^2 - (7x)^2\, dx + \pi \int_{1/2}^1 49 - (7x)^2\, dx = \frac{49}{3}\pi.$$

Problem 3.
a) area is

$$\int_{-2}^4 (\frac{x}{2}+2) - \frac{x^2}{4}\, dx = 9.$$

b) Calculate the following things
i)

$$\pi \int_{-2}^4 \left\{ (\frac{x}{2}+2)^2 - \frac{x^4}{16} \right\} dx.$$

ii)

$$\int_{-2}^4 2\pi(5-x)(\frac{x}{2}+2-\frac{x^2}{4})\, dx.$$

iii)

$$\pi \int_{-2}^{4} \left\{ (\frac{x}{2}+3)^2 - (\frac{x^2}{4}+1)^2 \right\} dx.$$

Problem 4.

$$V = \int_0^3 2\pi(y+2)[(9-y^2)-(y^2-9)]\,dy = 2\pi(9y^2 - \frac{1}{2}y^4 + 54y - \frac{4}{3}y^3)\Big|_0^3 = 225\pi.$$

Problem 5. From the parametric equation $x = 2-3t$, $y = 7-6t$, we have $3t = 2-x$ and next plug it into $y = 7-6t$. Then we have $y = 7-2(3t) = 7-2(2-x) = 3+2x$. Similarly, from the parametric equation $x = t-1$, $y = 1+2t$, we have $t = x+1$ and next plug it into $y = 1+2t$. Then we have $y = 1+2t = 1+2(x+1) = 3+2x$. Thus these two equations produce the same line.

Problem 6. Arc length is

$$\int_0^3 \sqrt{1+(y')^2}\,dx = \int_0^3 \sqrt{1+(3\sqrt{x})^2}\,dx = \int_0^3 \sqrt{1+9x}\,dx = \frac{2}{27}(28\sqrt{28}-1).$$

Problem 7. $x^2+y^2 = 4^4$. So, the trajectory is (part of) the circle with radius 4 centered at the origin. When t varies from 0 to $\frac{2}{3}\pi$, the point (x, y) moves counter-clockwise starting and ending at (4,0). So, the length of the curve is 8π.

You may also use the formula for the arc length as follows.

$$\int_0^{\frac{2}{3}\pi} x'(t)^2 + y'(t)^2\,dt = \int_0^{\frac{2}{3}\pi} (12\sin 3t)^2 + (12\cos 3t)^2\,dt = \int_0^{\frac{2}{3}\pi} 12\,dt = 8\pi$$

Problem 8. The temperature function $T(x) = 4x$ has the average

$$\frac{\int_0^5 T(x)\,dx}{5-0} = 10\,(^\circ\mathrm{C})$$

over the interval $[0, 5]$.

Problem 9.

The total mass when the bucket is x meters above the bottom of the well is $0.5(30-x)+10$. So, the force at the instant is given to be

$$F(x) = 9.8\,(0.5(30-x)+10)\,.$$

Finally, the work required is

$$\int_0^{30} 9.8\,(0.5(30-x)+10)\,dx = 5145(\mathrm{J}).$$

Problem 10.

Let x be the height from the ground level. Then, part of the cone between the height x and $x+\Delta x$ can be approximated by a circular cylinder whose diameter is $100-x$ ft. and height is Δx. The work required to lift this slice from the ground level is $x \times 2 \times \pi(\frac{100-x}{2})^2 \Delta x$. Therefore, the work required to build the Cone is

$$\int_0^{100} 2x\pi \left(\frac{100-x}{2}\right)^2 dx$$

.

Problem 11.

The part of the gasoline between the height x and $x+\Delta x$ can be approximated by a thin rectangular plate with size $10 \times 2\sqrt{9-(3-x)^2} \times \Delta x = 20\sqrt{9-(3-x)^2}\Delta x$. So, the work required to pump the gas up is

$$\int_0^3 45(6-x)20\sqrt{9-(3-x)^2}\,dx$$

$$= \int_0^3 45(3+t)20\sqrt{9-t^2}\,dt$$

$$= 2700\int_0^3 \sqrt{9-t^2}\,dt + \int_0^3 900t\sqrt{9-t^2}\,dt$$

$$= 2700\frac{9\pi}{4} - 450(9-t^2)^{3/2}\Big]_0^3 = 6075\pi + 12150.$$

Problem 12.

(a) Find the total volume of this object.

$$\int_0^1 \pi(\sqrt{y})^2\,dy = \int_0^1 \pi y\,dx = \frac{\pi}{2}$$

(b) Suppose that the density of this object varies with height according to the rule $\rho(y) = 8(2-y)$ (grams per cubic centimeters) at height y, find the total weight of the object.

$$\int_0^1 \pi(\sqrt{y})^2 8(2-y)\,dy = \int_0^1 8\pi y(2-y)\,dy = \frac{16\pi}{3}$$

Problem 13.

(a) The percentage of students with scores between 50 and 70 is

$$\int_{50}^{70} p(x)dx = \frac{1}{2}(70-50)6\% = 60\%.$$

The percentage of students with scores between 80 and 90 is

$$\int_{80}^{90} p(x)dx = \frac{1}{2}(90-80)8\% = 40\%.$$

So there are more students with scores between 50 and 70.

