Math C37-1 Final Exam
December 10, 1998

1. (20 pts) Give the definition of *vector space* and *subspace.*

2. (25 pts) Let P_n denote the vector space of real polynomials of degree n or less. Let E denote the subset of polynomials which satisfy $f(x) = f(-x)$ for all x.
 a) Prove that E is a subspace of P_n.
 b) Define $T_1 : P_n \to P_n$ by $T_1(f)(x) = (f(x) + f(-x))/2$. Prove that $E = Image(T_1)$.
 c) Define $T_2 : P_n \to P_n$ by $T_2(f)(x) = (f(x) - f(-x))/2$. Prove that $E = ker(T_2)$.

3. (20 pts) Consider the set of vectors

$$S = \{(-1, 5, 0, 10), (0, 2, 3, 4), (-1, 2, -3, 4), (0, 1, 0, 2)\}$$

in R^4. Find a basis for the span of S.

4. (20 pts)
 a) Give the definition of *eigenvalue, eigenvector,* and *characteristic polynomial.*
 b) Give an example of a linear transformation $T : V \to V$ with the kernel and image of T having an intersection consisting of more than the zero vector.
 c) State the Cayley-Hamilton theorem.

5. (25 pts) Let P_3 be the vector space of all polynomials with real coefficients and degree 3 or less. Define the function $T : P \to P$ by $T(f) = f + f'$ where f' denotes the derivative of the polynomial f.
 a) Prove that the function T is linear.
 b) Find the matrix for T associated to the ordered basis $B = \{1, x, x^2, x^3\}$.
 c) Find all eigenvalues and eigenvectors of T.
 d) Find a polynomial $p(t)$ such that $p(T)$ is the zero transformation.

6. (25 pts) An $n \times n$ matrix A is called *idempotent* if A^k is the identity matrix for some $k > 0$.
 a) Prove that a real idempotent matrix has determinant ± 1.
 b) Prove that if an idempotent matrix has only real eigenvalues then those eigenvalues must be ± 1.
 c) Give an example of a 2×2 idempotent matrix which is not I, the identity matrix, or $-I$.
 d) Prove that if A is an $n \times n$ idempotent matrix with only real eigenvalues then A^2 is the identity matrix.

7. (20 pts) Find the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

8. (25 pts) Define *normal transformation.* Suppose A is an $n \times n$ complex matrix which is normal. The characteristic polynomial of such an A splits, so you may use this fact without proof. Prove that there is an orthonormal basis of C^n (using the standard inner product) which consists of eigenvectors of A. If you use Schur's theorem you must prove it.

9. (20 pts)

a) Define *unitary transformation.*

b) Suppose A is an $n \times n$ unitary matrix and x is an eigenvector of A with eigenvalue λ. If $f : C^n \to C$ is defined by $f(y) = \langle x, A^{-1}y \rangle$ prove that $f(y) = \lambda \langle x, y \rangle$.