Math B14-3 | Final Exam | December 7, 1998

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This test has 8 question and 100 points; point values are listed after each problem. Show all your work in your blue book. No credit will be given without work. Your need not work in order, but clearly mark which question is where. **Put a box around your answers!**

1. Find the work done moving through the force field

$$\mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + x^2\mathbf{k}$$

along the the path consisting of the two line segments: one from $(2, -1, 3)$ to $(2, 2, -1)$ and one from $(2, 2, -1)$ to $(4, 2, -1)$. (12 points)

Answer: $-41/2$

2. Consider the surface $z = x^2 + y^2$.

a) Find a parametric equation for the line perpendicular to the tangent plane of this surface at a typical point $(a, b, a^2 + b^2)$.

b) Show this line intersects the z-axis. (12 points)

Answer: $(a+2at, b+2bt, a^2+b^2-t)$. When $t = -1/2$ this line is at $(0, 0, a^2+b^2+1/2)$ on the z-axis.

3. An observer stands at the end of a runway and watches a light plane take off. At a given instant, the line of sight from the observer to the plane is at an angle of 10° with the ground and increasing at a rate of 15° per second. Also, the distance between the plane and the observer along the line of sight is 1000m and decreasing at a rate of 100m/sec. How fast is the plane climbing? (13 points)

Answer: 240.45 m/sec

4. A surface in 3-space is defined by the equation

$$x^3 + y^3 + z^3 + xyz = 0.$$

a) Find $\partial z/\partial x$ and $\partial z/\partial y$.

b) Find the tangent plane to this surface at $(1, 1, -1)$.

c) Find an approximate value of c if $(1.1, 0.99, c)$ is on this surface. (13 points)

Answer: $\partial z/\partial x = \partial z/\partial y = -1/2$. The equation of the tangent plane is $x+y+2z = 0$. The approximate value for c is -1.045.

5. Suppose that the temperature (in degrees Celsius) at the point (x, y, z) in space and time t in minutes is given by the formula

$$T = 100 - e^{-t}(x^2 + y^2 + z^2).$$

Here t is

a) Find the rate of change of temperature at the point $P(3, -4, 5)$ is the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. Your answer will depend on time.

b) In what direction does T increase most rapidly at P? Given your answer as a unit vector. It will not depend on time. (12 points)

Answer: The gradient is

$$\nabla T = (-2xe^{-t}, -2ye^{-t}, -2ze^{-t}).$$

The directional derivative is $-(170/13)e^{-t}$. The direction of greatest increase is the unit vector in the direction of the gradient, which is

$$(-\frac{3\sqrt{2}}{10}, -\frac{2\sqrt{2}}{5}, \frac{-\sqrt{2}}{2}).$$

6. Estimate $\displaystyle\int_0^1 \frac{1 - \cos(x)}{x^2} dx$ to three decimal places. (12 points)

Answer: 0.486.

7. Consider a right triangle with sides of length x, y, and z respectively and fixed perimeter P. (See picture below). Show that this triangle has maximum area when it is an isoceles triangle; that is, when $x = y$. Note that there are two constraints: $x + y + z = P$ and $x^2 + y^2 = z^2$. (13 points)

This is a Lagrange multiplier problem. The method gives two answers, and you must decide which is a maximum and which is a minimum. You might as well minimize $f(x, y) = xy$. The three equations one gets are

$$y = \lambda_1 + 2\lambda_2 x$$
$$x = \lambda_1 + 2\lambda_2 y$$
$$0 = \lambda_1 - 2\lambda_2 z$$

Subtracting the second from the first gives $x = y$ or $\lambda_2 = -1/2$. In the second case the first equation gives $\lambda_1 = x + y$ and the last equation gives $x + y + z = 0$, which happens only if $x = y = z = 0$. This is the minimum.

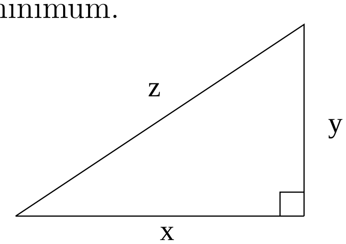

Figure for Problem 7

8. A wheel of radius 1 meter travels from left to right, turning at a rate of .25 revolutions per second. At time $t = 0$ it goes over a cliff 100m high (see drawing) and picks up a pebble in the tread as it does so.

a) Write down a parametric equation for the position of the center of the wheel at time t.

b) Write down a parametric equation for the position of the pebble at time t.

c) Find the slope of the tangent line to the path traced out by the pebble at $t = 2$ secs.

d) Is the pebble ever headed straight down? If so, when? (13 points)

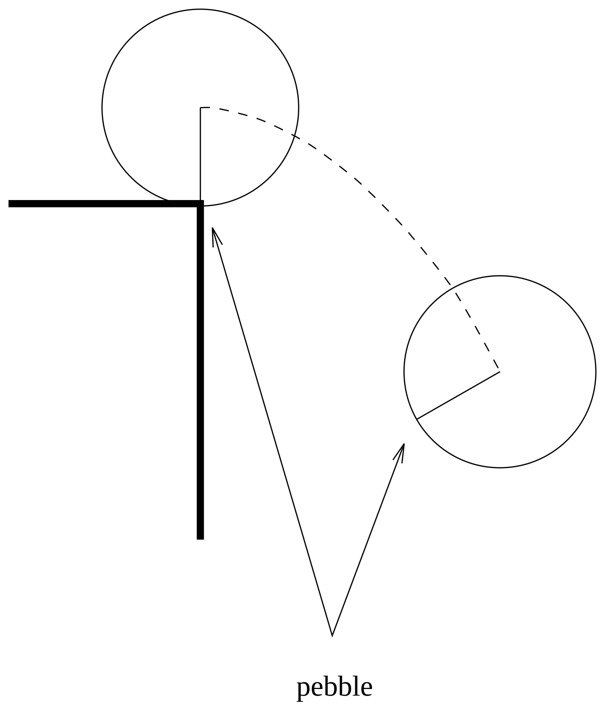

Figure for Problem 8

Answers: The parametric expression for the center is

$$(\pi t, 100.25 - 4.9t^2).$$

For the pebble the parametric expression is

$$(\pi t - .25 \sin(4\pi t), 100.25 - 4.9t^2 - .25 \cos(4\pi t)).$$

At $t = 2$, $dx/dt = \pi - \pi \cos(4\pi t) = 0$, so that tangent line is vertical. In, fact the pebble has vertical direction (and hence is headed straight down) when $dx/dt = 0$ and $dy/dt \neq 0$. We have $dx/dt = 0$ when t is $0, 1/2, 1, 3/2, 2$ etc. At $t = 0$, $dy/dt = 0$ and the pebble is instantaneously stationary.