(b)

$$\begin{aligned} P(X<50) &= \int_{-\infty}^{50} p(x)dx = 0\% \\ P(X>70) &= \int_{70}^{\infty} p(x)dx = \int_{80}^{90} p(x)dx = \frac{1}{2}(90-80)8\% = 40\% \\ P(50<X<70) &= \int_{50}^{70} p(x)dx = \frac{1}{2}(70-50)6\% = 60\% \end{aligned}$$

(c)

$$\begin{aligned} \text{mean } \mu &= \int_{-\infty}^{\infty} xp(x)dx \\ &= \int_{50}^{60} x\cdot\frac{0.06}{10}(x-50)dx + \int_{60}^{70} x\cdot(-\frac{0.06}{10})(x-70)dx \\ &\quad + \int_{80}^{85} x\cdot\frac{0.08}{5}(x-80)dx + \int_{85}^{90} x\cdot(-\frac{0.08}{5})(x-90)dx \\ &= 17+19+\frac{31.25}{3}+\frac{52}{3} = 63.75 \end{aligned}$$

By definition, the median m is a number such that

$$\int_{-\infty}^{m} p(x)dx = \int_{m}^{\infty} p(x)dx = 1/2.$$

Since $\int_{-\infty}^{70} p(x)dx = \int_{50}^{70} p(x)dx = 0.6$ and $\int_{50}^{60} p(x)dx = 0.3$, we conclude that m should be somewhere between 60 and 70 such that $\int_{60}^{m} p(x)dx = 0.5-0.3 = 0.2$. That is,

$$0.2 = \int_{60}^{m} (-\frac{0.06}{10})(x-70)dx = -0.006(\frac{1}{2}x^2-70x)\Big|_{60}^{m}$$

Hence we have the equation for m:

$$3m^2 - 420m + 14600 = 0.$$

Using the quadratic formula to solve this equation, we have $m = 70 \pm 10\sqrt{3}/3$. Since m should be between 60 and 70, we have $m = 70 - 10\sqrt{3}/3 \approx 64.23$.

Problem 14.

(a) $f(t)$ is indeed a probability density function for two reasons. First $f(t) \geq 0$ for all t; Secondly $\int_{-\infty}^{\infty} f(t)dt = \int_0^{\infty} ce^{-ct}dt = 1$. Here we assume $c > 0$.
(b) $\int_4^{\infty} f(t)dt = \int_4^{\infty} ce^{-ct}dt$.
(c) First we need to evaluate $\int_4^{\infty} ce^{-ct}dt$.

$$\int_4^u ce^{-ct}dt = -e^{-ct}\Big|_4^u = -(e^{-cu} - e^{-4c}).$$

Let $u \to \infty$, we have $\int_4^{\infty} = e^{-4c}$. Set $e^{-4c} = 0.2$ and solve for c. We have

$$c = \frac{\ln 0.2}{-4} \approx 0.4.$$

Problem 15.

$$\begin{aligned} \frac{dy}{dx} &= (x^2 - 1)y^2 \\ \int \frac{dy}{y^2} &= \int (x^2 - 1)dx \\ -\frac{1}{y} &= \frac{1}{3}x^3 - x + C \\ y &= \frac{-1}{\frac{1}{3}x^3 - x + C} \end{aligned}$$

Plugging in the initial condition $y(0) = 1$, we have $C = -1$.

Problem 16.

Let $P(t)$ be the size of the population at time t (in hours). Then

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad P = P(0)e^{kt}.$$

Initial count was 400 means $P(0) = 400$. When $t = 4$, $P = 25000$, that is

$$25000 = 400e^{4k} \quad \Rightarrow \quad k = \frac{\ln 62.5}{4} \approx 1.034.$$

Now set up the equation $800 = 400e^{1.034t}$ and solve for t. We have

$$t = \ln 2/1.034 \approx 0.52 \text{ hours } \approx 31 \text{ minutes.}$$

Problem 17.
(a) $\frac{dH}{dt} = k(8 - H), \quad H(0) = 27, \quad H(1) = 21.$
(b)

$$\int \frac{dH}{8 - H} = \int k dt \quad \Rightarrow \quad -\ln|8 - H| = kt + C \quad \Rightarrow \quad H = 8 + Ce^{-kt}.$$

$$H(0) = 27 \quad \Rightarrow \quad 27 = 8 + Ce^0 = 8 + C \quad \Rightarrow \quad C = 19.$$

$$H(1) = 21 \quad \Rightarrow \quad 21 = 8 + 19e^{-k} \quad \Rightarrow \quad k \approx 0.38.$$

That is, $H(t) = 8 + 19e^{-0.38t}$.
(c) Set up the equation $H(t) = 8 + 19e^{-0.38t} = 37$ and solve for t. We have

$$t = \frac{\ln(\frac{29}{19})}{-0.38} \approx -1.113 \text{ hours}.$$

That is, the corpse has been dead for approximately 1.113 hours at the moment of its discovery.

Problem 19. (I)-(c), (II)-(a), (III)-(b).