Mathematics B21-0 Final Examination March 18, 1991

Instructions: Do all work in the blue book

1. TRUE or FALSE (no proofs required)

a. The differential equation $y' = y^2$ has a solution defined for $-\frac{1}{2} < x < \frac{1}{2}$ and satisfying $y(0) = 1$.

b. The differential equation $y' = \sin(x^2)y + e^{x^3}\cos(x^2)$ has a solution defined for $-27 < x < 35$ and satisfying $y(0) = 16$.

c. If $y' = y^{\frac{5}{3}}$ and $y(0) = 0$, then $y(x) = 0$ for all x.

d. If $y' = y^{\frac{1}{3}}$ and $y(0) = 0$, then $y(x) = 0$ for all x.

e. If two solutions y_1, y_2 of the equation $y'' + e^x y' + (\sin x)y = 0$ satisfy $y_1y_2' - y_1'y_2 = 0$ for all x, then there are constants c_1, c_2, not both zero, so that $c_1y_1(x) + c_2y_2(x) = 0$ for all x.

f. If two differentiable functions y_1, y_2 satisfy $y_1y_2' - y_1'y_2 = 0$ for all x, then there are constants c_1, c_2, not both zero, so that $c_1y_1(x) + c_2y_2(x) = 0$ for all x.

g. If A is an $n \times n$ matrix with $\det A = 0$ then there is a non-zero vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{0}$.

h. If A is an $n \times n$ matrix with $\det A \neq 0$ then there is a non-zero vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{0}$.

i. If the 2×2 matrix A has both of its eigenvalues $r = 5$, then $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

j. If the 2×2 matrix has eigenvalues $r = 4$ and $r = 6$, then it is diagonalizable.

k. If the 2×2 matrix is symmetric ($a_{12} = a_{21}$) then it has two eigenvectors which are perpendicular.

2a. Find the general solution of the first-order linear equation $xy' + 5y = 3x$

b. Solve the initial-value problem with $y(1) = 2$.

c. On what interval is the solution valid?

3. a. Find the general solution of the 3×3 linear system of differential equations

$$(d/dt)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

b. Solve the initial-value problem with $y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$

c. Solve the inhomogeneous system $\mathbf{y}' = A\mathbf{y} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Any particular solution will suffice (A is the matrix from part a).

4a. Find the general solution of the equation $y'' + 8y' + 25y = 0$.

b. Find a particular solution of the equation $y'' + 8y' + 25y = e^{-4t}\cos 5t$

c. Solve the initial-value problem for the equation $y'' + 8y' + 25y = e^{-4t}\cos 5t$ with the initial conditions $y(0) = 0, y'(0) = 0$.

5. The equation $x^2y'' - 3xy' + 4y = 0$ is known to have the solution $y_1(x) = x^2$.

a. If y_2 is another linearly independent solution of this equation, find a formula for the Wronskian $W(y_1, y_2)$.

b. Find by any method a second solution y_2, linearly independent of y_1.

c. Find by any method a particular solution of the equation $x^2y'' - 3xy' + 4y = 5x$.

6. Consider the system of differential equations

$$x' = dx/dt = (x + 4 - y)(x - y)$$
$$y' = dy/dt = x(x + y - 6)$$

a. Find all of the critical points.

b. For each of the critical points, determine its type and stability.

c. On the basis of the information obtained in part b, carefully sketch the phase portrait in the **vicinity of each of the critical points**. (Do not attempt to sketch the global phase portrait)

GOOD LUCK