Northwestern University

Math 224 SELF PLACEMENT TEST

Instructions:

This test is very much like a final exam in Math 224, the second quarter of the basic sequence of calculus at Northwestern University. It covers topics in integral calculus. If you can pass it, you would be well advised to begin your math courses with Math 230 or a more advanced course. Work out as much of this exam as you can. Give yourself up to 100 minutes. Work carefully, writing out your work and stating your answers clearly, just as if this was an ordinary (not self-placement) exam. Do not look at the answers until you are finished.

Grade the exam following the attached grading scheme. If you make a simple numerical error, but judge your method to be correct, you may allot yourself some partial credits. If your score is higher that 75 (out of 110) with a minimum of 35 points (out of 50) on **Part I** and a minimum of 26 points (out of 40) on **Part II** , you have passed. If your total score is between 65 and 75, you have nearly passed. You can either retake 224 or start 230. However in your decision, please bear in mind that you would have done better on the test if you had reviewed the materials. If you are still in doubt, consult the Director of Calculus in the Mathematics Department.

Topic List:

Techniques of Integration

Fundamental theorem of calculus, integration by substitution and by parts, trigonometric integrals & substitutions, partial fractions, numerical integration, improper integrals, differential equations and separable equations, exponential growth and decay

Applications of Integration

Areas, volumes, arc length, work or probability (in this placement test you can choose between a work or probability problem)

Sequences and Series:

Sequences, Series, Geometric Series, Power Series, Radius of Convergence, Taylor Series, Applications of Taylor Series.

Part I: Techniques of Integration (50 points)

1. (20 points) Evaluate the following integrals:

   (a) (3 pts.) $\int (3\,\cos(\pi t) + \cos(3\pi t))dt$

   (b) (3 pts.) $\int (7\sqrt{x} + \frac{2}{\sqrt{x}} - 1)dx$

   (c) (5 pts.) $\int_0^1 x^2\sqrt{1-x^3}\,dx$

   (d) (5 pts.) $\int_0^{\frac{\pi}{2}} \cos^3 x\ \ \sin x\,dx$

   (e) (4 pts.) $\int 2xe^x\,dx$

2. (30 points)

   (a) (5 pts.) Solve the differential euqation $\frac{dy}{dx} = \frac{1}{3y^2}$ that satisfies the initial condition $y_0 = 2$.

   (b) (7 pts.) Use the trigonometric substitution $x = 2\tan\theta$ to evaluate the integral: $\int \frac{dx}{x^2\sqrt{x^2+4}}$

   (c) (6 pts.) Evaluate the following integral if it converges: $\int_0^\infty \frac{dx}{1+x^2}$

   (d) (6 pts.) Use partial fractions to evaluate the integral: $\int \frac{dx}{x^2-1}$

   (e) (3 pts.) Compute the derivative $f'(x)$ where $f(x) = \int_1^x \frac{dt}{t}$.

   (f) (3 pts.) Use either Trapezoidal or Simpson's rule to approximate $\int_0^4 f(x)\,dx$.

x	0	1	2	3	4
f(x)	0	0	2	6	12

Part II: Applications of Integration (40 points)

3. (6 pts.) First sketch the region bounded by the given curves $y = x^2 - 1$ and $y = x + 1$ and then find the area of that region.

4. Consider the plane region R enclosed by the curves $y = x$ and $y = x^2$. **Set up and do not evaluate** the volume of the solid obtained by revolving R

   (a) (5 pts.) about the x -axis.

   (b) (5 pts.) about the y -axis.

   (c) (6 pts.) about the line $y = 2$.

5. (6 pts.) **Set up and do not evaluate** the integral for the arc length of the curve $y = 2x^{3/2}$ between $x = 0$ and $x = 3$.

6. (6 pts.) Calctown has a fixed population of $10,000$ people. In January, 1000 people have had the flu. Two months later, 2000 people have had it. Assume that the rate of increase of the number $N(t)$ of people who have had the flu is proportional to the number who have not had it; i.e. $\dfrac{dN}{dt} = k(10,000 - N)$. How many people will have had the disease five months later, i.e., in June?

Solve either problem 7 or problem 8, but not both!

7. (6 pts.) How much work is done to bring a load of ore to the surface if a miner uses a cable weighing 2 lb/ft to haul a 100 lb bucket of ore up a mine shaft 800 feet deep?

8. (6 pts.) The distribution of people's ages in the United States is essentially constant, or uniform, from age 0 to age 60, and from there decreases linearly until age 100. This distribution $p(x)$ is shown below, where x is age in years, and p measures probability density and $b = 1/80$.

   (a) According to this simplified model of the distribution of people's ages in the United States, what fraction of the population is between 50 and 100 years old?

   (b) Find the median age of the United States population.

Part III: Sequences and Series (20 points)

9. (4 pts.) Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{4 \cdot (-3)^n}{5^n}$.

10. (4 pts.) Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{3^n} x^n$.

11. (12 pts.)

    (a) (6 pts.) Find the Taylor Polynomial of degree 6 about $a = 0$ for $\sin(x^2)$. Hint: Use the Taylor Series for $\sin y$ about $a = 0$ to generate the Taylor Ploynomial for $\sin(x^2)$.

    (b) (6 pts.) Using the Taylor Polynomial from part (a) find an estimate of $\displaystyle\int_0^1 \sin(x^2)\, dx$.

