SOLUTIONS OF THE REVIEW PROBLEMS FOR THE FIRST MIDTERM

Problem 1.

a)

$$\begin{aligned}\int_1^4 (\sqrt{x} - \frac{1}{\sqrt{x}} + \frac{1}{x})\,dx &= \int_1^4 (x^{1/2} - x^{-1/2} + \frac{1}{x})\,dx \\ &= \frac{2}{3}x^{3/2} - 2\sqrt{x} + \ln|x|\Big|_1^4 = \frac{8}{3} + \ln 4.\end{aligned}$$

b) Let $u = \sqrt{p+2}$ ($du = \frac{1}{2\sqrt{p+2}}\,dp$). Thus

$$\int \frac{e^{\sqrt{p+2}}}{\sqrt{p+2}}\,dp = \int 2e^u\,du = 2e^{\sqrt{p+2}} + C.$$

c)

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = \int \frac{3x}{\sqrt{1-x^2}}\,dx + \int \frac{2}{\sqrt{1-x^2}}\,dx.$$

Note that in the second integral on the right hand side, $\sin^{-1} x$ is the antiderivative of $\int \frac{1}{\sqrt{1-x^2}}dx$. In the first integral on the right hand side, by substitution rule, letting $u = 1-x^2$, $du = -2xdx$. Then

$$\int \frac{3x}{\sqrt{1-x^2}}\,dx = \int -\frac{3}{2}\frac{1}{\sqrt{u}}\,du = -\frac{3}{2}\cdot 2\cdot\sqrt{u} = -3\sqrt{1-x^2}.$$

Thus the given integral is $-3\sqrt{1-x^2} + 2\sin^{-1} x + C$.

d) By substitution rule, let $\sqrt{t} = u$ ($dt = 2\sqrt{t}\,du$). Thus

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = 2\int \cos u\,du = 2\sin\sqrt{t} + C.$$

e) Note

$$\int \arcsin x\,dx = \int 1\cdot\arcsin x\,dx.$$

To apply the integration by parts, let $f = \arcsin x$, $g' = 1$. Then $f' = 1/\sqrt{1-x^2}$, $g = x$. Then

$$\int 1\cdot\arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx.$$

To evaluate the integral in the right hand side, let $u = 1-x^2$ ($-2x\,dx = du$). Then

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2\sqrt{u}}\,du = -\sqrt{u} + C = -\sqrt{1-x^2} + C.$$

Thus

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + C.$$

f) By partial fractions,

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

In determining the constants A, B, C, we have

$$Ax(x-1) + B(x-1) + Cx^2 = x + 1$$

which yields $A + C = 0, -A + B = 1, -B = 1$. Thus $A = -2, B = -1$, and $C = 2$. Then

$$\int \left\{ -\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \right\} dx = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.$$

g) Note

$$\int \ln(1 + x^2) \, dx = \int 1 \cdot \ln(1 + x^2) \, dx.$$

To use the integration by parts, let $f = \ln(1 + x^2)$, $g' = 1$. Then $f' = 2x/(1 + x^2)$, $g = x$. Then we have

$$\int 1 \cdot \ln(1 + x^2) \, dx = x \ln(1 + x^2) - \int \frac{2x^2}{1 + x^2} \, dx.$$

To evaluate the integral on the right hand side, we use the long division to use the partial fractions. Then

$$\int \frac{2x^2}{1 + x^2} \, dx = \int \left(2 + \frac{-2}{1 + x^2}\right) dx = 2x - 2\tan^{-1} x.$$

Thus the indefinite integral is

$$\int \ln(1 + x^2) \, dx = x \ln(1 + x^2) - 2x + 2\tan^{-1} x + C.$$

h) Let $x = 2\sin t$ with $-\pi/2 \leq t \leq \pi/2$. Then $dx = 2\cos t \, dt$. Then we have

$$\begin{aligned} \int \frac{2\cos t}{\sqrt{4 - 4\sin^2 t}} \, dt &= \int \frac{2\cos t}{2|\cos t|} \, dt \\ &= \int dt = t + C = \sin^{-1}(x/2) + C \end{aligned}$$

i) By substitution rule,

$$\int e^{2t} \cos(2t) \, dt = \frac{1}{2} \int e^x \cos x \, dx.$$

By the integration by parts, let $f = e^x$, $g' = \cos x$ which yield $f' = e^x$, $g = \sin x$. Then

