Math B21 Final December 11, 1996 C. Robinson

1. (20 Points) Find the general solution (implicit if necessary, explicit if convenient) of $\frac{dy}{dx} = 3\sqrt{xy}$.

2. (25 Points) Consider the differential equation $\frac{dy}{dt} = 3y - y^2$.
   (a) Determine all the equilibria (critical points) and their stability or instability.
   (b) Determine where $3y - y^2$ is positive and where it is negative.
   (c) Use the above information to construct a sketch of typical solutions. Plot your solutions in (t, y) space. (You do not need to include the slope field or find the explicit solutions.)

3. (25 Points) Find the general solution of $\ddot{y} + 2\dot{y} + 5y = \sin(2t)$

4. (25 Points) Apply the **improved Euler method** with step size $h = 0.1$ and initial conditions $y(0) = 2$ to the system of equations $\frac{dy}{dt} = ty$ to calculate an approximation for $y(0.2)$

5. (21 Points) Consider the three equations

   $$\text{(i)} \qquad xy'' + (x+3)y' + (1-x^2)y = 0$$
   $$\text{(ii)} \qquad x^2y'' + (x+3)y' + (1-x^2)y = 0$$
   $$\text{(iii)} \qquad xy'' + x^2y' + x^4y = 0$$

   (a) For which of the three equations above is $x = 0$ an ordinary point, a regular singular point, or an irregular singular point?
   (b) Also indicate the form of the solution that is appropriate in each case.

6. (9 Points) The indicial equation for the equation

   $$2x^2y'' + 5xy' + (x-2)y = 0$$

   is $2r^2 + 3r - 2 = 0$. Is either solution defined at $x = 0$? For which r?

7. (25 Points) Consider the solution of the equation

   $$(1-x^2)y'' - 2xy' + 6y = 0$$

   near $x = 0$.
   (a) Find the recurrence relation for the coefficients.
   (b) Find the first three terms of the solution with $y(0) = 0$ and $y'(0) = 1$.

8. (25 Points) Consider the system of equations

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 0 \end{pmatrix} \mathbf{x}.$$

(a) Express the general solution of the above system of equations in terms of real-valued functions.

(b) Show that the solutions found in part (a) form a fundamental set of solutions (they are independent solutions).

9. (25 Points) Consider the linear system

$$\dot{\mathbf{x}} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 10 & -4 & -3 \end{pmatrix} \mathbf{x}.$$

The eigenvalues of the coefficient matrix are $\lambda = 3, -1, -1$. The eigenvector for the eigenvalue 3 is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

One eigenvector for the eigenvalue -1 is

$$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Find the general solution.