Part I: Techniques of Integration (50 points)

1. (a) (3 pts.) $\int (3\,\cos(\pi t) + \cos(3\pi t))dt = \dfrac{3}{\pi}\sin(\pi t) + \dfrac{1}{3\pi}\sin(3\pi t) + c$

   (b) (3 pts.) $\int (7\sqrt{x} + \dfrac{2}{\sqrt{x}} - 1)dx = \dfrac{14}{3}x^{3/2} + 4\sqrt{x} - x + c$

   (c) (5 pts.) (substitution: $u = 1 - x^3$) $\int_0^1 x^2\sqrt{1-x^3}\,dx = -1/3\int_1^0 \sqrt{u}\,du = 2/9$

   (d) (5 pts.) (substitution: $u = \cos x$) $\int_0^{\frac{\pi}{2}} \cos^3 x\ \sin x\,dx = -\int_1^0 u^3\ du = 1/4$

   (e) (4 pts.) (integration by parts) $\int 2xe^x\,dx = 2xe^x - 2e^x + c$

2. (30 points)

   (a) (5 pts.) First separate the variables x and y: $\int 3y^2dy = \int dx$, thus $y^3 = x + c$. Since $y_0 = 2$, it follows that $c = 2$ and therefore $y = \sqrt[3]{x+2}$.

   (b) (7 pts.) (trig substitution $x = 2\tan\theta$ followed by the substitution $u = \sin\theta$) $\int \dfrac{dx}{x^2\sqrt{x^2+4}} = \int \dfrac{\cos\theta}{4\sin^2\theta}\,d\theta = \dfrac{1}{4u} + c = \dfrac{1}{4\sin\theta} + c = \dfrac{\sqrt{x^2+4}}{x} + c$

   (c) (6 pts.) $\int_0^\infty \dfrac{dx}{1+x^2} = \lim_{b\to\infty} \arctan b = \dfrac{\pi}{2}$

   (d) (6 pts.) $\int \dfrac{dx}{x^2-1} = 1/2\int \dfrac{1}{x-1} - 1/2\int \dfrac{1}{x+1} = 1/2\ln|x-1| - 1/2\ln|x+1| + c$

   (e) (3 pts.) $f'(x) = d/dx \int_1^x \dfrac{dt}{t} = 1/x$ (Fundamental Theorem of Calculus)

   (f) (3 pts.) $\int_0^4 f(x)\,dx \approx 1/2(0 + 0 + 2\cdot 2 + 2\cdot 6 + 12) = 14$ (Trapezoidal rule)

   $\int_0^4 f(x)\,dx \approx 1/3(0 + 0 + 2\cdot 2 + 4\cdot 6 + 12) = 40/3$ (Simpson's rule)

Part II: Applications of Integration (40 points)

3. (6 pts.) The region bounded by $y = x^2 - 1$ and $y = x + 1$:

   Area $= \int_{-1}^{2} (x+1) - (x^2 - 1)\, dx = 27/6$

4.

   (a) (5 pts.) (Washer with outer radius $R = x$ and inner radius $r = x^2$)

   Volume $= \int_0^1 \pi(x^2 - x^4)\, dx$

   (b) (5 pts.) (Shell of radius $r = x$ and height $h = x - x^2$)

   Volume $= \int_0^1 2\pi x(x - x^2)\, dx$

   (c) (6 pts.) (Washer with outer radius $R = 2 - x^2$ and inner radius $r = 2 - x$)

   Volume $= \int_0^1 \pi((2 - x^2)^2 - (2 - x)^2)\, dx$

5. (6 pts.) Arc length $= \int_0^1 \sqrt{1 + 9x}\, dx$

6. (6 pts.) Solve the differential equation: $\frac{dN}{dt} = k(10{,}000 - N)$, $N_0 = 1000$ &$N_1 = 2000$

   $|10{,}000 - N| = Ce^{-kt}$

   $N_0 = 1000$ implies $C = 9{,}000$ and $N_1 = 2000$ implies $k = \ln(3\sqrt{2}/4)$ thus five month later $N = 10{,}000 - 9{,}000(2/3\sqrt{2})^5 \approx 3296$ people will have had the disease.

7. (6 pts.) Work on bucket $= 80{,}000$ foot pounds, work on rope $= \int_0^{800} 2(800 - h)\, dh =$ $640{,}000$ foot pounds; altogether $720{,}000$ foot pounds of work is done to bring the load of ore to the surface.

8. (6 pts.) (a) Fraction of the population between 50 and 100 years old = area under the density graph for $50 \leq x \leq 100 = \int_{50}^{100} \rho(x)\, dx = 3/8 = 37.5\%$

   (b) If M = median age of the United States population then $\int_0^M \rho(x)\, dx = 1/2$ thus $M = 40$.

Part III: Sequences and Series (20 points)

9. (4 pts.) $\sum_{n=0}^{\infty} \frac{4 \cdot (-3)^n}{5^n} = \frac{4}{1-(-3/5)} = \frac{5}{2}$

10. (4 pts.) To find the radius of convergence we use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1} \cdot x^{n+1}|}{|a_n \cdot x^n|} = \lim_{n\to\infty} \frac{2(n+1)+1}{3^{n+1}} \cdot |x^{n+1}| \cdot \frac{3^n}{2n+1} \cdot \frac{1}{|x^n|} = \frac{1}{3}|x|$$

The series converges for $\frac{1}{3}|x| < 1$, thus $|x| < 3$. The radius of convergence is thus 3.

11. (12 pts.)

(a) $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$

$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$

$P_6(x) = x^2 - \frac{x^6}{6}$ (Taylor polynomial of degree 6)

(b) $\int_0^1 \sin(x^2)\, dx \approx \int_0^1 (x^2 - \frac{x^6}{6})\, dx = \frac{13}{42}$