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx = e^x(\sin x + \cos x) - \int e^x \cos x\,dx$$

after using one more integration by parts. Thus,

$$\begin{aligned}\frac{1}{2}\int e^x \cos x\,dx &= \frac{1}{2}\cdot\frac{1}{2}e^x(\sin x + \cos x) + C = \frac{1}{4}e^x(\sin x + \cos x) + C \\ &= \frac{1}{4}e^{2t}(\sin(2t) + \cos(2t)) + C.\end{aligned}$$

j) To use the integration by parts, let $f = \ln x$, $g' = x^3$. Then $f' = 1/x$, $g = x^4/4$. Thus

$$\begin{aligned}\int x^3 \ln x\,dx &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\,dx = \frac{1}{4}x^4[\ln x - 1/4] + C.\end{aligned}$$

k) We apply integration by parts three times to the original integral. Then the integral is equal to

$$\begin{aligned}&-(p^3 + 6p)\cos p + \int (3p^2 + 6)\cos p\,dp \\ =\ &-(p^3 + 6p)\cos p + (3p^2 + 6)\sin p - \int 6p \sin p\,dp \\ =\ &-(p^3 + 6p)\cos p + (3p^2 + 6)\sin p + 6p\cos p - \int 6\cos p\,dp \\ =\ &-(p^3 + 6p)\cos p + (3p^2 + 6)\sin p + 6p\cos p + 6\sin p + C.\end{aligned}$$

l) $\int (1 + \tan^2\theta)\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$.

m) Let $u = 9 - x^2$. Then $du = -2x\,dx$. So the integral is equal to

$$\int_9^8 -\frac{1}{2}\sqrt{u}\,du = -\frac{1}{3}u^{3/2}\Big]_9^8 = -\frac{16\sqrt{2}}{3} - 9.$$

n) The integral is equal to $\int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx$. Let $u = \cos x$ so that $du = -\sin x\,dx$. The the integral becomes

$$\begin{aligned}\int -(1 - u^2)u^2\,du &= \int (-u^2 + u^4)\,du \\ &= -u^3/3 + u^5/5 + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C\end{aligned}$$

o) The integral is equal to

$$\int \frac{\cos^4 x}{\cos^2 x}\,dx = \int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = x/2 + \sin(2x)/4 + C.$$

p) Using partial fractions, we set

$$\frac{x+9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}.$$

By multiplying both sides by $x(x^2+9)$, we have $x+9 = A(x^2+9) + (Bx+C)x = (A+B)x^2 + Cx + 9A$. Compare the coefficients and get $A = 1, B = -1, C = 1$. So the integral is equal to

$$\int \frac{1}{x} + \frac{-x+1}{x^2+9} = \int \frac{1}{x} - \frac{x}{x^2+9} + \frac{1}{x^2+9}$$

$$= \ln|x| - \frac{\ln(x^2+9)}{2} + \frac{1}{3}\tan^{-1}(\frac{x}{3}) + K,$$

where K is a constant.

q) Let $\theta^2 = t$. Then $2\theta\,d\theta = dt$. If $\theta = 0$, then $t = 0$ and if $\theta = 1$, then $t = 1$. We have

$$\int_0^1 \theta^3 e^{\theta^2}\,d\theta = \frac{1}{2}\int_0^1 te^t\,dt = \frac{1}{2}\Big\{te^t\Big|_0^1 - \int_0^1 e^t\,dt\Big\} = \frac{1}{2}.$$

r) Use the trigonometric substitution $x = 2\tan t$ with $-\pi/2 < t < \pi/2$. Then the integral becomes

$$\begin{aligned}\int \frac{1}{4\tan^2 t\sqrt{4\sec^2 t}}2\sec^2 t\,dt &= \int \frac{1}{(4\tan^2 t)(2\sec t)}2\sec^2 t\,dt \\ &= \frac{1}{4}\int \frac{\cos^2 t}{\sin^2 t}\frac{1}{\cos t}\,dt = \frac{1}{4}\int \frac{\cos t}{\sin^2 t}\,dt\end{aligned}$$

Make another substitution $u = \sin t$, then the integral becomes

$$\begin{aligned}\frac{1}{4}\int \frac{1}{u^2}\,du &= -\frac{1}{4u} + C = -\frac{1}{4\sin(\tan^{-1}(x/2))} + C \\ &= -\frac{1}{4\sqrt{x^2+4}/x} + C = -\frac{x}{4\sqrt{x^2+4}} + C\end{aligned}$$

Problem 2.

$$\int_0^5 f(t)\,dt = -\frac{7}{2},\ \int_5^6 f(t)\,dt = \frac{1}{2},\ \int_0^6 f(t)\,dt = -3,\ \int_0^6 |f(t)|\,dt = 4.$$

Problem 3. Let

$$g(x) = \int_{x^2}^{\pi} \frac{\sin t}{t}\, dt.$$

Since π is constant, we change the upper and lower limits to apply the fundamental theorem of calculus. Then

$$g(x) = -\int_{\pi}^{x^2} \frac{\sin t}{t}\, dt$$

which yield

$$g'(x) = -\frac{\sin(x^2)}{x^2} \cdot 2x = -\frac{2\sin(x^2)}{x}.$$

Problem 4.

(a) $\int_0^{60} r(t)dt$

(b) $L_6 = 10(5 + 6.2 + 7 + 7.6 + 8.2 + 8.7) = 427$ gallons (lower estimate),
$R_6 = 10(6.2 + 7 + 7.6 + 8.2 + 8.7 + 9) = 477$ gallons (upper estimate).

(c) $S_6 = \frac{10}{3}[5 + 4(6.2) + 2(7) + 4(7.6) + 2(8.2) + 4(8.7) + 9] = 448$

Problem 5.

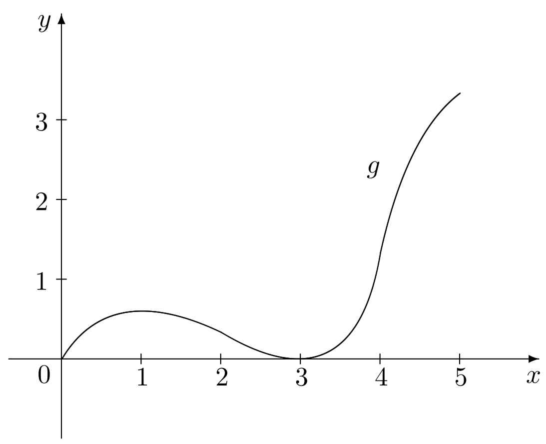
(a) According to the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Then g has a local max at $x = 1$ and local min at $x = 3$.

(b) g is increasing on $(0, 1)$ and $(3, 5)$ and decreasing on $(1, 3)$.

(c) g is concave upward on $(2, 4)$ and concave downward on $(0, 2)$ and $(4, 5)$.

(d) g attains its absolute max at $x = 5$.

(e)



Problem 6.

a) By the areas, the amount of oil at $t = 2$ is $10+(4*2+(2*(6-4))/2) = 20$, the amount of oil at $t = 3$ is $20 + (3 - 2) * 6/2 = 23$, and the amount of oil at $t = 5$ is $23 - (5 - 3) * 3/2 = 20$.

b) By the sign of the given function(rate of change of the amount of oil), there is the most oil at $t = 3$.

c) The graph is increasing on $(0, 3)$ and decreasing on $(3, 13)$. The graph is concave up on $(0, 2), (5, 7), (10, 13)$ and concave down on $(2, 5), (7, 10)$.